Aleksandr I. Veksler

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FUNCTIONAL CHARACTERISTICS OF P'-SPACES

A.I. VEKSLER, Leningrad

Abstract: A topological space T is said to be P-space (resp. P'-space) iff $t \in \operatorname{int} E$ (resp. $t \in \operatorname{cl} \operatorname{int} E$) for any $t \in T$ and G_δ-set $E \ni t$. N. Onuchic [1] - K. Iseki [2] theorem states that T is P-space iff a pointwise limit of any sequences of real-valued continuous functions on T is a real-valued continuous function on T. In this paper there are given the functional characteristics of P'-spaces.

Key words: P-space, P-point, P'-point, P'-space, upper (lower) semicontinuous function.

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All the considered spaces are supposed to be completely regular. We recall that a point $t \in T$ is a P-point [3],[4], iff $t \in \operatorname{int} E$ for any G_δ-set $E \ni t$. A space T is P-space iff any point in T is a P-point. A point $t \in T$ is a P'-point [5] iff $t \in \operatorname{cl} \operatorname{int} E$ for any G_δ-set $E \ni t$. A space T is P'-space iff any point in T is a P'-point.

P'-spaces have a good deal of significant properties. For instance, in any P'-space, any meager set is nowhere dense and a non-empty open set cannot be covered by a family of $\aleph_1$ nowhere dense sets. If B is compact P'-space and the weight of B is $\aleph_1$, then B contains P-points. The most important case of compact P'-space is $\beta N \setminus N$; the corresponding results for $\beta N \setminus N$ were obtained by I.I. Parovichenko

[6] and W. Rudin [7]. Some topological characteristics of P'-spaces were studied in [5]. Besides in [5] using properties of the vector lattice C(B), some characteristics of a compact P'-space B were presented.

Note that the class of P'-spaces is much wider than the one of P-spaces. Any compact P-space is finite, whereas all $\beta D \setminus D$ (for discrete D), all one-point compactifications αD of uncountable discrete D, all $\beta T \setminus T$ (for locally compact, realcompact, but not compact T), all the boundaries of zero-sets in compact F-spaces (in particular all nowhere dense zero-sets in basically disconnected compact spaces) are compact P'-spaces.

Let f be an extended real-valued function on T. Let

$$f_{min}(t) = \sup_{G(t)} \inf_{t' \in G(t)} f(t')$$

$$f_{max}(t) = \inf_{G(t)} \sup_{t' \in G(t)} f(t')$$

(where $\{G(t)\}$ is the family of all the open neighbourhoods of the point t). A function f is said to be lower (upper) semicontinuous iff $f = f_{min}$ (resp. $f = f_{max}$). f is normally lower (upper) semicontinuous iff $f = (f_{max})_{min}$ (resp. $f = (f_{min})_{max}$).

<u>Theorem</u>. For any completely regular space T the following conditions are equivalent:

1) T is P'-space;

2) if $\{f_n\}$ is a sequence of real-valued continuous functions on T and f is its pointwise limit, then

$$(f_{max})_{min} \leq f \leq (f_{min})_{max};$$

3) if $\{f_n\}$ is an increasing (resp. decreasing) sequence of real-valued continuous functions, then its pointwise limit f is a normally lower (resp. upper) semicontinuous function.

Proof. 2) $\Longrightarrow$ 3). Let $f(t) = \lim f_n(t)$ and $\{f_n\}$ is increasing. Then $f(t) = \sup f_n(t)$ and f is lower semicontinuous (cf. [8]), i.e. $f = f_{min}$. It means $(f_{max})_{min} \geq f_{min} = f$; 2) implies $(f_{max})_{min} = f$. Therefore 3) holds.

3) $\Longrightarrow$ 1). Let us suppose that T is not a P'-space. In virtue of [5] there is a nowhere dense zero-set E. Let $E = \bigcap \{G_n : n \in N\}$, where G_n are open and decreasing, and $t_0 \in E$. Then let us construct a sequence $\{f_n\}$ of increasing continuous functions on T such that

$$f_n(T \setminus G_n) = \{1\},\ f_n(t_0) = 0 \text{ and } 0 \leq f_n(t) \leq 1 \quad (t \in T).$$

Let $f(t) = \lim f_n(t)$. Then $f(t_0) = 0$, $f(T \setminus E) = \{1\}$, but $(f_{max})_{min}(t) = 1$ for any $t \in T$. It means $(f_{max})_{min} > f$.

1) $\Longrightarrow$ 2). Let T be a P'-space, $f(t) = \lim f_n(t)$. Let us fix up a point $t \in T$. Then

$$\forall \varepsilon > 0\ \exists n_0 \in N\ \forall n \geq n_0\ \exists G_n(t)\ \forall t' \in G_n(t)\ [f_n(t') \leq f(t) + \varepsilon].$$

Let $G_0 = \operatorname{int} \bigcap \{G_n(t)\} : n \in N$. Since t is a P'-point, then $t \in \operatorname{cl} G_0$ and $f_n(t') \leq f(t) + \varepsilon$ for all $n \geq n_0$, $t' \in G_0$.

It means $f(t') \leq f(t) + \varepsilon$ and $f_{max}(t') \leq f(t) + \varepsilon$. Since $(f_{max})_{min}(t) = \sup\limits_{G(t)} \inf\limits_{t' \in G(t)} f_{max}(t')$ and $t \in \operatorname{cl} G_0$, then $G(t) \cap G_0 \neq \emptyset$ and $\inf\limits_{t' \in G(t)} f_{max}(t') \leq f(t) + \varepsilon$. It implies $(f_{max})_{min}(t) \leq f(t) + \varepsilon$ and $(f_{max})_{min}(t) \leq f(t)$, $(f_{max})_{min} \leq f$. Likewise, $(f_{min})_{max} \geq f$. It means 2) holds.

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Кафедра математики

ЛИТЛП им. С.М. Кирова

СССР, 191065, Ленинград

ул. Герцена 18

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