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THE INSERTION OF G_δ SETS AND FINE TOPOLOGIES

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Abstract: A simple proof of the Fuglede's theorem asserting that any finely continuous function in an abstract harmonic space is of the first class of Baire is given. Some applications of our method to the density topology are also exhibited.

Key words: Functions of Baire class one, fine topology in potential theory, density topology.

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In 1974, B. Fuglede proved that any finely continuous function in an abstract harmonic space is of the first class of Baire (see [2]). The simplified proof of this assertion suggests certain procedure described below as "the method of insertion of G_δ sets".

Let us consider any abstract $\mathfrak{P}$-harmonic space. By this we mean a locally compact space X with countable base equipped with a sheaf of so called hyperharmonic functions and satisfying certain axioms (see, e.g.,[1]). The fine topology on X is defined as the coarsest topology on X which is finer than the initial topology and which makes any hyperharmonic function on X continuous. For any set $A \subset X$ there is always a set of type G_δ (in the initial topology) containing the fine interior of A and contained in the fine closure of A. Indeed, if we denote by b(A) the set of all points of X where A is not thin, then b(A) has all these properties. Now, let f be a function continuous in the fine topology, and let c be a real. Then

$$\{x \in X;\ f(x) \geqq c\} \subset \{x \in X;\ f(x) > c - n^{-1}\} \subset b\{x \in X;\ f(x) > c - n^{-1}\} \subset \{x \in X;\ f(x) \geqq c - n^{-1}\},$$

and, therefore

$$\{x \in X;\ f(x) \geqq c\} = \bigcap_{n=1}^{\infty} b\{x \in X;\ f(x) > c - n^{-1}\}$$

is a G_δ set. Similarly $\{x \in X;\ f(x) \leqq c\}$ is of type G_δ. Thus, f is of the Baire class one in the initial topology.

The just explained idea can be generalized as indicated in the following theorem.

Theorem. Given a metric space (P, ρ) equipped with another topology τ assume that for any subset A of P there is a set A^* of type G_δ satisfying

τ-interior of $A \subset A^* \subset \tau$-closure of A.

Then any τ-continuous function on P is of the Baire class one.

As we stated above, the fine topology on any harmonic space has the mentioned property. It is not difficult to prove that also the ordinary density topology on an euclidean space R^k fulfils the assumptions of the theorem. In fact, given any set $A \subset R^k$, the set

$A^* = \{x \in R^k$; for any natural n there is $m > n$ such that $\dfrac{\mu(A\cap K(x,m^{-1}))}{\mu K(x,m^{-1})} > \dfrac{1}{2}$ }

(μ denotes the outer Lebesgue measure and $K(a,r)$ is open ball with center a and radius r) is of type G_δ and it is "inserted" between the density-interior and the density-closure of A.

Using the similar ideas on insertion of G_δ sets combined with the Jarník-Snyder method, it can easily be proved, for example, that any approximate derivative (possibly infinite) is of the Baire class one.

To close this short note we give a negative answer to one problem possed by F.D. Tall. Even though the density topology on the real line is completely regular, it is not normal. On the other hand, any two disjoint countable sets can be separated by open sets in density topology. F.D. Tall in [3], p. 279 asked for the "pseudonormality" of the density topology, i.e. if disjoint closed sets, one of which is countable, can be separated by disjoint open sets. We construct an example that this is not the case.
Let F_1 be a G_δ residual subset of R of measure zero, and let $F_2 \subset R \setminus F_1$ be countable and dense in R. Suppose that there are disjoint open sets (in density topology) $F_1 \subset G_1$, $F_2 \subset$ $\subset G_2$. As mentioned above, there is a G_δ set F^* inserted between G_2 and density closure of G_2. Thus, F^* is a residual subset of R disjoint with F_1, which is a contradiction.

The details and more informations can be found in our paper "When finely continuous functions are of the first class of Baire".

References

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