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A NOTE ON CLOSED CATEGORIES

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Abstract: For an adjoint situation

$$\mathcal{V}_0(A \otimes B, C) \approx \mathcal{V}_0(A, [BC])$$

in a category $\mathcal{V}_0$, the paper gives a description in terms of the left adjoint $\otimes$ of those closed category structures in the sense of Eilenberg-Kelly on $\mathcal{V}_0$ that have $[-,-]$ for the internal hom-functor. It turns out that $\otimes$ need not really be (even up to an isomorphism) associative.

Key-Words: Adjoint situation, closed category, internal hom-functor, natural associativity.

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Introduction. Although the concept of a closed category is the minimal one of the enrichments of category theory treated in [1], it already provides enough framework for some interesting applications (the study of $\mathcal{V}$-categories, $\mathcal{V}$-functors, etc.). Of course, it facilitates the calculus considerably if the internal hom-functor

$$[-,-] : \mathcal{V}_0^* \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$$

has a left adjoint

$$\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$$

Nevertheless, once an adjoint to $[-,-]$ is considered it is always required to be (up to a specified natural isomorphism) associative.

Since there exist closed categories in which the internal hom-functor has a non-associative adjoint (an example will be given in Section 2), we can ask what it is on the side of $\otimes$ that exactly corresponds to an extension of $[-,-]$ to a closed category structure on $\mathcal{V}_0$.

To settle this question we first analyze, on similar lines as in [1], Chapter II, §§ 3, 4, the interaction between properties of $\otimes$ and those of $[-,-]$ induced by the adjunction. This time, however, we shall emphasize whatever independence there is between individual couples of corresponding data or axioms and we shall go as far as possible without normalization of the couple $\langle \mathcal{V}_0, [-,-] \rangle$. As for the statements 1.2, 2.2, 3.2, and 3.3 of Section 1, this results in a certain restriction on the proof techniques available and the proofs are, consequently, longer than those in [1]. Because, on the other hand, their complexity is due only to complexity of the calculus involved and they are based on a very simple idea, we shall illustrate the idea by carrying out one of the proofs in question and omit all others. The proofs of 1.1, 2.1, and 3.1 will be also omitted - the reader can be referred to [1], Chapter II, Lemma 3.1.

Convention: The identity morphism of an object A will often be also denoted by A. We denote by $\bar{f}$ the inverse of an isomorphism f.

We shall constantly refer to diagrams MC1 - MC4 on p. 472 and to diagrams CC1 - CC4 on p. 429 in [1]. When we say, for instance, that some transformations a and ℓ satisfy MC1 it means that every diagram of the sort labelled on p.472 by MC1 commutes.

1. Relations between data and axioms. Throughout this section we shall deal with the following basic situation: we assume given bifunctors

$$\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \dashrightarrow \mathcal{V}_0 \quad \text{and} \quad [-,-] : \mathcal{V}_0^* \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$$

together with a natural isomorphism

$$\pi_{ABC} : \mathcal{V}_0(A \otimes B, C) \approx \mathcal{V}_0(A, [BC]).$$

We shall also use the alternative description of π by its unit: $e_{AB} = \pi_{A,B,A\otimes B}(A \otimes B) : A \longrightarrow [B, A \otimes B]$ natural in A and dinatural in B, and

counit: $e_{AB} = \pi_{[AB],A,B}([AB]) : [AB] \otimes A \longrightarrow B$ natural in B and dinatural in A.

1.1. Given a transformation

(1.1) $$a_{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

natural in A, B, C, the formula

(1.2) $$L^A_{BC} = \pi_{[BC]\otimes[AB],A,B} \cdot \pi_{[BC][AB][AC]}\{e_{BC} \cdot ([BC] \otimes e_{AB}) \cdot a_{[BC][AB]A}\}$$

defines a transformation

(1.3) $$L^A_{BC} : [BC] \longrightarrow [[AB][AC]]$$

natural in B, C and dinatural in A.

Conversely, given (1.3), the formula

(1.4) $$a_{ABC} = \overline{\pi}_{A,B,[C,A\otimes(B\otimes C)]} \cdot$$

$$\cdot \bar{\pi}_{A\otimes B, C, A\otimes(B\otimes C)} \{ [\partial_{BC} [C, A\otimes(B\otimes C)]] \cdot \cdot L^{C}_{B\otimes C, A\otimes(B\otimes C)} \cdot \partial_{A, B\otimes C} \}$$

defines a transformation (1.1).

Moreover, the procedures (1.2) and (1.4) are mutually inverse and thus establish a 1-1 correspondence between (1.1) and (1.3).

(We shall speak about π-corresponding couples of transformations $\langle a, L \rangle$.)

1.2. Let $\langle a, L \rangle$ be a π-corresponding couple. Then a satisfies MC3 iff L satisfies CC3.

<u>Proof</u> of CC3 $\Longrightarrow$ MC3 (a shortened version): We have to show that under the assumption CC3 the equality

$$(1.5) \quad (A\otimes a_{BCD}) \cdot a_{A(B\otimes C)D} \cdot (a_{ABC} \otimes D) = a_{AB(C\otimes D)} \cdot a_{(A\otimes B)CD}$$

holds for all A, B, C, $D \in \text{obj } \mathcal{V}_0$. Since π is an isomorphism we can as well prove (1.5) with

$$\pi_{A,B,[C[DE]]} \cdot \pi_{A\otimes B, C, [DE]} \cdot \pi_{(A\otimes B)\otimes C, D, E},$$

where $E = A\otimes (B\otimes (C\otimes D))$, applied to both sides. Now

$$\pi\, \pi\, \pi\, \{ (A\otimes a_{BCD}) \cdot a_{A(B\otimes C)D} \cdot (a_{ABC} \otimes D) \} =$$

$$= \pi\, \pi \{ [D, (A\otimes a) \cdot a \cdot (a \otimes D)] \cdot \partial_{(A\otimes B)\otimes C, D} \} =$$

$$= \pi\, \pi \{ [D, (A\otimes a) \cdot a] \cdot [D, a\otimes D] \cdot \partial \}$$

which by the naturality of ∂ equals

$$\pi\pi\{[D,(A\otimes a)\cdot a]\cdot \partial_{A\otimes(B\otimes C),D}\cdot a_{ABC}\} =$$

$$= [B[C[D,(A\otimes a)\cdot a]]]\cdot[B[C,\partial_{A\otimes(B\otimes C),D}]]\cdot$$

$$\cdot[B[C,a]]\cdot[B,\partial_{A\otimes B,C}]\cdot\partial_{AB}\cdot$$

We apply (1.4) for a_{ABC} and obtain

$$[B[C[D,(A\otimes a)\cdot a]]]\cdot[B[C,\partial]]\cdot$$

$$\cdot[\partial_{BC}[C,A\otimes(B\otimes C)]]\cdot L^{C}_{B\otimes C,A\otimes(B\otimes C)}\cdot\partial_{A,B\otimes C}\cdot$$

By the naturality of Θ, the naturality of L (applied three times), and (1.4) for $a_{A(B\otimes C)D}$, the last line can be rewritten as

$$[\partial_{BC}[D,A\otimes(B\otimes(C\otimes D))]]\cdot L^{C}_{B\otimes C,[D,A\otimes((B\otimes C)\otimes D)]}\cdot$$

$$\cdot[B\otimes C[D,A\otimes a]]\cdot[\partial_{B\otimes C,D}[D,A\otimes((B\otimes C)\otimes D))]]\cdot$$

$$\cdot L^{D}_{(B\otimes C)\otimes D,A\otimes((B\otimes C)\otimes D)}\cdot\partial_{A,(B\otimes C)\otimes D}\cdot$$

Next we use the naturality of L, dinaturality of ∂, again the naturality of L (three times), then (1.4) for a_{BCD} applied in the first variable of $[-,-]$, and the dinaturality of L, and obtain

$$[\partial_{B(C\otimes D)},1]\cdot[1,[\partial_{CD},1]]\cdot[L^{D}_{C\otimes D,B\otimes(C\otimes D)},1]\cdot$$

$$\cdot L^{[D,C\otimes D]}_{[D,B\otimes(C\otimes D)],[D,A\otimes(B\otimes(C\otimes D))]}\cdot L^{D}_{A\otimes(B\otimes C),A\otimes(B\otimes(C\otimes D))}\cdot$$

$$\cdot\partial_{A,B\otimes(C\otimes D)}\cdot$$

By CC3, this equals

$$[\partial,1]\cdot[1[\partial_{CD},1]]\cdot[1,L^{D}_{C\otimes D,A\otimes(B\otimes(C\otimes D))}]\cdot$$

$\cdot L^{C\otimes D}_{B\otimes(C\otimes D), A\otimes(B\otimes(C\otimes D))} \cdot \partial$.

Using three times the fact that $[-,-]$ is a functor and by (1.4) for $a_{AB(C\otimes D)}$ we obtain

$$[B[\partial_{CD},1]] \cdot [B,L^D] \cdot [B[C\otimes D, a_{AB(C\otimes D)}]] \cdot$$

$$\cdot [B, \partial_{A\otimes B, C\otimes D}] \cdot \partial_{AB}$$

which, by the naturality of L applied in the second variable of $[-,-]$, the description of π via ∂, and by (1.4) for $a_{(A\otimes B)CD}$, yields

$$\pi\{[C[D, a_{AB(C\otimes D)} \cdot a_{(A\otimes B)CD}]] \cdot [C, \partial_{(A\otimes B)\otimes C, D}] \cdot$$

$$\cdot \partial_{A\otimes B, C}\} = \pi\pi\pi\{a_{AB(C\otimes D)} \cdot a_{(A\otimes B)CD}\} .$$

2.1. Let $I \in \operatorname{obj} \mathcal{V}_0$. Then the formulas

(2.1) $\qquad i_A = \pi_{AIA}(r_A) = [I, r_A] \cdot \partial_{AI}$

(2.2) $\qquad r_A = \bar{\pi}_{AIA}(i_A) = e_{IA} \cdot (i_A \otimes I)$

establish a 1-1 correspondence between natural transformations

(2.3) $\qquad r_A : A \otimes I \longrightarrow A$

and

(2.4) $\qquad i_A : A \longrightarrow [IA]$

Moreover, r is an isomorphism iff i is.

2.2. Given π-corresponding couples $\langle a, L \rangle$ and $\langle r, i \rangle$. Then a, r satisfy MC4 iff L,i satisfy CC4.

3.1. Let $I \in \text{obj}\ \mathcal{V}_0$. Then the formulas

(3.1) $$j_A = \pi_{IAA}(\ell_A) = [A, \ell_A] \cdot \partial_{IA}$$

(3.2) $$\ell_A = \overline{\pi}_{IAA}(j_A) = e_{AA} \cdot (j_A \otimes A)$$

establish a 1-1 correspondence between

(3.3) natural transformations $\ell_A : I \otimes A \longrightarrow A$

and

(3.4) dinatural transformations $j_A : I \longrightarrow [AA]$

3.2. Given π-corresponding couples $\langle a, L \rangle$ and $\langle \ell, j \rangle$. Then a, ℓ satisfy MC1 iff L,j satisfy CC1.

3.3. Given π-corresponding couples $\langle a, L \rangle$, $\langle r, i \rangle$, and $\langle \ell, j \rangle$. Then a, ℓ, r satisfy MC2 iff L,i,j satisfy CC2.

3.4. Given π-corresponding couples $\langle r, i \rangle$ and $\langle \ell, j \rangle$. Then $r_I = \ell_I$ iff $i_I = j_I$.

4.1. Given a transformation (3.4) put for any $\xi : A \longrightarrow B$ in $\mathcal{V}_0$

(4.1) $$\tau_{AB}(\xi) = [A, \xi] \cdot j_A$$

We obtain a natural transformation

$$'\tau_{AB} : \mathcal{V}_0(A,B) \longrightarrow \mathcal{V}_0(I, [AB]).$$

4.2. Let $\langle \ell, j \rangle$ be a π-corresponding couple and let τ be defined by (4.1). Then

a) if ℓ is an isomorphism, so is τ and its inverse is determined by

$$(4.2) \qquad \bar{\tau}_{AB}(\eta) = e_{AB} \cdot (\eta \otimes A) \cdot \bar{\ell}_A$$

where $\eta : I \longrightarrow [AB]$ in $\mathcal{V}_o$.

b) If τ is an isomorphism, so is ℓ and we have

$$(4.3) \qquad \bar{\ell}_A = \bar{\tau}_{A,I\otimes A}(\partial_{IA})$$

<u>Proof</u>: All the verifications are straightforward except perhaps that of 4.2 b. Assume τ is an isomorphism and put

$$(4.4) \qquad \hat{\ell}_A = \bar{\tau}_{A,I\otimes A}(\partial_{IA}).$$

We show that $\hat{\ell}_A$ is inverse to ℓ_A. For every $A \in \mathcal{V}_o$ we have

$$\ell_A \cdot \hat{\ell}_A = \{\mathcal{V}_o(A, \ell_A) \cdot \bar{\tau}_{A,I\otimes A}\} \partial_{IA} =$$

(by the naturality of $\bar{\tau}$) $= \{\bar{\tau}_{AA} \cdot \mathcal{V}_o(I, [A, \ell_A])\} \partial_{IA} =$

$= \bar{\tau}_{AA} \{[A, \ell_A] \cdot \partial_{IA}\}$

(by (3.1)) $= \bar{\tau}_{AA}\{j_A\} =$

(by (4.1)) $= 1_A$.

To complete the proof it now suffices to show that each $\hat{\ell}_A$ is an epimorphism. Suppose that

$$A \xrightarrow{\ \hat{\ell}_A\ } I \otimes A \underset{\varphi_1}{\overset{\varphi_0}{\rightrightarrows}} B$$

commutes. Then

$$\pi_{IAB}\ \varphi_0 = [A, \varphi_0] \cdot \partial_{IA} = [A, \varphi_0] \cdot [A, \hat{\ell}_A] \cdot j_A =$$

$= [A, \varphi_1] \cdot [A, \hat{\ell}_A] \cdot j_A = [A, \varphi_1] \cdot \partial_{IA} = \pi_{IAB}\, \varphi_1$

whence $\varphi_0 = \varphi_1$.

2. The comparison theorem.

Proposition. In the basic situation of Section 1, the following statements are equivalent:

(T) $\otimes$ can be extended to a structure on $\mathcal{V}_0$ whose definition is obtained from the concept of a monoidal category as defined in [1] by weakening the associativity of $\otimes$ to a_{ABC} being just morphisms in $\mathcal{V}_0$ natural in A, B, C,

(H) $[-,-]$ can be extended to a closed category structure on $\mathcal{V}_0$ as defined in [1].

Moreover, the structures on $\mathcal{V}_0$ mentioned in (T) and (H), respectively, determine each other (up to some freedom we have when defining the basic functor V in the transition from (T) to (H)) uniquely.

Proof: a) (H) $\longmapsto$ (T). Given a closed category structure $\langle V, [-,-], I, L, i, j \rangle$ on $\mathcal{V}_0$, use (1.4), (2.2), and (3.2) to obtain transformations a, r, ℓ satisfying MC1 - MC4; r is an isomorphism. By Propositions 2.4 and 2.7 of Chapter I in [1],

$$Vi_{[AB]} = \tau_{AB},$$

where τ is defined by (4.1), holds for any A, $B \in \text{obj}\, \mathcal{V}_0$ and

$$i_I = j_I.$$

Hence ℓ is an isomorphism and $r_I = \ell_I$ (MC5 in [1]).

b) (T) $\longmapsto$ (H). Given $\langle \otimes, I, a, r, \ell \rangle$ such that a, r, ℓ satisfy MC1 - MC5 and r, ℓ are isomorphisms, use (1.2), (2.1), (3.1) and (4.1) to obtain transformations L, i, j, τ such that L, i, j satisfy CC1 - CC4, i, τ are isomorphisms and $i_I = j_I$. It remains to normalize the couple $\langle \mathcal{V}_0, [-,-] \rangle$ (cf. [1], p. 491). To this end, we prove the following

__Lemma.__ Given i, j such that $i_I = j_I$ and τ is an isomorphism. For any $C \in \text{obj } \mathcal{V}_0$,

$$\begin{cases} \text{if } C = [AB] \text{ put } VC = \mathcal{V}_0(AB), \quad \iota_C = \tau_{AB} \\ \text{otherwise put } VC = \mathcal{V}_0(I,C), \quad \iota_C = 1_{\mathcal{V}_0(I,C)}. \end{cases}$$

For any $f\colon C \longrightarrow D$ in $\mathcal{V}_0$ define a mapping $Vf\colon VC \longrightarrow VD$ by

$$Vf = \bar{\iota}_D \cdot \mathcal{V}_0(I,f) \cdot \iota_C$$

We obtain a functor $V\colon \mathcal{V}_0 \longrightarrow \text{Set}$ and a natural isomorphism $\iota\colon V \approx \mathcal{V}_0(I,-)$ such that

(CC0) (i) $\quad V \cdot [-,-] = \mathcal{V}_0(-,-)$

(ii) $\quad Vi_{[AB]} = \tau_{AB}$ holds for any $A, B \in \text{obj } \mathcal{V}_0$, in particular,

(CC5) $\quad Vi_{[AA]}(1_A) = j_A$

__Proof:__ (i) is clearly true on objects. Next, for any

$$A' \xrightarrow{f} A \xrightarrow{\xi} B \xrightarrow{g} B' \text{ in } \mathcal{V}_0$$

we have $\tau_{A'B'}\{V[f,g]\xi\} = \{\tau_{A'B'} \cdot \bar{\tau}_{A'B'} \cdot \mathcal{V}_0(I, [fg]) \cdot \tau_{AB}\}\xi = \{\mathcal{V}_0(I, [fg]) \cdot \tau_{AB}\}\xi = [fg] \cdot [A\xi] \cdot j_A = [f, g\xi] \cdot j_A = [A', g\xi] \cdot [f, A] \cdot j_A = [A', g\xi] \cdot [A', f] \cdot j_{A'} = [A', g\xi f] \cdot j_{A'} = \tau_{A'B'}(g\xi f) = \tau_{A'B'}\{\mathcal{V}_0(f,g)\xi\}$, hence $V[fg]\xi = \mathcal{V}_0(f,g)\xi$.

(ii) For any $A \xrightarrow{\xi} B$ in $\mathcal{V}_0$ we have

$$Vi_{[AB]}\xi = \{\bar{\tau}_{I,[AB]} \cdot \mathcal{V}_0(I, i_{[AB]}) \cdot \tau_{AB}\}\xi =$$

$$= \bar{\tau}_{I[AB]}\{i_{[AB]} \cdot [A\xi] \cdot j_A\} = \bar{\tau}_{I[AB]}\{[I[A\xi]] \cdot i_{[AA]} \cdot j_A\} =$$

$$= \bar{\tau}_{I[AB]}\{[I[A\xi]] \cdot [I, j_A] \cdot i_I\} = \bar{\tau}_{I[AB]}\{[I[A\xi]] \cdot$$

$$\cdot [I, j_A] \cdot j_I\} = \bar{\tau}_{I[AB]}\{\tau_{I[AB]}\{\tau_{AB}\xi\}\} = \tau_{AB}\xi \ .$$

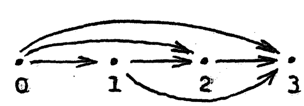
Example. To satisfy ourselves that there exist closed categories in which the internal hom-functor has a non-associative adjoint, let us turn to the following special case.

Consider a partially ordered set (i.e. a small thin skeletal category) $\langle P, \leq \rangle$. A closed category structure on $\langle P, \leq \rangle$ boils down to a couple $\langle [-,-], I \rangle$, where $I \in P$ and $[-,-] : P \times P \longrightarrow P$ is an operation order reversing in the first and order preserving in the second variable, such that

$$[y,z] \leq [[xy][xz]]$$
$$x = [Ix]$$
$$I \leq [xx]$$
$$x \leq y \text{ iff } I \leq [x,y]$$

hold for any $x, y, z \in P$.

Now take the closed category structure $\langle [-,-], 3 \rangle$ on the category

$\mathbf{4}$ = 

where $[-,-]$ is defined by Table 1. Table 2 shows the value of its adjoint $\otimes$.

3	3	3	3	3
2	3	3	3	2
1	3	3	2	1
0	3	1	1	0
	0	1	2	3

Table 1

3	0	1	2	3
2	0	0	1	2
1	0	0	1	1
0	0	0	0	0
	0	1	2	3

Table 2

Observe that $(2 \otimes 2) \otimes 2 = 1 \otimes 2 = 0 < 1 = 2 \otimes 1 = 2 \otimes (2 \otimes 2)$.

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