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SOME REMARKS ON SUBSPACES OF WEAKLY COMPACTLY GENERATED BANACH SPACES

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Abstract: Some hereditary properties of weakly compactly generated Banach spaces are shown.

Key words: Weakly compactly generated Banach space, Eberlein compact, densities property of a Banach space.

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Introduction. We work through the paper only with real Banach spaces. A Banach space X is said to be weakly compactly generated (WCG) if there exists a weakly compact set $K \subset X$ which generates X, i.e. the closed linear span of K is X.

Recently, Rosenthal [7] has shown that a closed subspace of a WCG space need not be WCG. Such a subspace (even with an unconditional basis) was found in the space $L_1(\mu)$ for a finite measure μ. We have remarked in this paper some properties of WCG spaces which are hereditary to general closed linear subspaces, e.g. a certain densities property (Proposition 5).

We mention our notation. For a Banach space X we denote B_X^* the unit ball of X^* with the w^* topology. For a topological (completely regular Hausdorff) space T $C_b(T)$ denotes the space of real-valued functions on T under the supremum norm.

By a subspace of a Banach space we mean always a closed linear subspace.

We quote at first a few marked properties of WCG spaces which are (some of them less evidently) kept by general subspaces of WCG spaces.

<u>Proposition 1</u>. Let X be a subspace of a WCG space. Then there holds:

(i) X has an equivalent norm which is LUR;

(ii) X has an equivalent norm such that X^* is strictly convex;

(iii) X has a Markuševič basis;

(iv) if $c_0 \subset X$ then there exists a linear projection P of X onto c_0 with $\|P\| \leq 2$.

<u>Proof</u>: Properties (i) and (ii) are hereditary, (iii) can be proved using the method of [5] and the decomposition of subspaces of WCG spaces in [3]. (iv) holds by the results of [8] and [3].

We use the following easy characterization of subspaces of WCG spaces for the next.

<u>Lemma</u>. A Banach space X is a subspace of a WCG space if and only if the unit ball of X^* with the w^* topology is a continuous image of an Eberlein compact.

<u>Proof</u>: Let X be a subspace of a WCG space Y. The

restriction mapping $R: Y^* \longrightarrow X^*$ defined by $Rf = f/X$ for $f \in Y^*$ is $w^* - w^*$ continuous and $R(B^*_Y) = B^*_X$ by Hahn-Banach theorem. The space B^*_Y (with the w^* topology) is an Eberlein compact ([1]) and B^*_X is a continuous image of it.

On the other hand, let X be a Banach space and B^*_X a continuous image of an Eberlein compact K. Then we can suppose the inclusions $X \subset C(B^*_X) \subset C(K)$ and the latter space is WCG ([1]).

There is observed in [4] that if $T: X \longrightarrow Y$ is a linear continuous mapping with the range dense in Y and X is WCG, then so is Y. Indeed, if K is a weakly compact set generating X, then $T(K)$ is a weakly compact set generating Y. We make an analogy to this within subspaces of WCG spaces.

<u>Proposition 2</u>. Let both X, Y be Banach spaces and $T: X \longrightarrow Y$ a continuous linear mapping with $\overline{TX} = Y$. Suppose X is a subspace of a WCG space. Then so is Y.

<u>Proof</u>: The mapping $T^*: Y^* \longrightarrow X^*$ is $w^* - w^*$ continuous and one-to-one. Accordingly, T^* is a homeomorphism on B^*_Y and we can assume the inclusion $B^*_Y \subset \|T^*\| \cdot B^*_X$. Since the property "to be a continuous image of an Eberlein compact" is closed hereditary, our assertion is a consequence of the Lemma.

<u>Remark</u>. Each Eberlein compact K has the following property due to Kaplansky: if $A \subset K$ and $x \in \overline{A}$, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in A such that $x_n \longrightarrow x$. It

is easy to verify that this property is kept by continuous Hausdorff images of Eberlein compacts.

<u>Proposition 3</u>. Let X be a subspace of a WCG space and $K \subset X^*$ a w^*-sequentially closed set which is either bounded or convex. Then K is w^*-closed.

<u>Proof</u>: For any $r > 0$ the set $K \cap \{x \in X^* ; \|x\| \leqq r\}$ is w^*-closed by the Remark and Lemma.

<u>Corollary</u>. Let f be a convex function on X^* where X is a subspace of a WCG space. Then f is w^*-lower semicontinuous if it is sequentially w^*-lower semicontinous.

<u>Proposition 4</u>. Let X be a topological Hausdorff completely regular space. Suppose $C_b(X)$ is a subspace of a WCG space. Then there holds:

(a) X is pseudocompact;

(b) X is compact if it is normal.

<u>Proof</u>: Suppose X is not pseudocompact. Then there exists an infinite discrete set $\Gamma \subset X$ which is C-embedded into X, i.e. the restriction mapping $R: C_b(X) \longrightarrow m(\Gamma)$ (defined by $Rf = f/\Gamma$ for $f \in C_b(X)$) is onto $m(\Gamma)$. The space $m(\Gamma)$ cannot be a subspace of a WCG space by Proposition 1. Consequently, the space $C_b(X)$ cannot be a subspace of a WCG space by Proposition 2, a contradiction. Let X be now moreoever normal. Denote B_C^* the unit ball of $C_b^*(X)$ with the w^* topology. As for the Čech-Stone compactification βX of X we can assume $\beta X \subset B_C^*$, βX is a continuous image of an Eberlein compact by the Lemma. Thus βX has the property of Kaplansky from the Remark.

It implies easily (provided X is normal) that X must be compact.

D. Preiss and P. Simon have shown recently that if K is a pseudocompact subset of an Eberlein compact, then K is compact ([6]) . Consequently, if for a Hausdorff completely regular space X the space $C_b(X)$ is WCG, then X is compact.

For a topological space X dX (density of X) is the smallest cardinal number $\aleph$ such that there exists a subset A dense in X with card $A = \aleph$.

The next property and also Corollary 1 are proved in [4] for WCG spaces, but the method used there cannot be utilized in our case.

<u>Proposition 5</u>. Let X be a subspace of a WCG space. Then for the densities of X and X^* we have the equality $dX = d(X^*, w^*)$.

<u>Proof</u>: For any normed linear space there holds $d(X^*, w^*) \leq dX$. Thus for X separable our assertion is evident. So supposae X is a non-separable subspace of a WCG space Y . We can assume $dX = dY$ ([1]) . Suppose the inequality $d(X^*, w^*) = dX$ is false, i.e. let A be a w^*-dense subset of X^* with card $A < dX$. Since X is non-separable we can assume that card $A \geq \aleph_0$.

Let $\tilde{f} \in Y^*$ be an extension of f for each $f \in A$ and denote $\tilde{A} = \{\tilde{f} ; f \in A\}$. By [3] there is a continuous linear projection $P: Y \longrightarrow Y$ with $PX \subset X$, $P^* \tilde{f} = \tilde{f}$ for $\tilde{f} \in \tilde{A}$ and $d(PY) \leq$ card $\tilde{A}$. We define the projection $S: X \longrightarrow X$

by the restriction of P on X. Clearly, $S^* f = f$ for $f \in A$. Since A is w^*-dense in X^* and S^* is w^*-w^* continuous we have $S^* = id_{X^*}$. Consequently, $S = id_X$ and hence $X \subset PY$. But for the densities we have $d(PY) \leq \text{card } A < dX$, thus $d(PY) < dX$, a contradiction.

Corollary 1. Let X be a Banach space such that X^* is a subspace of a WCG space. Then $dX = dX^*$.

Proof: For any normed linear space there holds $d(X^{**}, w^*) \leq dX \leq dX^*$, and the first member of the inequality is equal to the last one by Proposition 5.

Corollary 2 (cf.[2]). Let X be as in Corollary 1. Then X has the densities property, i.e. for each subspace $Y \subset X$ there is $dY^* = dY$. Thus X^* has the Radon-Nikodym property.

Proof: Suppose Y is a subspace of X. Then Y^* is a continuous linear image of X^* and thus Y^* is a subspace of a WCG space by Proposition 2. Consequently, $dY = dY^*$ by Corollary 1.

If X has the densities property, then X^* has the Radon-Nikodym property, see e.g. [2].

References

[1] D. AMIR, J. LINDENSTRAUSS: The structure of weakly compact sets in Banach spaces, Ann. of Math. 88(1968), 35-46.

[2] J. DIESTEL, J.J. UHL: The Radon-Nikodym theorem for Banach space valued measures, to appear.

[3] K. JOHN, V. ZIZLER: Smoothness and its equivalents in weakly compactly generated Banach spaces, Journ. Funct. Anal. 15(1974), 1-11.

[4] J. LINDENSTRAUSS: Weakly compact sets, their topological properties and Banach spaces they generate, Ann. of Math. Stud. 69, Princeton Univ. Press (1972), 235-273.

[5] J. LINDENSTRAUSS: Decomposition of Banach spaces, Indiana Univ. Journ. 20(1971), 817-819.

[6] D. PREISS, P. SIMON: A weakly pseudocompact subspace of a Banach space is weakly compact, Comment. Math.Univ. Carolinae 15(1974), 603-611.

[7] H.P. ROSENTHAL: The heredity problem for weakly compactly generated Banach spaces, Comp. Math. 28(1974), 83-111.

[8] A. SOBCZYK: Projection of the space (m) on its subspace (c_0), Bull. Amer. Math. Soc. 47(1941),938-947.

[9] C. STEGALL: The Radon-Nikodym property in conjugate Banach spaces, to appear.

[10] S. TROJANSKI: On locally uniformly convex and differentiable norms in certain non-separable Banach spaces, Studia Math. 37(1971), 173-180.

[11] W.A. VEECH: Short proof of Sobczyk theorem, Proc. Amer. Math. Soc. 28(1971), 627-628.

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