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BAIRE FUNCTIONS AND CLASSES BOUNDED BY FILTERS

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Abstract: In [1],[2], certain classes of functions generated by filters have been examined. In the present note, we consider classes of spaces bounded (weakly bounded) by a filter on a countable set. Weakly bounded classes turn out to coincide with classes such that there are "not too many" metrizable images of spaces in the class. It is shown that, on weakly bounded classes, Baire functions coincide with those generated by a suitable filter, depending on the class. This result corrects an error in [1], see 4.1 below.

Key words: Baire functions, filter-generated function, descriptively bounded class.

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1.1. We use the standard terminology and notation with slight modifications. The power of a set M is denoted by $|M|$. The countable infinite cardinal is denoted by ω, the first uncountable one by ω^+. If α is a cardinal, $\exp \alpha$ stands for 2^{α}.

1.2. Conventions. "Space" always means a completely regular Hausdorff space. "Mapping" means a mapping (continuous or not) of a space into a space or of a set into a set, "function" means a mapping into R, the space of reals. The set of all natural numbers is denoted by N or ω. Letters i, j, k, n denote natural numbers; $\mathcal{F}$, $\mathcal{G}$,

$\mathcal{H}$, $\mathcal{K}$, possibly with subscripts, denote filters; $\mathcal{N}$ denotes the Fréchet filter on N ; P, S, T denote spaces. If $f: S \longrightarrow T$ is continuous, then βf denotes the extension to a continuous mapping of βS into βT .

1.3. The domain of a mapping f is denoted by Df. If $X \subset Df$, then $f \upharpoonright X$ denotes the restriction of f to X . If $f: S \longrightarrow T$, $g: U \longrightarrow V$ are mappings of spaces, then the composition $f \circ g$ is defined iff V is a subspace of S . If T is a space, then $F(T)$ denotes the set of all functions on T , $C(T)$ that of all continuous $f \in F(T)$. The set $F(T)$ is endowed with the topology of the product R^T .

1.4. The term "filter" has its usual meaning. If $\mathcal{F}$ is a filter on A , called the support of $\mathcal{F}$, and $|A| \leqq \alpha$ ($|A| = \alpha$) , then we shall say that $\mathcal{F}$ is an $(\leqq \alpha)$-filter (an α-filter). (Observe that, in [1],[2], "filter" means what is called a free ω-filter here.) - If $\mathcal{F}$ is a filter on A , and $M \subset A$ intersects all $F \in \mathcal{F}$, then the filter $\{F \cap M \mid F \in \mathcal{F}\}$, denoted by $\mathcal{F} \upharpoonright M$, is called the trace of $\mathcal{F}$ on M .

1.5. A morphism (cf. [2], 1.9) from $\mathcal{F}$ (on A) to $\mathcal{G}$ (on B) is, by definition, a triple $\langle \varphi, \mathcal{F}, \mathcal{G} \rangle$, where $\varphi : A \longrightarrow B$ is a mapping such that $Y \in \mathcal{G}$ implies $\varphi^{-1} Y \in \mathcal{F}$. If there exists a morphism from $\mathcal{F}$ to $\mathcal{G}$, we shall write $\mathcal{F} \geqq \mathcal{G}$ or $\mathcal{G} \leqq \mathcal{F}$. The class of all filters will be considered as quasi-ordered by the relation $\leqq$.

1.6. Let $\mathcal{F}$ be a filter on A . Let T be a space. If $\{x_a \mid a \in A\}$ is a family of points of T , $x \in T$ and, for every neighborhood V of x in T, there is a set $F \in \mathcal{F}$ such that $a \in F$ implies $x_a \in V$, we shall say that x is the $\mathcal{F}$-limit of $\{x_a\}$ (in T) and we shall write $x = \mathcal{F}\text{-lim}\{x_a \mid a \in A\}$ or $x = \mathcal{F}\text{-lim}\, x_a$. If $X \subset T$, then $\mathcal{F}\text{-Lim}\, X$ denotes the set of all $x \in T$ such that $x = \mathcal{F}\text{-lim}\, x_a$ for some $\{x_a\} \in X^A$.

1.7. Let $\mathcal{F}$ be a filter on A. A function f will be called $\mathcal{F}$-generated if $f \in \mathcal{F}\text{-Lim}\, C(Df)$. The class of all $\mathcal{F}$-generated functions will be denoted by $C\ell(\mathcal{F})$.

1.8. If f is $\mathcal{F}$-generated, then every $f \circ \varphi$, where φ is a continuous mapping, is $\mathcal{F}$-generated. If $\mathcal{F}$ is a filter on A, f is $\mathcal{F}$-generated, then there is a continuous $\psi : Df \longrightarrow R^A$ and an $\mathcal{F}$-generated g on $\psi[Df]$ such that $f = g \circ \psi$.

Proof. Let $f = \mathcal{F}\text{-lim}\, f_a$, $f_a \in C(Df)$. Clearly, $f \circ \varphi = \mathcal{F}\text{-lim}\, f_a \circ \varphi$. Put $\psi x = \{f_a x\}$, for every $x \in Df$. Then $\psi : Df \longrightarrow R^A$ is continuous. Put $Y = \psi[Df]$. For $y = \{y_a\} \in Y$, put $g_a(y) = y_a$ and let $g(y)$ be defined by $g(\psi(x)) = fx$. Clearly, $g_a \in C(Y)$, $g = \mathcal{F}\text{-lim}\, g_a$, $f = g \circ \psi$.

1.9. If $\mathcal{F}$ is a filter on A, $|A| = \omega$, and S contains a dense countable set, then $|\mathcal{F}\text{-Lim}\, C(S)| \leq \exp \omega$.

Proof. $|C(S)| \leq \exp \omega$, hence $|(C(S))^A| \leq \exp \omega$.

1.10. If $\{M_a \mid a \in A\}$ is a family of sets, then $\sum \{M_a \mid a \in A\}$ or $\sum M_a$ denotes the set $\{\langle a,x \rangle \mid a \in A, x \in M_a\}$.

2.1. Definition. Let $\{\mathcal{F}_a \mid a \in A\}$ be a non-void family of filters with supports M_a. The filter on $\sum M_a$ consisting of all $\sum F_a$, where $F_a \in \mathcal{F}_a$ for every a, will be called the sum of $\{\mathcal{F}_a\}$ and will be denoted by $\prod\{\mathcal{F}_a \mid a \in A\}$ or by $\sum \mathcal{F}_a$. The filter on $\prod M_a$ with a subbase consisting of all $\pi_a^{-1}[F_a]$, where $a \in A$, $F_a \in \mathcal{F}_a$, π_a is the projection of $\prod M_a$ onto M_a, will be called the cartesian product of $\{\mathcal{F}_a\}$ and will be denoted by $\sum\{\mathcal{F}_a \mid a \in A\}$ or $\prod \mathcal{F}_a$. We write $\mathcal{F}_1 + \mathcal{F}_2$ instead of $\sum\{\mathcal{F}_i \mid i = 1,2\}$, etc.

2.2. Proposition. If $\{\mathcal{F}_a\}$ is a non-void family of filters, then $\prod \mathcal{F}_a$ is a join and $\sum \mathcal{F}_a$ is a meet of $\{\mathcal{F}_a\}$ in the quasi-ordered class of all filters.

The proof is straightforward and may be omitted.

2.3. Definition. The least power of a collection $\mathcal{M} \subset \mathcal{F}$ such that $\bigcap \mathcal{M} = \bigcap \mathcal{F}$ will be called the pseudoweight of $\mathcal{F}$. - Clearly, the pseudoweight of $\mathcal{F}$ is ω iff $\mathcal{F} \geq \mathcal{N}$.

2.4. Proposition. Let α be an infinite cardinal. In the class of all $(\leq \alpha)$-filters, every set of power $\leq \exp \alpha$ is bounded. - Cf. [2], 1.11.

Proof. Let $|A| \leq \exp \alpha$, $A \neq \emptyset$. For every $a \in A$, let $\mathcal{F}_a$ be a filter on B_a, $|B_a| \leq \alpha$. By a well known theorem, the product $B = \prod B_a$ of discrete spaces B_a contains a dense set H of power $\leq \alpha$. The injection $H \to B$ is a morphism from $(\prod \mathcal{F}_a) \upharpoonright H$ to $\prod \mathcal{F}_a$.

2.5. Theorem. Let α be an infinite cardinal. Every countable family of $(\leqq \alpha)$-filters of pseudoweight ω has a joint which is an $(\leqq \alpha)$-filter.

Proof. Let $\mathcal{F}_k$, $k \in N$, be filters of pseudoweight ω on sets M_k, $|M_k| \leqq \alpha$. Put $M = \prod M_k$, $\mathcal{F} = \prod \mathcal{F}_k$. Choose a sequence $\{u_k\} \in M$. Let H consist of all $\{x_k\} \in M$ such that $x_k = u_k$ for almost all k. Clearly, $|H| \leqq \alpha$. Put $\mathcal{H} = \mathcal{F} \upharpoonright H$. Then every projection π_k is a morphism from $\mathcal{H}$ to $\mathcal{F}_k$. Hence $\mathcal{H} \geqq \mathcal{F}$ (see 2.2).

Since $\mathcal{F}_k$ are of pseudoweight ω, there exist mappings $f_k : M_k \longrightarrow N$ such that every $\{x \mid x \in M_k, f_k(x) \geqq q\}$ is in $\mathcal{F}_k$. For every $x = \{x_k\} \in M$ let $p(x)$ be the largest $p \in N$ such that $f_k(x_k) \geqq p$ whenever $0 \leqq k \leqq p$. Put $\varphi(x) = \{y_k\}$ where $y_k = x_k$ for $k \leqq p(x)$, $y_k = u_k$ for $k > p(x)$. To prove that φ is a morphism from $\mathcal{F}$ to $\mathcal{H}$, it is enough to show that, for every $q \in N$ and every $F \in \mathcal{F}_q$, there is a set $U \in \mathcal{F}$ such that if $x \in U$, $\varphi(x) = \{y_k\}$, then $y_q \in F$. Let U consists of all $x = \{x_k\} \in M$ such that $x_q \in F$, $f_k(x_k) \geqq q$ for $k \leqq q$. Clearly, $U \in \mathcal{F}$. If $x \in U$, then $p(x) \geqq q$, hence, with $\{y_k\} = \varphi(x)$, we have $y_k = x_k$ for $k \leqq q$. This proves the theorem.

2.6. <u>Corollary</u>. Every countable family of ω-filters has a join which is an ω-filter. - Cf. [2], 1.11.

2.7. <u>Remark</u>. The note [2] contains a statement (4.5), which may be re-formulated as follows: Let $\mathcal{F}_n$, $n \in N$, be ω-filters. Put $\mathcal{D}_n = C\ell(\mathcal{F}_n)$. Then (1) in the class of all ω-filters, $\{\mathcal{F}_n\}$ has a join, (2) there exists a class $\mathcal{D}$ of the form $C\ell(\mathcal{G})$, where $\mathcal{G}$ is an ω-filter, such that (i) $C\ell(\mathcal{G}) \supset \mathcal{D}_n$ for all n, (ii) if $\mathcal{H}$ is an ω-

filter and $C\ell(\mathcal{H}) \supset \mathcal{D}_n$ for all n , then $C\ell(\mathcal{H}) \supset C\ell(\mathcal{G})$, (3) the class $\mathcal{D}$ is equal to $C\ell(\mathcal{F})$ where $\mathcal{F}$ is a join of $\{\mathcal{F}_n\}$ in the class of all ω -filters.

We have proved the first assertion (in fact, slightly more). As for assertions (2),(3), the intended proof fails, and the question remains open whether (2) and (3) are valid.

3.1. Definition. If $S \subset T$, then $\chi(S,T)$ or simply χS will denote the characteristic function of S in T . We shall say that $S \subset T$ is $\mathcal{F}$-generated in T if $\chi(S,T)$ is $\mathcal{F}$-generated. We shall say that $\mathcal{F}$ bounds a class $\mathfrak{M}$ of spaces if every $X \in \mathfrak{M}$ is $\mathcal{F}$-generated in some compact T .

3.2. It is easy to see that a space S is $\mathcal{F}$-bounded iff it is $\mathcal{F}$-generated in βS .

3.3. A continuous mapping $f: S \longrightarrow T$ is called perfect if (1) all $f^{-1}y$, $y \in fS$, are compact, (2) fM is closed in fS whenever M is closed in S .

3.4. The following facts are well known: (1) every continuous mapping of a compact space is perfect; (2) if $f: X \longrightarrow Y$ is perfect, $M \subset Y$, then $f \upharpoonright (f^{-1}M)$ is perfect; (3) if $f: X \longrightarrow Y$ is continuous, Z is dense in X , $f \upharpoonright Z$ is perfect, then $f[Z] \cap f[X - Z] = \emptyset$.

3.5. Conventions. If there exists a continuous (perfect) mapping of S onto T , we shall say that S is a continuous (perfect) counterimage of T and that T is a continuous (perfect) image of S . If, in addition, e.g., S is metrizable, we shall say that S is a metrizable continuous (perfect) counterimage of T , etc.

3.6. Every perfect counterimage of an $\mathcal{F}$-bounded space is $\mathcal{F}$-bounded.

Proof. If $f: X \to Y$ is perfect onto, then, by 3.4, (3), $(\beta f)^{-1}[Y] = X$, hence, by 1.8, X is $\mathcal{F}$-generated in βX.

3.7. Let $\mathcal{F}$ be an $(\leqq \alpha)$-filter. Every $\mathcal{F}$-bounded space is a perfect counterimage of a space $\mathcal{F}$-generated in a compact space of weight $\leqq \alpha$.

Proof. Let S be $\mathcal{F}$-bounded and let $\chi(S, \beta S) = \mathcal{F}\text{-}\lim \{f_a \mid a \in A\}$. For $x \in \beta S$ put $\varphi x = \{f_a x\}$. Then $\varphi: \beta S \to R^A$ is continuous, $S = \varphi^{-1}[\varphi S]$. By 3.4, (2), $\varphi \restriction S$ is perfect. Clearly, φS is $\mathcal{F}$-generated in $\varphi[\beta S]$.

3.8. Let $\mathfrak{M}$ be a class of separable metrizable spaces such that (1) if $X \in \mathfrak{M}$, $Y \subset X$ is closed in X, then $Y \in \mathfrak{M}$, (2) if K is compact metrizable, $X \in \mathfrak{M}$, then $K \times X$ is homeomorphic to a space in $\mathfrak{M}$. Let S be a perfect counterimage of a space in $\mathfrak{M}$. If $g: S \to T$ is continuous and T is separable metrizable, then gS is a continuous image of a space in $\mathfrak{M}$.

Proof. We may assume that T is compact. There exists a perfect $f: S \to K$ such that K is compact metrizable, $Y = fS$ in $\mathfrak{M}$, $S = f^{-1}Y$. For $x \in \beta S$, put $\varphi x = \langle (\beta f)x, (\beta g)x \rangle$. Then $\varphi: \beta S \to K \times T$ is continuous. Put $Z = \varphi[\beta S]$. By 3.4, (3), $(\beta f)[\beta S - S] \cap fS = \emptyset$, hence $\varphi S = Z \cap (Y \times T)$, φS is closed in $Y \times T$, and therefore $\varphi S \in \mathfrak{M}$. Clearly, the projection $K \times T \to T$ maps φS onto gS.

3.9. Definition. A class $\mathfrak{M}$ of spaces will be called ω-filter-bounded or descriptively bounded it there is an ω-filter bounding $\mathfrak{M}$.

3.10. Proposition. A space X is descriptively bounded if and only if it is a perfect counterimage of a separable metrizable space.

Proof. "Only if" follows from 3.7. If $f: X \to Y$ is perfect onto a metrizable separable Y, let $K \supset Y$ be compact metrizable. Clearly, $C(K)$ endowed with the sup-norm is separable. Hence, for some ω-filter $\mathcal{F}$, Y is $\mathcal{F}$-generated in K. By 3.6, X is $\mathcal{F}$-bounded.

3.11. Remark. It is well known that perfect counterimages of metrizable spaces coincide with paracompact M-spaces, introduced by K. Morita, and with paracompact p-spaces, introduced by A. Ahangelskii (for this theorem and further references see e.g. [4]). It is easy to show that descriptively bounded spaces coincide with Lindelöf M-spaces (p-spaces).

3.12. Let $\mathfrak{M}$ be a collection of separable metrizable spaces, $|\mathfrak{M}| \leq \exp \omega$. Let $\mathfrak{N}$ consist of all metrizable continuous images of spaces in $\mathfrak{M}$. Then $\mathfrak{N}$ is descriptively bounded.

Proof. Clearly, every $X \in \mathfrak{N}$ is separable. It is easy to see that $|\{X \mid X \subset R^N, X \in \mathfrak{N}\}| \leq \exp \omega$. For every $X \in \mathfrak{N}$, $X \subset R^N$, choose an ω-filter $\mathcal{F}(X)$ bounding X. By 2.4, there is an ω-filter $\mathcal{F}$ such that $\mathcal{F} \geq \mathcal{F}(X)$ for all $X \in \mathfrak{N}$, $X \subset R^N$. Clearly, $\mathcal{F}$ bounds $\mathfrak{N}$.

3.13. For every ω-filter $\mathcal{F}$, there are exactly $\exp \omega$ $\mathcal{F}$-bounded subspaces of R^N. - This follows easily from 3.7 and 1.9.

3.14. <u>Proposition</u>. For every ω-filter $\mathcal{F}$ there exists an ω-filter $\mathcal{G}$ such that every descriptively bounded continuous image of an $\mathcal{F}$-bounded space is $\mathcal{G}$-bounded.

Proof. Consider the class $\mathfrak{M}$ of all $\mathcal{F}$-bounded spaces $S \subset R^N$, and the class $\mathfrak{N}$ of all metrizable continuous images of spaces in $\mathfrak{M}$. By 3.13 and 3.12, there exists an ω-filter $\mathcal{G}$ which bounds $\mathfrak{N}$. Assume that S is $\mathcal{F}$-bounded, $\varphi : S \longrightarrow P$ is continuous onto, P is descriptively bounded. Then there exists, by 3.10, a surjective perfect $h: P \longrightarrow T$, where $T \subset R^N$. By 3.7, S is a perfect counterimage of some space in $\mathfrak{M}$. Put $g = h \circ \varphi$. By 3.8, $T = hP = gS$ is a continuous image of a space in $\mathfrak{M}$, hence T is in $\mathfrak{N}$, T is $\mathcal{G}$-bounded. By 3.6, P is $\mathcal{G}$-bounded.

3.15. <u>Theorem</u>. The class of all descriptively bounded continuous images of spaces from a given descriptively bounded class is descriptively bounded. - This follows at once from 3.14.

3.16. <u>Examples</u>. 1) Compact spaces are bounded by every filter. 2) The class of all σ-compact completely metrizable spaces consists exactly of all $\mathcal{N}$-bounded metrizable spaces. 3) The class of all projective spaces, in the sense of N. Lusin, see e.g. [3], § 38, and their perfect counterimages is descriptively bounded.

4) Consider the smallest class $\mathfrak{P}$ of separable metrizable spaces such that: (1) R^N is in $\mathfrak{P}$, (2) if $X = \bigcup X_k$ is separable metrizable, X_k are in $\mathfrak{P}$, then $X \in \mathfrak{P}$ (3) if X, Y are in $\mathfrak{P}$, $X \supset Y$, then $X - Y$ is in $\mathfrak{P}$, (4) if $X \in \mathfrak{P}$, then every metrizable continuous image of X is in $\mathfrak{P}$. Let a space be called σ-projective if it is a perfect counterimage of a space in $\mathfrak{P}$. It can be shown that the class of all σ-projective spaces is descriptively bounded.

3.17. Descriptively bounded classes possess various nice properties. However, descriptive boundedness is not preserved, in general, under continuous mappings (example: $N \cup (x)$, where $x \in \beta N - N$). Therefore, we introduce broader classes. It will be shown that, on these classes (and, of course, on all narrower ones) Baire functions and suitable filter-generated ones do coincide.

3.18. <u>Definition</u>. Let $\mathcal{F}$ be an ω-filter. Let $\mathfrak{M}$ be a class of spaces. If for every $X \in \mathfrak{M}$ and every descriptively bounded S such that $X \subset S \subset \beta X$ there exists an $\mathcal{F}$-bounded P such that $X \subset P \subset S$, we shall say that $\mathcal{F}$ weakly bounds $\mathfrak{M}$. A class of spaces will be called weakly descriptively bounded if it is weakly bounded by some ω-filter.

3.19. <u>Theorem</u>. A class $\mathfrak{M}$ of spaces is weakly descriptively bounded if and only if the class of all separable metrizable continuous images of spaces from $\mathfrak{M}$ contains $\exp \omega$ topologically distinct spaces at most.

Proof. I. Let $\mathfrak{M}$ be weakly bounded by an ω-filter $\mathcal{F}$. Let $\mathcal{G}$ possess properties described in 3.14. If $Y \subset R^N$, $X \in \mathfrak{M}$, $f: X \longrightarrow Y$ is continuous onto, put $g = \beta f$. By 3.4, (2), and 3.10, $g^{-1}Y$ is descriptively bounded, hence there is an $\mathcal{F}$-bounded Z such that $X \subset Z \subset g^{-1}Y$. By 3.14, $Y = gZ$ is $\mathcal{G}$-bounded. By 3.13, this proves the "only if" part. II. Let $\mathfrak{P}$ be a maximal collection of topologically distinct separable metrizable continuous images of spaces from $\mathfrak{M}$. Assume $|\mathfrak{P}| \leq \exp \omega$. By 3.12, there is an ω-filter $\mathcal{G}$ which bounds $\mathfrak{P}$. Let $X \in \mathfrak{M}$ and let S be descriptively bounded, $X \subset S \subset \beta X$. By 3.10, there exists a perfect $f: S \longrightarrow R^N$. Put $Y = fS$. By 3.4, (3), $S = g^{-1}Y$, where $g = \beta f$. Since fX is $\mathcal{G}$-bounded, $g^{-1}[fX]$ is also $\mathcal{G}$-bounded. Clearly, $X \subset g^{-1}[fX] \subset S$.

3.20. Examples. 1) The class of all σ-compact spaces is weakly descriptively bounded (cf. the example in 3.17). - 2) A discrete space of infinite power α is weakly descriptively bounded iff $\exp \alpha = \exp \omega$.

4.1. In [1], 5.4, it was asserted that, under the continuum hypothesis (CH), there exists an ω-filter $\mathcal{F}$ such that (*) Baire functions coincide with $\mathcal{F}$-generated ones. This is false, as the following elementary example shows. Let $\mathcal{F}$ be an ω-filter generating all Baire functions. Let $T = \{\xi \mid \xi < \omega^+\}$ be endowed with the discrete topology. For every $\xi \in T$ let $f_\xi \in F(R)$ be exactly of Baire class ξ . For $\langle \xi, x \rangle \in T \times R$ put $f\langle \xi, x \rangle = f_\xi x$. Clearly, f is $\mathcal{F}$-generated, but is not a Baire function.

However, it can be shown that the assertion (*) is true (under CH) if the class of spaces considered is suitably restricted.

4.2. We recall the following lemma stated and proved in [1], 5.1 (for the definition of a Souslin filter, see [1], 3.1, 3.2). - Let $\mathfrak{M}$ be a collection of spaces, $|\mathfrak{M}| \leq \exp \omega$. For every $S \in \mathfrak{M}$ let $Q_S \subset S$, $Z_S \subset S$ be of power $\leq \exp \omega$. Assume that $Z_S \cap \mathcal{G}\text{-Lim}\, Q_S = \emptyset$ for every Souslin filter $\mathcal{G}$. Let $\mathcal{F}$ be a Souslin filter on a set A . If the continuum hypothesis is assumed, then there exists a family $\{\mathcal{F}_\xi \mid \xi < \omega^+\}$ such that (1) $\mathcal{F}_0 = \mathcal{F}$, (2) every $\mathcal{F}_\xi$ is a Souslin filter on A , (3) $\xi < \eta < \omega^+$ implies $\mathcal{F}_\xi \subset \mathcal{F}_\eta$, (4) if $\xi < \eta < \omega^+$, then every Baire function (on any space) of class ξ is $\mathcal{F}_\eta$-generated, (5) if $S \in \mathfrak{M}$, $z \in Z_S$, $x_a \in Q_S$ for every $a \in A$, then there is a neighborhood V of z in S and a set $M \in \in \bigcup \{\mathcal{F}_\xi \mid \xi < \omega^+\}$ such that $a \in M$ implies x_a non$\in V$.

4.3. Assume CH . Let $\mathfrak{P}$ be a collection of separable metrizable spaces, $|\mathfrak{P}| \leq \exp \omega$. Let $\mathcal{G}$ be an ω-filter generating all Baire functions on spaces $P \in \mathfrak{P}$. Then there is a filter $\mathcal{H}$ on N such that, on spaces in $\mathfrak{P}$, Baire functions coincide with $(\mathcal{G}+\mathcal{K})$-generated ones whenever $\mathcal{K}$ is a filter on N , $\mathcal{K} \supset \mathcal{H}$.

Proof. Let $\mathfrak{M}$ be the collection of all $F(P)$, $P \in \in \mathfrak{P}$. If $S = F(P)$, put $Q_S = C(P)$, and let Z_S consist of all those $f \in \mathcal{G}$-Lim $C(P)$ which are not Baire functions. Put $\mathcal{F} = \mathcal{N}$. Then the assumptions of 4.2 are satisfied (since, by [1], 3.10, a function generated by a Souslin filter is a Baire function). Hence, by 4.2, there is a family $\{\mathcal{F}_\xi\}$ with properties described in 4.2. Put $\mathcal{H} = \bigcup\{\mathcal{F}_\xi \mid \xi < \omega^+\}$. If $\mathcal{K} \supset \mathcal{H}$ is a filter on N, then every Baire function is $\mathcal{K}$-generated and no $\mathcal{G}$-generated function on a space $P \in \mathcal{P}$ is $\mathcal{K}$-generated unless it is a Baire function. This proves the assertion, since, clearly, $\mathrm{c}\ell(\mathcal{G}+\mathcal{K}) = \mathrm{c}\ell(\mathcal{G}) \cap \mathrm{c}\ell(\mathcal{K})$.

4.4. If f is a Baire function of class $\xi < \omega^+$, then there is a continuous $\varphi : X \to R^N$ such that $f = g \circ \varphi$, where g is a Baire function of class ξ on φX.

This follows from 1.8, since, by [1], 2.17, Baire functions of class ξ coincide with $\mathcal{N}^\xi$-generated ones (for the filters $\mathcal{N}^\xi$ see [1], 2.7).

4.5. Let $\mathcal{F}_i$, $i = 1,2$, be filters on a set A. Then (1) $\mathcal{F}_1 + \mathcal{F}_2 \geq \mathcal{F}_1 \cap \mathcal{F}_2$, (2) if there are $X_i \in \mathcal{F}_i$ such that $X_1 \cap X_2 = \emptyset$, then $\mathcal{F}_1 \cap \mathcal{F}_2 \geq \mathcal{F}_1 + \mathcal{F}_2$.

Proof. I. If $\varphi\langle i,x\rangle = x$, then φ is a morphism from $\mathcal{F}_1 + \mathcal{F}_2$ to $\mathcal{F}_1 \cap \mathcal{F}_2$. II. Let $X_i \in \mathcal{F}_i$ be disjoint. Put $\psi x = \langle 1,x\rangle$ if $x \in X_1$, $\psi x = \langle 2,x\rangle$ if $x \in A - X_1$. Then ψ is a morphism from $\mathcal{F}_1 \cap \mathcal{F}_2$ to $\mathcal{F}_1 + \mathcal{F}_2$.

4.6. <u>Theorem</u>. Assume the continuum hypothesis. Let $\mathfrak{M}$ be a weakly descriptively bounded class of spaces. Let $\mathcal{G}$ be a filter on a countable set A generating all Baire functions. Then there exists a filter $\mathcal{H}$ on A such that, for every filter $\mathcal{K} \supset \mathcal{H}$ on A, Baire functions and $(\mathcal{G} \cap \mathcal{K})$-generated ones coincide.

Proof. Let $\mathcal{P}$ consist of all those $S \subset R^N$ which are continuous images of some space $X \in \mathfrak{M}$. By 3.19, $|\mathcal{P}| \leqq \exp \omega$. Let $\mathcal{H}$ be a filter on N with properties described in 4.3. Let $\psi : N \longrightarrow A$ be injective and such that $A - \psi N \in \mathcal{G}$. The collection of all $X \subset A$ such that $X \supset \psi H$ for some $H \in \mathcal{H}$ is a filter on A, which will be still denoted by $\mathcal{H}$. It follows from 4.5 that, for every filter $\mathcal{K} \supset \mathcal{H}$ on A, $(\mathcal{G} \cap \mathcal{K})$-generated functions on spaces $S \in \mathcal{P}$ coincide with $(\mathcal{G} + \mathcal{K})$-generated ones, hence, by 4.3, with Baire functions. By 4.4 and 1.8, this holds for every $S \in \mathfrak{M}$.

4.7. Proposition. There exists an ω-filter generating all Baire functions.

Proof. By 2.4, there is an ω-filter $\mathcal{F}$ such that $\mathcal{F} \geqq \mathcal{N}^{\xi}$ for all $\xi < \omega^{+}$.

4.8. Theorem. Assume the continuum hypothesis. Let $\mathfrak{M}$ be a weakly descriptively bounded class of spaces. Then there exist ω-ultrafilters $\mathcal{F}$, $\mathcal{G}$ such that, on spaces in $\mathfrak{M}$, Baire functions and $(\mathcal{F} \cap \mathcal{G})$-generated ones coincide.

This is an immediate consequence of 4.7, 4.6.

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Matematický ústav

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