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TRANSFORMATIONS DETERMINING UNIQUELY A MONOID IV
WEAK DETERMINANCY

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Dedicated to Prof. Š. Schwarz to his 60th-birthday

Abstract: This paper is a direct continuation of the paper [7].

Key words: Algebraic monoid, Caleye's representation, left translation, right translation, algebraic isomorphism.

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In this paper we shall use all conventions, notions and results given in [7].

First we are going to give an answer to the question of the form of a connected weakly determining translation with a bijective kernel.

Theorem 1. A connected translation $f: X \longrightarrow X$ with a bijective kernel is weakly determining if and only if one of the following conditions holds:

1) f is a determining translation;

2) f is a bijective translation;

3) if $Q_f \neq Z_f$, then $|X \setminus Q_f| \leq 2$ and for all $x \in Q_f$ it is $|f^{-1}(x)| \leq 2$;

4) for $|Z_f| = p$, e being a top element, $u(e) = 1$ and

the following three conditions are fulfilled:

a) for all $x \in Z_f$ it is $|f^{-1}(x)| \leq 2$;

b) there are no elements $x, y \in A$, $z \in K$ such that $(d(x) - d(y) = d(z)) \equiv \bmod p$;

c) for all $x \in A$ there exists an integer $r(x)$ relatively prime to p and a set $\{x_1, \ldots, x_n\} \subset A \cup \{e\}$ such that the system

$\{\{(d(x_i)r^{\ell} + d(x)(r(x)^{\ell-1} + r(x)^{\ell-2} + \ldots + 1)) \bmod p\}_{\ell=0,1,2,\ldots}\}_{i=1,\ldots,n}$

forms the decomposition of the set $\{d(y) \mid y \in A \cup \{e\}\}$.

Remark. If Condition 3) or 4) is fulfilled for $e \in X$, then it is fulfilled for all $x \in A$. In this case $A \cup \{e\}$ is the set of all top elements of f .

Proof: Evidently if f is a bijection, then f is a weakly determining translation. Consider f for which $u(e) \geq 1$. Using constructions in [5] and [6] and the fact given in [1] and [3] that every connected translation with a bijective kernel is a left translation of commutative monoid, we get the following assertions:

(A) If either $Q_f \neq \emptyset$ and $u(e) \geq 2$ or $Q_f = \emptyset$, then f is a weakly determining translation if and only if f is a determining one.

(B) If f is a weakly determining translation and $u(e) = 1$ then for all $x \in X$ it is $|f^{-1}(x)| \leq 2$ and there are no elements $x, y, z, u \in A \cup \{e\}$ with $d_x(y) = d_z(u)$.

Assume f is a translation with $Q_f \neq \emptyset$ and $u(e) = 1$ for which (B) holds. It is easy to show that in this case for a given top element e there is exactly one Cayley's T-monoid $(X,L(M))$ for which e is an exact source.

Now let $Q_f = Z_f$. For an isomorphism φ between M_e

and M_x , M_e , M_x being the only monoids containing f as a left translation and having e , x as the identity element, resp., we have $\varphi(e) = x$. Designate $f(e) = a$, $\varphi(a) = b \in Z_f$. (It holds that φ is an isomorphism, thus $\varphi(Z_f) = Z_f$ and therefore $b = a^s$ in M_e for some $s > 0$.) Hence $b = f^{s-d_e(x)}(x)$ and further for all $n \geq 0$, b^n in M_x is equal to $f^{nr}(x)$, where for simplicity we write r instead of $s - d_e(x)$. So $\varphi(a^n) = f^{nr}(x) = f^{nr+d_e(x)}(e)$. Take $y \in A$, then it holds $a.y = f^{d_e(y)+1}(e)$ (in M_e), thus $b \circ \varphi(y) = f^{(d_e(y)+1)r+d_e(x)}(e)$ and also by the definition of M_x it is $b \circ \varphi(y) = f^r(\varphi(y)) = f^{r+d_e(\varphi(y))}(e)$.

Hence $d_e(\varphi(y)) = ((d_e(y)r + d_e(x)) \bmod p$. $(\ast)$

If $(\ast)$ holds for all $y \in A$, φ is a homomorphism between M_e and M_x . As M_e and M_x are finite (see 4)b)) and of the same cardinality, it is sufficient for φ to be an isomorphism to have $|\varphi(M_e)| = |M_x|$. And it is fulfilled iff r is relatively prime to p and f satisfies Condition 4) for e and x .

Take f with $Z_f = \emptyset$, $Q_f \neq \emptyset$, $u(e) = 1$. If f fulfils Condition 3) then either f is a determining translation or $A = \{x\}$. Designate again $f(e) = a$ and define $\varphi : M_e \longrightarrow M_x$ as follows:

$\varphi(x) = e$, $\varphi(e) = x$, $\varphi(a^n) = f^{d_e(x)-n+1}(e) \cap Q_f$ and $\varphi(y) = f^{d_e(x)-n+1}(e) \cap Q_f$ for $y \in Q_f$ with $a^n y = a$. Such φ is an algebraic homomorphism, moreover it is a bijection, thus it is an isomorphism.

Suppose Condition 3) does not hold. Let e, x be two distinct top elements, φ an isomorphism between M_e and M_x, $f(e) = a$. As the left translation of a is connected, so is the left translation of $\varphi(a)$ in M_x. But there are only two elements of M_x with connected left translation: $b = f(x)$ and $c = f^{-2}(f(x)) \cap Q_f$.

First consider $\varphi(a) = b$. As $b = a^{d_e(x)+1}$ (in M_e) we have $\varphi(b) = b^{d_e(x)+1}$ in M_x. Further $b^{d_e(x)+1} = f^{d_e(x)+1}(x) = f^{2d_e(x)+1}(e)$. On the other hand, $\varphi(x) = z \notin Q_f$ (φ is a bijection), thus $d_e(z) = 2d_e(x)$ and we have $d_e(x) = d_x(z)$, a contradiction with (B).

Similarly it can be shown that if $\varphi(a) = c$, the conditions from (B) do not hold for $e, \varphi(u), x, u$, where $u \in T \setminus \{x, e\}$.

Thus Theorem 1 has been proved.

<u>Theorem 2</u>. Let f be a connected non-surjective translation with an increasing kernel. Then f is weakly determining if and only if f is determining.

<u>Proof</u>: Evidently if f is a determining translation, thus it is also weakly determining.

Let f be a weakly determining translation. Using constructions in [6] and Construction 1, we get that either f is a determining translation or f has more than one top element and satisfies Conditions (ii) - (vi) from Theorem 3 in [6].

Suppose f has two distinct top elements e_1, e_2. $f^{u(e_1)}(e_1) = f^{u(e_2)}(e_2)$ contradicts Condition (iv), $f^{u(e_2)}(e_2) \subset T_{1,1} \cap Q_f$ contradicts Condition (vi) from

Theorem 3 given in [6]. Therefore f has exactly one top element, i.e. f is a determining translation.

Now we shall deal with a connected surjective translation which has an increasing kernel.

To formulate the necessary and sufficient conditions for f to be a weakly determining translation we introduce other notions. For a given $x \in X$, $N_x = f^{-1}(x) \setminus P_f(e)$, define $N = \{x \in X; |N_x| > 1\}$.

Let $x, y \in X$, define an equivalence $\sim$ as follows:

$x \sim y$ iff $\mathcal{L}_x$ is isomorphic to $\mathcal{L}_y$.

By $[z]$ we shall mean the set

$[z] = \{y \in X; y \in N_{f(z)}$ and $y \sim z\}$.

To simplify the proof of the following theorem we give two assertions.

<u>Lemma 11</u>. Given $x \in N$, $x \sim y$ and g_1, g_2 translations with (3). Then there exist bijections φ_1 from N_x onto N_y and φ_2 from $N_{g_1(x)}$ onto $N_{g_2(y)}$ satisfying the following properties: $g_2 \varphi_1 = \varphi_2 g_1$, (9)

$\varphi_i(z) \sim z$, $i = 1,2$, for all z (10)

if and only if for every $z \in N_x$ it holds

$|\{u \in [z]; g_1(u) \sim g_1(z)\}| = |\{u \in [\bar{z}]; g_2(u) \sim g_1(z)\}|$,

where $\bar{z} \in N_y$ and $\bar{z} \sim z$.

The proof is obvious.

<u>Convention 1</u>. Given $x \in N$, g translation having Property (3). Denote by B_i^x, $i = 1,2,3$, subsets of N_x as follows

for all $u \in B_1^x$ there is no $z \in N_{g(x)}$ with $g([u]) \subset [z]$;

for all $u \in B_2^x$ there is $z \in N_{g(x)}$ with $g([u]) = [z]$;

for all $u \in B_3^x$ there is $z \in N_{g(x)}$ with $g([u]) \subsetneq [z]$. Denote by $C_i^x = g(B_i^x)$.

Lemma 12. Given $x \in N$, $x \sim y$ and g_1, g_2 translations with (3). Let there exist bijections φ_1, φ_2 from Lemma 11. Denote by B_i^x , C_i^x the sets defined relative to g_1, $\bar{B}_i^y$, $\bar{C}_i^y$ the sets defined relative to g_2. Let B_0 be a subset of B_2^x having the following property:

for all $u \in B_0$, $v \in B_2^x$, $v \sim u$ it is $v \in B_0$,

set $C_o = C_2^x \setminus g_1(B_o)$.

Then for every bijections φ from $B_3^x \cup B_o$ into $\bar{B}_3^y \cup \bar{B}_2^y$ and φ' from $C_1^x \cup C_o$ into $\bar{C}_1^y \cup \bar{C}_2^y$ satisfying (10) there exists exactly one bijection ψ from $N_x \cup N_{g_1}(x)$ onto $N_y \cup N_{g_2(y)}$ satisfying (10) and such that

$$g_2 \psi = \psi g_1 ,$$

$$\psi | B_3^x \cup B_o = \varphi, \psi | C_1^x \cup C_o = \varphi' .$$

The proof is obvious.

Theorem 3. Given a connected surjective translation f with an increasing kernel. Then f is a weakly determining translation if and only if there is $e \in T$ (T being the set of all top elements of f), g having (3) for which the following holds:

1) $\mathcal{J}\mathcal{C}(f) | T$ is a transitive group.

2) For all $x \in X$ and $y \in N_x$ $[y]$ is a finite set.

3) For all $x \in N$, $y_1, y_2 \in N_x$, $y_1 \not\sim y_2$ such that $g(y_1) \sim g(y_2)$ it holds: for all g_1 with (3) and k being an integer

$$g_1^k(g(y_1)) \sim g_1^k(g(y_2)) .$$

4) For all $x \in N$, $y \in N_x$ such that there is g_1 with (3)

and $g^{-1}(g_1(y)) = \emptyset$ it holds: if $z \in [y]$ then for all g_2 with (3) and $i = 1,2,\dots$ it is

$$g^i(u) \sim g_2^i(y) .$$

5) For no $x \in N$, $y_1,\dots,y_n \in N_x$ with $y_i \not\sim y_j$, $g(y_i) \not\sim g(y_1)$ for $i \neq j$, $i,j = 1,\dots,n-1$ and $\mathcal{L}_{y_i}$ embeddable into $\mathcal{L}_{g(y_{i+1})}$ for $i = 1,\dots,n-1$, $\mathcal{L}_{y_n}$ cannot be embedded into $\mathcal{L}_{g(y_1)}$.

6) For all $x \in N$ and $y \in N_x$ such that $g^{-1}(y) = \emptyset$ it holds if $y_1, y_2 \in L_y \cap T_{m,n}$, then $|f^{-1}(y_1)| = |f^{-1}(y_n)|$.

7) For all $T_{o,n} \cap N \neq \emptyset$, $n \geq 0$, it is $N \cap T_{o,n+1} = \emptyset$.

8) Let $x \in N \cap (X \setminus H_o)$, $y \in N_x$ with $g^{-1}(y) = \emptyset$, let m be the smallest integer with $g^{-1}(f^m(x)) \neq \emptyset$; then

$$|f^{-1} g^{-1}(h^{m-1} f^m(x)) \setminus P_f(e)| = 1 .$$

9) For all $x \in T_{m,1}$, $m > 1$ such that $\mathcal{H}_o$ can be embedded into $\mathcal{L}_x$ it is $g^{-1}(x) \neq \emptyset$.

10) If for some elements $x_i \neq e$, $i = 1,2,3$ it holds $g^{-1}(x_i) = h^{-1}(x_i) = \emptyset$, $x_i \in T_{m_i,n_i}$, $i = 1,2,3$ and for $n_2 > m_1 f(x_3) = h^{n_1-1} f^{m_1}(x_2)$, for $n_2 \leq m_1$ $f(x_3) = g^{m_2} k^{n_2}(f(x_1))$, then only some of the following possibilities may hold:

a) $x_2 = x_3$, $x_1 \neq x_2$ and $n_1 = n_2 = m_2$,

b) $x_1 = x_3$, $x_1 \neq x_2$ imply $n_2 = m_1$, $n_2 \geq n_1$ and $f(x_3) = f(g^{m_2} k^{n_2}(f(x_3)))$.

<u>Proof of Theorem 3</u>: In the first part of the proof we show that every weakly determining translation satisfies Conditions 1 - 10.

Denote by g, h, k the translations having Properties (3) and (4) for a top element e.

For the proof of necessity of Conditions 1 - 8 one can use Lemma 3; assuming the contrary of any of these conditions we get two quadruples e_i, g_i, h_i, k_i, $i = 1,2$ satisfying (3) and (4) for which a bijection

$$\varphi : X \to X, \ \varphi(e_1) = e_2, \ \varphi g_1 = g_2 \varphi, \ \varphi h_1 = h_2 \varphi, \ \varphi k_1 = k_2 \varphi \qquad (11)$$

does not exist.

The necessity of Condition 9 follows from Construction 3.

The necessity of Condition 10b) follows from Construction 5. The only fact which is not evident is the following:

if Condition 10a) or b) is not fulfilled, then the assumptions of Construction 4 hold for some $\overline{x}_i$, $i = 1,2,3$.

Suppose $x_2 = x_3$, then for $n_2 > m_1$ it must be $m_1 = n_1 = 1$. See $m_1 = 0$ implies $n_1 = 0$ and it contradicts $x_1 \neq e$. Assume $n_2 \leq m_1$, then $x_3 \in T_{m_1-n_2+m_2, n_1}$, hence from $x_2 = x_3$ we have $m_1 = n_2 = n_1$. Thus if $x_2 = x_3$ and $x_1 \neq T_{1,1}$, then $x_1 = x_2 = x_3$. In all cases we can set $\overline{x}_i = x_1$, $i = 1,2,3$.

The second part of the proof is to show that the conditions of Theorem 3 are also sufficient, i.e. that every f satisfying Conditions 1 - 10 is weakly determining. First we show that for every e_1, g_1, h_1, k_1 fulfilling Conditions (3) and (4) and such that $h_1(e) = k_1(e)$, there exists a bijection $\varphi \in \mathscr{C}(f)$ with (11).

Using Lemma 3 from this it follows that all monoids given by Construction 2 are isomorphic.

Let us prove that there exists exactly one $\overline{k}$ (and thus

$\bar{k} = k$) such that $e, g, h, \bar{k}$ satisfy (3) and (4) and $h(e) = \bar{k}(e)$. Using the induction on n, x being an element of $T_{m,n}$, and Condition 7 for $m = 0$ and Condition 8 for $m > 0$, we get $\bar{k}(x) = h\bar{k}f(x)$ for $g^{-1}(x) = \emptyset$. Thus for given e, g, h there is exactly one k with (4).

Obviously from Condition 6 we have for e, g and $h_1 \in \mathscr{C}(g)$, $fh_1 = 1_X$ a bijection $\psi \in \mathscr{C}(f,g)$ such that $\psi(e) = e$, and $\psi h_1 = h\psi$.

Now we prove that if Conditions 2 - 5 hold, then for one fixed e, g and arbitrary g_1 with (3) there is a bijection $\varphi \in \mathscr{C}(f)$ with $\varphi(e) = e$ and $\varphi g = g_1\varphi$. The proof of this assertion is divided into two steps: first we show that there are isomorphisms $\varphi_m : \mathscr{H}_m \longrightarrow \mathscr{H}_m$, $\varphi'_m : \mathscr{H}_{m+1} \longrightarrow \mathscr{H}_{m+1}$, $m = 0,1,\ldots$ such that $g_1 \varphi_m = \varphi'_m g$.

This is proved by induction on k , x being an element of $T_{m,k}$. Suppose we have defined φ_m for all $x \in \bigcup_{i<k} T_{m,i}$, φ'_m for all $x \in \bigcup_{i<k} T_{m+1,i}$ and moreover $x \sim \varphi_m(x)$, $\varphi'_m(g(x)) \sim g(x)$ for all x . Evidently $\varphi_m(f^m(e)) = f^m(e)$, $\varphi'_m(f^{m+1}(e)) = f^{m+1}(e)$ have the required property. Let us construct φ_m for elements of $T_{m,k}$, φ'_m for elements of $T_{m+1,k}$.

Take $x \in T_{m,k-1}$; if $x \notin N$ it can be easily shown that there is only one extending of φ_m on N_x , φ'_m on $N_{g(x)}$ (use $N_{g(x)} \sim N_x$) with the required properties.

b) Let $x \in N$. By Lemma 11 it is sufficient to show that for every $y \in N_x$ the following holds: if $\bar{y} \in N_{\varphi_m(x)}$, $y \sim \bar{y}$, then

$|\{z \in [y] \, ; \, g(z) \sim g(y)\}| = |\{z \in [\bar{y}] \, ; \, g_1(z) \sim g(y)\}|$.

Assume the contrary; we shall construct a sequence $\{y_i\}_{i=0}^{\infty}$ with the following properties: $y_i \in N_x$, $y_i \not\sim y_j$, $g(y_i) \not\sim g(y_j)$,

$\mathcal{L}_{y_i}$ embeddable into $\mathcal{L}_{g(y_{i+1})}$, $i \neq j$, $i,j = 0, 1, \dots$, and

$|\{z \in [y_i] ; g(z) \sim g(y_i)\}| > |\{z \in [\bar{y}_i] ; g_1(z) \sim g(y_i)\}|$, where $\bar{y}_i \in N_{\varphi_m}(x)$, $\bar{y}_i \sim y_i$. By the assumption we know that there is an element y with the required properties, put $y_0 = y$. Use an induction, let $\{y_i\}_{i<k}$ be constructed and construct y_k. For y_{k-1} it holds

$|\{z \in [y_{k-1}] ; g(z) \sim g(y_{k-1})\}| > |\{z \in [\bar{y}_{k-1}] ; g_1(z) \sim g(y_{k-1})\}|$, $\bar{y}_{k-1} \in N_{\varphi_m}(x)$, $\bar{y}_{k-1} \sim y_{k-1}$.

By Condition 2 there exists $z_{k-1} \in [y_{k-1}]$ such that

$|\{z \in [y_{k-1}] ; g(z) \sim g_1(z_{k-1})\}| < |\{z \in [\bar{y}_{k-1}] ; g_1(z) \sim g_1(z_{k-1})\}|$

and moreover we can suppose that for this z_{k-1} it is $g(z_{k-1}) \not\sim g(y_i)$ for $i < k-1$ (use $y_1, \dots, y_k$ fulfil Condition 5). Using Condition 2 and the induction assumption we get that there is y_k such that $g_1(y_k) \sim g_1(z_{k-1})$ and $|\{z \in [y_k] ; g(z) \sim g(y_k)\}| > |\{z \in [\bar{y}_k] ; g_1(z) \sim g(y_k)\}|$, $\bar{y}_k \in N_{\varphi_m}(x)$, $\bar{y}_k \sim y_k$ and $\mathcal{L}_{y_{k-1}}$ embeddable into $\mathcal{L}_{g(y_k)}$. Assuming that $y_k \sim y_i$ for some $i < k$ we get that $y_{i+1}, \dots, y_k$ do not satisfy Condition 5. Hence we have constructed the sequence $\{y_i\}_{i=0}^{\infty}$.

Now define g_2 as follows:

for $z \in X \setminus \bigcup_{i=0}^{\infty} L_{y_i}$ put $g_2(z) = g(z)$,

$g_2 | L_{y_i}$ is an embedding of $\mathscr{L}_{y_i}$ into $\mathscr{L}_{g(y_{i+1})}$. Evidently g_2 has (3) and $g_2^{-1}(g(y_0)) = \emptyset$. By Condition 4 it is $g(y_0) \sim g_1(y_1)$, a contradiction.

Thus we have shown that bijections φ_m, φ'_m can be extended to N_x, $x \in N$.

Now let us construct a bijection $\varphi : X \longrightarrow X$ with the required properties.

The bijection will be constructed if we have a sequence of bijections $\{\psi_k\}_{k=0}^{\infty}$, $\psi_k : \bigcup_{i=0}^{k+1} H_i \longrightarrow \bigcup_{i=0}^{k+1} H_i$ such that

$$\psi_k f = f \psi_k \quad \text{and} \quad g_1 \psi_k = \psi_k g$$

and moreover for all $x \in X$ there is an integer k_x such that for all $k > k_x$ it is $\psi_k(x) = \psi_{k_x}(x)$.

We shall construct a sequence $\{\psi_k\}_{k=0}^{\infty}$ by an induction on k. Take $\psi_0 = \varphi_0 \cup \varphi'_0$. Suppose we have ψ_i for all $i < k$; the sequence $\{\psi_i\}_{i<k}$ has the following property:

if $\psi_i(z) \neq \psi_{i+1}(z)$, $z \in T_{r,s}$, then there is $u \in T_{r,q} \cap N$, $q < s$ such that

$$g_1([y]) \nsubseteq [v] \quad \text{for any} \quad v, y \in N_u .$$

Let us define ψ_k by an induction on n, x being an element of $T_{i,n}$, $i \leq k + 1$.

Assume ψ_k is defined for all $x \in T_{i,j}$, $i \leq k + 1$, $j \leq n$, define $\psi_k \mid \bigcup_{i=0}^{k+1} T_{i,n+1}$. Take $x \in T_{k,n}$, if $x \notin N$, then evidently there is only one possibility of extending ψ_k to N_x with the required properties.

Assume $x \in N$, divide N_x into three parts B_1^x, B_2^x, B_3^x (see Convention 1) as in Lemma 12. Take $\psi_{k-1} \mid B_3^x$,

$\psi_{k-1} | B_2^x$ and ω a bijection from C_1^x onto $\overline{C}^{\psi_k(x)}$ such that for all $z \in C_1^x$ it is $z \sim \omega(z)$. Using Lemma 12 we get $\psi_k | N_x$ such that $\psi_k | B_2^x \cup B_3^x = \psi_{k-1} | B_2^x \cup B_3^x$ and $\psi_k | C_1^x = \omega$.

Moreover, it holds:

for all $u \in B_1^x$ there is an integer r_u such that $g^{-r_u}(y) \neq \emptyset$ and $g^{-(r_u+1)}(y) = \emptyset$ for all $y \in B_1^x$, $g(u) \sim g(y)$. This assertion follows from Condition 4. Denote $r = \max_{u \in B_1^x} (r_u)$ (evidently $g^{-r}(x) \neq \emptyset$). Put $z = g^{-1}(x)$, B_i^z, $i = 1,2,3$. As Condition 4 holds we have $g^{-1}(B_1^x) \subset B_1^z \cup B_2^z$. Thus there is exactly one extending of $\psi_k | B_1^x$, $\psi_{k-1} | B_3^z \cup \cup (B_2^z \setminus g^{-1}(B_1^x))$ to $\psi_k | N_z$ (use Lemma 12). The proof goes by the induction up to $N_{g^{-r}(x)}$.

Given $x \in T_{m,n}$, suppose $\psi_s(x) \neq \psi_{s+1}(x)$ for some s. By construction of ψ_{s+1} it means that there is $y \in N \cap \cap T_{s+1,q}$, $q < n$ and there are $u_1, u_2 \in g^{s+2-m}([x])$, $u_1 \not\sim u_2$. As $[x]$ is a finite set (use Condition 2) so is $N \cap T_{p,q}$ for $q < n$, hence there is only a finite number of s with $\psi_s(x) \neq \psi_{s+1}(x)$. Now $k_x = \max s$ has the required property.

Hence the existence of a bijection φ with (11) has been shown.

Let e_1, g_1, h_1, k_1 satisfy Conditions (3) and (4) and $h_1(e_1) = k_1(e_1)$. From Condition 1 the existence of a bijection $\varphi_1 \in \mathcal{C}(f)$ with $\varphi_1(e_1) = e$ follows. Denote $g' = \varphi_1 g_1 \varphi_1^{-1}$, $h' = \varphi_1 h_1 \varphi_1^{-1}$, $k' = \varphi_1 k_1 \varphi_1^{-1}$. Translations g', h', k' with e have the property (3) and (4); thus we have a bijection $\varphi_2 \in \mathcal{C}(f)$ such that $\varphi_2(e) = e$ and

$\varphi_2 g' = g\varphi_2$. Put $\bar{h} = \varphi_2 h' \varphi_2^{-1}$, $\bar{k} = \varphi_2 k' \varphi_2^{-1}$. Also $e, g, \bar{h}, \bar{k}$ satisfy (3) and (4) and hence there is a bijection $\varphi_3 \in \mathcal{C}(f,g)$ for which $\varphi_3(e) = e$, $\varphi_3 \bar{h} = h\varphi_3$. Put $\tilde{k} = \varphi_3 \bar{k} \varphi_3^{-1}$. Define $\varphi = \varphi_3 \varphi_2 \varphi_1$, φ is a bijection with (11). But we have proved that $\tilde{k} = k$ (there is only one k with the property (4)), hence we have a bijection φ for which we can use Lemma 3.

Let now M' be an arbitrary monoid with $f \in L(M')$, e' its identity element. In [2] it has been proved that there exist $g', k' \in R(M')$, $h' \in L(M')$ such that e', g' satisfy (3), $fh' = k'g' = 1_M$, and $k'(e') = h'(e')$. Further in [2] it has been shown that there exists k'' such that $k''(e') = h'(e')$ and e', g', h', k'' satisfy (4). So as we have shown in the previous part of the proof, there exists a bijection ψ with (11). Therefore f fulfils Conditions 1 - 10 for e', g', h'. So it holds $m \geq 1$, $k'(T'_{m,1}) \subset T'_{m-1,1}$ (the sets $T'_{m,n}$ are defined relative to e'). Assume the contrary, i.e. there is $x \in T'_{m,1}$ and $k'(x) = f^{m-2}(e)$, hence the translation g_x is injective, but this is not possible because of Condition 9 . Thus also e', g', h', k' have the properties (4) and so $k'' = k'$. The bijection ψ induces an isomorphism φ of M' onto $\bar{M}$ such that $f, h \in L(\bar{M})$, $g, k \in R(\bar{M})$. Denote by M the monoid given by Construction 2 and containing e, f, g, h, k . The proof will be finished if we show that $M = \bar{M}$.

We show even more, we give the proof of the following assertion: Let e, f, g, h, k be translations as above, then for every algebraic monoid $\bar{M}$ with $f, h \in L(\bar{M})$, $g, k \in R(\bar{M})$

and e identity element of $\overline{M}$, it holds $\overline{f}_x = f_x$ where f_x are translations given in Construction 2 .

Define an ordering $\preceq$ as follows: $(m,n) \preceq (m',n')$ if $m < m'$ or $m = m$ and $n \leq n'$. Evidently $\preceq$ is a well-ordering. We shall use an induction on (m,n) with the ordering $\preceq$, $x \in T_{m,n}$. Evidently $T_{o,o} = \{e\}$ and $\overline{f}_e = 1_X = f_e$.

Suppose $\overline{f}_u = f_u$ for all $u \in T_{m,n}$, $(m',n') \prec (m,n)$. Take $x \in T_{m,n}$. Consider three cases:

1) Let $x = h(y)$, then $y \in T_{m,n-1}$, and $\overline{f}_x = h\overline{f}_y = hf_y = f_x$; use $\overline{f}_x(e) = h\overline{f}_y(e)$ and e is an exact source of $L(\overline{M})$.

2) Let $x = g(y)$, then $y \in T_{m-1,n}$, and $\overline{f}_x = \overline{f}_y f = f_y f = f_x$; use $\overline{f}_x(e) = \overline{f}_y f(e)$ and e is an exact source of $L(\overline{M})$.

3) Consider $g^{-1}(x) = h^{-1}(x) = \emptyset$. For the proof that for such x it holds $\overline{f}_x(t) = f_x(t)$ we shall need an induction on (p,q), t being an element of $T_{p,q}$. Evidently $\overline{f}_x(e) = f_x(e)$. Assume for all $u \in T_{p',q'}$, $(p',q') \prec (p,q)$, it is $\overline{f}_x(u) = f_x(u)$; take $t \in T_{p,q}$. Again we have three possibilities:

a) Consider $t = h(v)$, then $\overline{f}_x(t) = hk\overline{f}_{f(x)}(v) = hkf_{f(x)}(v) = f_x(t)$, use $\overline{f}_x(h(e)) = \overline{f}_{k(x)}(e)$, $k(x) = hkf(x)$ and the induction assumption.

b) Consider $t = g(v)$, then $\overline{f}_x(g(v)) = g\overline{f}_x(v) = gf_x(v) = f_x(t)$, as $v \in T_{p-1,q}$.

c) Consider $g^{-1}(t) = h^{-1}(t) = \emptyset$. Let us suppose $\overline{f}_x(t) = z$. We know that $f\,\overline{f}_x(t) = \overline{f}_{f(x)}(t) = f_{f(x)}(t)$, hence $\overline{f}_x(t) \in f^{-1}(f_{f(x)}(t))$. If $h^{-1}(z) \neq \emptyset$ then it is $z = f_x(t)$, for $f_{f(x)}(t) = g^p k^q(f(x))$, use the property of k . Further $k\bar{f}_x(t) = khk\bar{f}_{f(x)}(f(t)) = khkf_{f(x)}(f(t)) = kf_x(t)$. Hence if $g^{-1}(z) \neq \emptyset$, then $z = f_x(t)$.

Therefore the only possibility of $z = \bar{f}_x(t)$ to be $z \neq f_x(t)$ is z with $g^{-1}(z) = h^{-1}(z) = \emptyset$. Consider there are three elements x, t, z with $g^{-1}(a) = h^{-1}(a) = \emptyset$, $a = x,t,z$ and $z \in f^{-1}(f_{f(x)}(t))$. As $f(x) \in T_{m,n-1}$, $n-1 \geq 0$ (use $g^{-1}(x) = \emptyset$), we have for $q > m$, $z \in f^{-1}(h^{n-1}f^m(t))$ for $q \leq m$, $z \in f^{-1}(g^p k^q f(x))$. Using Condition 10 we know that there may be only two possibilities:

α) $x \neq t$, $t = z$ and $n = p = q$; in this case we have $t \in f^{-1}(g^m k^p f(t))$, thus $m = p$ and $t \in f^{-1}(g^p k^p f(t))$ means that Condition 10 is not fulfilled for $x_i = t$, $i = 1,2,3$.

Consider $x \neq t$ and $x = z$. Assume $q > m$, then $x \in f^{-1}(h^{n-1}f^m(t))$ implies $x \in T_{p,q-m+n}$; therefore $q = m$, a contradiction. So $q \not\leq m$ and $x \in f^{-1}(g^p k^q f(x))$, i.e. $p = q$.

Suppose $\bar{f}_x(t) = x$, then $\bar{f}_x\bar{f}_t(t) = \bar{f}_{\bar{f}_x(t)}(t) = \bar{f}_x(t) = x$, thus $\bar{f}_t(t) \neq f_t(t)$. From this it follows $q = m = p$ and $p > n$ (use the induction assumption and $\bar{f}_t \neq f_t$). Take $\bar{z} = f^b(t)$, $b \geq 0$ such that $\bar{f}_{\bar{z}}(t) \neq f_{\bar{z}}(t)$ and $\bar{f}_{f(\bar{z})} = f_{f(\bar{z})}$. (Such element $\bar{z}$ exists because $f^p(t) = f^p(e)$.) Suppose $\bar{f}_t(t) = v$, then $\bar{f}_{\bar{z}}(t) = f^b\,\bar{f}_t(t) = f^b(v)$. Further $g^{-1}(\bar{z}) = h^{-1}(\bar{z}) = \emptyset$ (use the induction assumption and $\bar{f}_{\bar{z}} \neq f_{\bar{z}}$. Moreover, $g^{-1}(\bar{f}_{\bar{z}}(t)) = h^{-1}(\bar{f}_{\bar{z}}(t)) = \emptyset$, the proof is exactly the same as the proof that $g^{-1}(z) = h^{-1}(z) = \emptyset$.

Therefore either $f^b(v) \neq \bar{z}$ and $\bar{z}, t, f^b(v)$ do not fulfil Condition 10b) or $\bar{z} = f^b(v)$ and $\bar{z} = t$ and again $\bar{z}, t, f^b(v)$ do not fulfil Condition 10, ($\bar{z} \neq t$ and $\bar{z} = f^b(v)$ implies $f(\bar{z}) = g^p k^p f(\bar{z})$).

We shall now deal with disconnected translations.

Theorem 4. A translation $f: X \longrightarrow X$ is weakly determining if and only if there is a top element e for which the following holds:

1) $f \mid E_f(e)$ is a weakly determining translation;

2) Y has at most one element and $|E_f(e)| > |Y|$ or $f \mid Y$ is a disconnected permutation with $Y \subset Z_f$, $r(x)$ does not divide $r(y)$ for any $x, y \in Y$, $x \notin E_f(y)$.

3) If $q \neq 1$ is a common division of all $r(x)$, $x \in Y$ then there exists $x_0 \in Y$ such that for all p relatively prime to $\frac{r(x_0)}{q}$ the expression $\frac{r(x_0)p - q}{q^2}$ is not an integer.

Proof: Let e, e' be two top elements of f; from Condition 2 it follows that $e' \in E_f(e)$. Suppose $f \in L(M)$, M being an algebraic monoid. It can be seen that for f satisfying Conditions 2 and 3 it holds $f_x(y) = x$ for all $x \in Y$, $y \in X$. Moreover if Condition 2 or 3 does not hold then there are two non-isomorphic monoids (see constructions in [6] and Construction 2).

Let M_1, M_2 be two monoids with $f \in L(M_i)$, $e_1 \in E_f(e_2)$, e_i identity element of M_i, $i = 1,2$, and the left translations of M_i corresponding to elements of Y be constants, then for every bijection $\bar{\varphi}: E_f(e_1) \longrightarrow E_f(e_1)$ such that $\bar{\varphi}\, {}^{1}f_x(y) = {}^{2}f_{\bar{\varphi}(x)}(\bar{\varphi}(y))$, $x, y \in E_f(e)$, ${}^{i}f_x \in L(M_i)$, the mapping φ define by

$\varphi(x) = \bar{\varphi}(x)$ for $x \in E_f(e_1)$ and

$\varphi(x) = x$ for $x \in Y$

is an algebraic isomorphism between M_1 and M_2. On the other hand, if there is an isomorphism φ between M_1 and M_2, then $\varphi | E_f(e_1)$ is an isomorphism between monoids given by $L(\overline{M}_1) = \{ {}^1f_x | E_f(e_1) , x \in E_f(e_1) \}$ and $L(\overline{M}_2) = \{ {}^2f_x | E_f(e_1) ; x \in E_f(e_1) \}$.

Thus the proof has been finished.

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