Commentationes Mathematicae Universitatis Carolinae

Václav Slavík

On h-primitive lattices

Commentationes Mathematicae Universitatis Carolinae, Vol. 16 (1975), No. 3, 505--514

Persistent URL: http://dml.cz/dmlcz/105643

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

16,3 (1975)

ON H-PRIMITIVE LATTICES

Václav SLAVÍK, Praha

Abstract: This paper is concerned with h-primitive lattices. There are shown infinitely many primitive classes of lattices which are h-characterizable by means of a single lattice and are not characterizable.

Key words: Primitive class, splitting lattice, projective lattice, characterizable class, h-characterizable class.

AMS: 06A20 Ref. Ž.: 2.724.8

Given a set E of finite lattices, we shall denote by $N(E)$ the class of all lattices that contain no sublattice isomorphic to a lattice in E and by $N_h(E)$ ($N_{hf}(E)$) the class of all lattices L such that no homomorphic image of any sublattice (finite sublattice) of L belongs to E. A class K of lattices will be called characterizable (h-characterizable, hf-characterizable) if there exists a set E of finite lattices such that $K = N(E)$ ($K = N_h(E)$, $K = N_{hf}(E)$). If E is a finite set of finite lattices $\{L_1,\dots,L_n\}$, the classes $N(E)$, $N_h(E)$and NHf(E) will be denoted by $N(L_1,\dots,L_n)$, $N_h(L_1,\dots,L_n)$ and $N_{hf}(L_1,\dots,L_n)$, respectively. A finite lattice L is said to be primitive (see [3]) (h-primitive, hf-primitive) if the class $N(L)$ ($N_h(L)$, $N_{hf}(L)$) is primitive. It is evident that $N(E) \supseteq N_{hf}(E) \supseteq N_h(E)$. If $K = N(E)$ is a characterizable primitive class of lattices and a lattice L does not belong to $N_h(E)$ then there exists a homomorphism of a sublattice S of L onto a lattice A in E. Since $A \notin K$, $S \notin K$ and $L \notin K$. We have proved that K is hf-characterizable and h-characterizable and $K = N(E) = N_{hf}(E) = N_h(E)$. Similarly we can prove that any hf-characterizable primitive class of lattices $K = N_{hf}(E)$ is h-characterizable and $K = N_{hf}(E) = N_h(E)$. Especially, any primitive lattice L is hf-primitive and $N(L) = N_{hf}(L)$; any hf-primitive lattice L is h-primitive and $N_{hf}(L) = N_h(L)$. If K is an hf-characterizable primitive class of lattices, then a lattice L belongs to K iff any finite sublattice of L belongs to K and thus K is characterizable (see [1]). The purpose of the present paper is to show that there exist h-primitive lattices that are not hf-primitive, hf-primitive lattices that are not primitive and h-characterizable primitive classes of lattices that are not characterizable. Notice that Igošin ([2]) has shown that there exist h-characterizable primitive classes of algebras with one unary operation that are not characterizable.

McKenzie ([5]) investigates splitting lattices, i.e. finite subdirectly irreducible lattices B such that there exists an equation $p = q$ and any primitive class K of lattices satisfies precisely one of the following conditions: either K satisfies $p = q$ or $B \in K$.

<u>Theorem 1</u>. A finite lattice B is h-primitive if and only if B is a splitting lattice.

<u>Proof</u>. Let B be an h-primitive lattice. The class

$N_h(B)$ is finitely based ([2]) and thus it can be characterized by an equation $p = q$. Let K be a primitive class of lattices that does not satisfy the equation $p = q$. Then there is a lattice $L \in K$, $L \notin N_h(B)$. Since B is a homomorphic image of a sublattice of L, $B \in K$. The equation $p = q$ is not satisfied in B and thus B is a splitting lattice. But if B is a splitting lattice, then there exists (see [5]) a homomorphism f of $FL(k)$, the free lattice with k generators, onto B and $p, q \in FL(k)$ such that $\operatorname{Ker} f$ is the greatest congruence of $FL(k)$ that separates p, q. We shall show that $N_h(B)$ is the class of all lattices satisfying the equation $p = q$. If $L \notin N_h(B)$, then there is a homomorphism of a sublattice of L onto B and since $p = q$ is not satisfied in B, the equation $p = q$ is not satisfied in L. If a lattice L does not satisfy $p = q$, then there exists a homomorphism h of $FL(k)$ into L such that $h(p) \neq h(q)$. Since $\operatorname{Ker} h \subseteq \operatorname{Ker} f$, there exists a homomorphism g of L into B such that $g \circ h = f$ and thus $L \notin N_h(B)$. So the class $N_h(B)$ is primitive, i.e. B is h-primitive.

Theorem 2. Let B be a finite lattice. The following conditions are equivalent:

(1) B is hf-primitive.

(2) B is h-primitive and $N_h(B)$ is characterizable.

(3) B is subdirectly irreducible and there exists a homomorphism f of a finite sublattice L of a free lattice onto B.

Moreover, if B is hf-primitive, then $N_{hf}(B) = N_h(B) = N(E)$, where E is the set of all lattices A such that there exist homomorphisms g of L onto A and h of A onto B with $h \circ g = f$.

<u>Proof</u>. Assume (1). Then B is evidently h-primitive, $N_h(B) = N_{hf}(B)$ and since any hf-characterizable primitive class is characterizable, we have (2). Suppose (2). Let $N_h(B) = N(E)$. Since B is a homomorphic image of some $FL(k)$, $FL(k)$ does not belong to $N_h(B) = N(E)$, there exists a finite sublattice C of $FL(k)$ isomorphic to a lattice in E. $C \notin N(E) = N_h(B)$ and we get that there is a homomorphism of a sublattice L of C onto B. It is evident that L is a sublattice of $FL(k)$. Clearly, any h-primitive lattice must be subdirectly irreducible. Now, assume (3). McKenzie ([5]) has shown that any finite subdirectly irreducible lattice which is a homomorphic image of a free lattice is a splitting lattice, i.e, h-primitive. We shall show that $N_h(E) = N_{hf}(B)$. Clearly, $N_h B \subseteq N_{hf}(B)$. If a lattice $S \notin N_h(B)$, there exists a homomorphism h of a sublattice C of S onto B. Since L is projective ([4],[5]), there is a homomorphism g of L into C such that $h \circ g = f$. Since $g(L)$ is a finite sublattice of S, $S \notin N_{hf}(B)$. Thus $N_h(B) = N_{hf}(B)$ and so B is hf-primitive. The proof can now be finished easily.

Given a finite lattice L, define a lattice L^* in this way: L is a sublattice of L^*, $L^* \setminus L$ contains exactly three elements u, v, a; u is the smallest and v the greatest element of L^* and a is comparable with no element of L.

<u>Theorem 3</u>. Let L be a h-primitive lattice. Then L^*

is h-primitive, too. Moreover, the following holds:

(1) If $N_h(L)$ is the class of all lattices satisfying an equation $p(x_1,\dots,x_n) = q(x_1,\dots,x_n)$, then $N_h(L^*)$ is the class of all lattices satisfying the equation $p^*(x_1,\dots,x_{n+1}) = q^*(x_1,\dots,x_{n+1})$, where $p^*(x_1,\dots,x_{n+1}) = p(t_1,\dots,t_n)$, $q^*(x_1,\dots,x_{n+1}) = q(t_1,\dots,t_n)$ and $t_k = (x_k \wedge i) \vee o$ ($k = 1,2,\dots,n$), $o = (x_1 \wedge \dots \wedge x_n) \vee (x_{n+1} \wedge (x_1 \vee \dots \vee x_n))$, $i = (x_1 \vee \dots \vee x_n) \wedge (x_{n+1} \vee (x_1 \wedge \dots \wedge x_n))$.

(2) L^* is hf-primitive iff L is hf-primitive.

(3) L^* is primitive iff L is primitive.

<u>Proof</u>. Let $N_h(L)$ be the class of all lattices satisfying the equation $p(x_1,\dots,x_n) = q(x_1,\dots,x_n)$. Let $a_1,\dots,a_n$, d be elements of a lattice S such that $p^*(a_1,\dots,a_n,d) \neq q^*(a_1,\dots,a_n,d)$. Put $r = (a_1 \vee \dots \vee a_n) \wedge (d \vee (a_1 \wedge \dots \wedge a_n))$, $s = (a_1 \wedge \dots \wedge a_n) \vee (d \wedge (a_1 \vee \dots \vee a_n))$, $\ell_k = (a_k \wedge r) \vee s$ ($k = 1,2,\dots,n$). Since $p^*(a_1,\dots,a_n,d) = p(\ell_1,\dots,\ell_n)$ and $q^*(a_1,\dots,a_n,d) = q(\ell_1,\dots,\ell_n)$, the equation $p = q$ is not satisfied in the interval $[s,r]$. There exists a homomorphism f of a sublattice S' of $[s,r]$ onto L. Since $d \wedge r = d \wedge s$ and $d \vee r = d \vee s$, the set $S' \cup \{d, d \wedge r, d \vee r\}$ forms a sublattice of S that can be homomorphically mapped onto L^*. Thus $S \notin N_h(L^*)$. The equation $p = q$ is not satisfied in L and thus there exist elements $a_1,\dots,a_n$ of L such that $p(a_1,\dots,a_n) \neq q(a_1,\dots,a_n)$. Clearly, $p^*(a_1,\dots,a_n,a) = p(a_1,\dots,a_n)$ and $q^*(a_1,\dots,a_n,a) = q(a_1,\dots,a_n)$. The equation $p^* = q^*$ is not satisfied in L and we get that any

lattice satisfying $p^* = q^*$ belongs to $N_h(L^*)$. It is easy to show that L is a homomorphic image of a finite sublattice of a free lattice (L is a sublattice of a free lattice) iff L^* has the same property.

Now we shall show that there exist h-primitive lattices that are not hf-primitive.

<u>Lemma</u> 1. For any positive integer n , the lattice B_n in Fig. 1 is generated by the elements a,b,c , and there exists a homomorphism f_n of B_n onto the lattice B_o in Fig. 1 such that $f_n(a) = a$, $f_n(b) = b$, $f_n(c) = c$.

<u>Proof</u>. It is easy to verify that the elements o,d,e,f, k,h,g, ℓ ,p,i,r,s,t,u,v , are in the sublattice C of B_n generated by $\{a,b,c\}$. Since $t_1 = b \vee \ell$, $v_1 = \ell \vee c$, $s_1 = a \vee \ell$, $z_1 = s_1 \wedge v_1$, $u_1 = t_1 \wedge u$, we have $\{s_1,t_1,u_1,v_1,z_1\} \subseteq C$. Assume $\{s_i,t_i,u_i,v_i,z_i\} \subseteq C$. Since $s_i \vee u_i = s_{i+1}$, $v_i \vee u_i = v_{i+1}$, $s_{i+1} \wedge v_{i+1} = z_{i+1}$, $b \vee z_{i+1} = t_{i+1}$ and $t_{i+1} \wedge u = u_{i+1}$, we have $\{s_{i+1},t_{i+1},u_{i+1},v_{i+1},z_{i+1}\} \subseteq C$. Thus we get that $C = B_n$. One can easily verify that the mapping f_n of B_n into B_o defined by $f_n(s_k) = s$, $f_n(t_k) = t$, $f_n(u_k) = f_n(z_k) = u$, $f_n(v_k) = v$ for all k , $1 \leq k \leq n$, and $f_n(x) = x$ for all other $x \in B_n$, is a homomorphism of B_n onto B_o such that $f_n(a) = a$, $f_n(b) = b$, $f_n(c) = c$.

<u>Theorem 4</u>. The lattice B_o in Fig.1 is h-primitive and it is not hf-primitive.

<u>Proof</u>. McKenzie ([5]) has shown that B_o is a splitting lattice, i.e., by Theorem 1, B_o is h-primitive. Suppose that B_o is hf-primitive. By Theorem 2, there exists a homomorphism f of a sublattice C of a free lattice onto B_o . Since C

is projective ([4],[5]), there exist homomorphisms g_n of C into B_n such that $f_n \circ g_n = f$. There exist elements a', b', c' of C such that $g_n(a') = a$, $g_n(b') = b$, $g_n(c') = c$. Thus g_n are homomorphism of C onto B_n and so C cannot be finite; a contradiction.

Corollary 1. Any finite sublattice of a free lattice satisfies the inclusion

$$(a \vee (b \wedge c))(b \vee (a \wedge c))(c \vee (a \wedge b)) \leq (a \wedge (b \vee c)) \vee (b \wedge (a \vee c)) \vee$$
$$\vee (c \wedge (a \vee b)).$$

Proof. All finite sublattices of a free lattice belong to $N_h(B_o)$ and $N_h(B_o)$ is the class of all lattices satisfying this inclusion (see [5]).

Starting from the lattice B_o in Fig. 1, we can obtain by Theorem 3 an infinite sequence of h-primitive lattices that are not hf-primitive. Hereby we obtain infinitely many h-characterizable primitive classes of lattices that are not characterizable.

Finally we shall give a construction of hf-primitive lattices that are not primitive.

Let A be the lattice given in Fig. 1 and let L be a primitive lattice (i.e. a finite subdirectly irreducible sublattice of a free lattice) of cardinality greater than two. Define a lattice $A(L)$ in this way: $A(L) = A \cup L$, A and L are sublattices of $A(L)$, $x \wedge y = x \wedge a = x \wedge c$ and $x \vee y = x \vee a = x \vee c$ for all $x \in A$, $y \in L$.

Lemma 2. The lattice $A(L)$ is a sublattice of a free lattice.

Proof. We shall show that $A(L)$ is projective. Let f

be a homomorphism of a lattice S onto $A(L)$. Since A is projective (see [3],[5]), there exists a sublattice A' of S such that $f|_{A'}$ is an isomorphism of A' onto A. Let $a' \in A'$ and $b' \in A'$ be such that $f(a') = a$ and $f(b') = b$. If $c \in S$ and $f(c) \in L$, then $f((c \vee b') \wedge a) = f(c)$. The interval $[b', a']$ is mapped by f onto L. The lattice L is projective and thus there exists a sublattice L of $[b', a']$ such that $f|_{L'}$ is an isomorphism of L' onto L. The set $A' \cup L'$ forms a sublattice of S and $f|_{A' \cup L'}$ is an isomorphism of $A' \cup L'$ onto $A(L)$.

If we identify in $A(L)$ the greatest element v of L with a and the smallest element u of L with b, we get a subdirectly irreducible lattice $B(L)$ that is a homomorphic image of $A(L)$. Since v is join reducible, i.e. there are $v_1, v_2 \in L$ such that $v \neq v_1$, $v \neq v_2$, $v = b_1 \vee v_2$, we get $v_1 \vee v_2 = e \wedge f$ in $B(L)$ and since $e \wedge f \nleq v_1$, $e \wedge f \nleq v_2$, $e \nleq v_1 \vee v_2$, $f \nleq v_1 \vee v_2$ in $B(L)$, the lattice $B(L)$ is not a sublattice of a free lattice. Using Theorem 2 we obtain

<u>Theorem 5</u>. The lattice $B(L)$ is hf-primitive and $B(L)$ is not primitive.

Since the lattices L_n ($n = 1,2,\ldots$) in Fig. 1 are primitive (see [3],[5]) we have that lattices $B(L_n)$ ($n = 1,2,\ldots$) are hf-primitive and $B(L_n)$ are not primitive. Using Theorem 3 we can obtain other examples of such lattices.

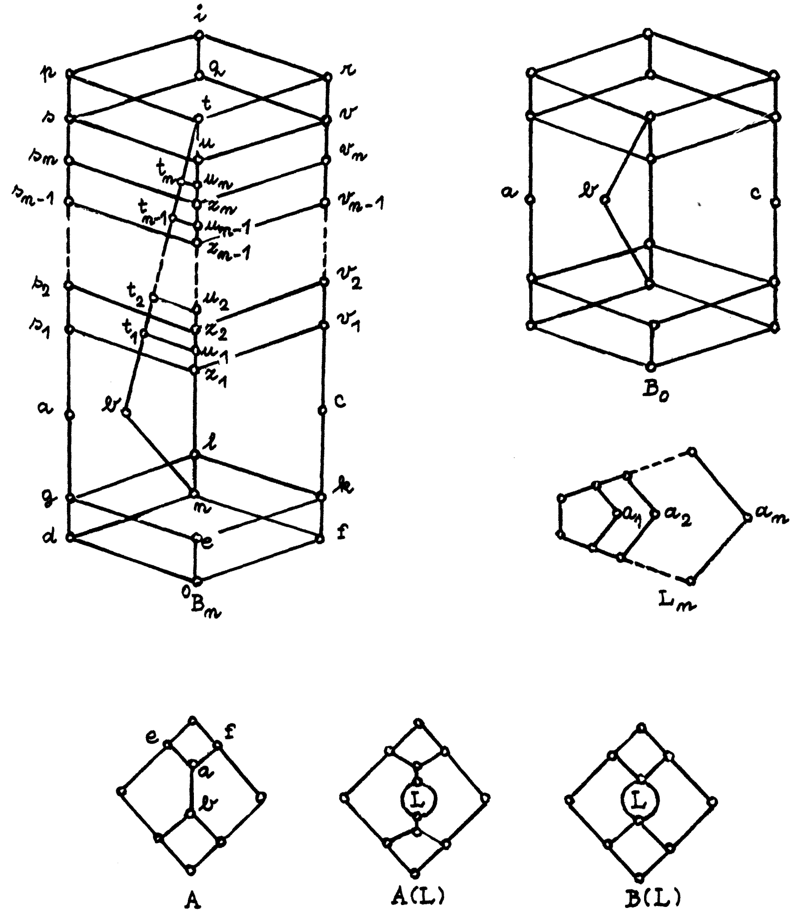

Fig. 1

R e f e r e n c e s

[1] V.I. IGOŠIN: Charakterizuemy klassy algebraičeskich sistem, Issledovanija po alg.,vyp.4,Saratov 1974, 27-42.

[2] V.I. IGOŠIN: Ob h-charakterizuemych klassach algebraičeskich sistem, Issl.po alg.,vyp.3,Saratov 1973, 14-19.

[3] J. JEŽEK, V. SLAVÍK: Some examples of primitive lattices, Acta Univ.Carolinae-Math.et Phys.14(1973),3-8.

[4] A. KOSTINSKY: Projective lattices and bounded homomorphisms, Pacif.J.Math.40(1972),111-119.

[5] R. McKENZIE: Equational bases and nonmodular lattice varieties, Trans.Amer.Math.Soc.174(1972),1-44.

Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 27.3.1975)