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QUASIGROUPS, ISOTOPIC TO A GROUP

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Abstract: This paper is concerned with some properties of the variety of all quasigroups which are isotopic to a group.

As to the terminology used in this paper, the reader is referred to [1] and [2]. Nevertheless, a bit of terminology should be mentioned. Namely, it will be convenient in some cases to consider quasigroups as algebras with three binary operations.

Key words: Quasigroup, isotopy, variety, equivalence, linear.

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1. Universal-algebraic preliminaries. By a type we mean a set Δ of symbols. Any symbol $F \in \Delta$ is associated with a non-negative integer n_F, called the arity of F. By an algebra of type Δ we mean an ordered pair $A = \langle X, \xi \rangle$, where X is a non-empty set (the underlying set of A) an ξ is a mapping, assigning to any $F \in \Delta$ and n_F-ary operation F_A on X. (If $n_F = 0$, then F_A is an element of X.) If there is no confusion, we write F instead of F_A and identify A with its underlying set.

Let K_1 be a class of algebras of type Δ_1 and K_2 a class of algebras of type Δ_2. A one-to-one mapping ε

of K_1 onto K_2 is called an equivalence between K_1 and K_2 if the following two conditions are satisfied:

(E1) If $A \in K_1$, then the algebras A and $\varepsilon(A)$ have the same underlying sets.

(E2) If $A, B \in K_1$, then a mapping of A into B is a homomorphism of A into B iff it is a homomorphism of $\varepsilon(A)$ into $\varepsilon(B)$.

If such an equivalence exists, the classes K_1 and K_2 are called equivalent.

By a parastrophy of a class K we mean an equivalence between K and K. Parastrophies of a variety V constitute a group. In particular, the group of parastrophies of the variety of quasigroups has exactly six elements.

1.1. Proposition. Let ε be an equivalence between K_1 and K_2.

(i) If $A, B \in K_1$, then A is a subalgebra of B iff $\varepsilon(A)$ is a subalgebra of $\varepsilon(B)$.

(ii) If $A \in K_1$ and $(B_i)_{i \in I}$ is a family of algebras from K_1, then A is the direct product of $(B_i)_{i \in I}$ iff $\varepsilon(A)$ is the direct product of $(\varepsilon(B_i))_{i \in I}$.

(iii) K_1 is closed with respect to direct products iff the same holds for K_2.

(iv) K_1 is closed with respect to isomorphic algebras iff the same holds for K_2.

Proof. (i) A is a subalgebra of B iff $A \subseteq B$ and the identical mapping of A is a homomorphism of A into B.

(ii) A is the direct product of $(B_i)_{i \in I}$ iff its underlying set is the cartesian product of the underlying sets of B_i and every projection is a homomorphism.

(iii) follows from (ii) and (iv) is obvious.

1.2. *Proposition*. Let ε be an equivalence between two varieties K_1 and K_2.

(i) If $A \in K_1$ and Z is a subset of A, then Z is closed in A iff it is closed in $\varepsilon(A)$.

(ii) If $A \in K_1$ and $Z \subseteq A$, then Z generates A iff it generates $\varepsilon(A)$.

(iii) If $A \in K_1$ and r is an equivalence on A, then r is a congruence of A iff it is a congruence of $\varepsilon(A)$.

(iv) If A is the free K_1-algebra, with K_1-basis Z, then $\varepsilon(A)$ is the free K_2-algebra with K_2-basis Z.

(v) A class $L \subseteq K_1$ is a variety iff the class $\{\varepsilon(A) ; A \in L\}$ is a variety.

(vi) The lattice of subvarieties of K_1 is isomorphic to the lattice of subvarieties of K_2.

Proof. (i) follows from 1.1(i), as K_1 and K_2 are closed with respect to subalgebras.

(ii) follows from (i).

(iii) An equivalence r is a congruence of A iff it is a closed subset of the algebra $A \times A$; we may apply

1.1(ii) and 1.2(i).

(iv) follows from (ii).

(v) and (vi) follow from 1.1(i), 1.1(iii) and 1.2(iii).

A class K which is equivalent to a variety, need not be a variety itself. The class of all groupoids with unique division is not closed with respect to subalgebras and homomorphic images; however, it is equivalent to the variety of quasigroups. The class of all semigroups with division is not closed w.r.t. subalgebras (and is closed w.r.t. homomorphic images); however, it is equivalent to the variety of groups.

1.3. Proposition. Let a class K of Δ -algebras be equivalent to a variety. Then K is a variety iff it is closed under subalgebras.

Proof. Only the converse implication requires to be proved. Let ε be an equivalence between K and a variety V . By 1.1(iii) and 1.1(iv) it is enough to show that $A \in K$ implies $A/r \in K$ for every congruence r of A . As the equivalence r is a closed subset of the algebra $A \times A$ and K is closed under subalgebras, r is closed in $\varepsilon(A \times A) = \varepsilon(A) \times \varepsilon(A)$, too, so that r is a congruence of $\varepsilon(A)$ and $\varepsilon(A)/r \in V$. It is easy to see that $\varepsilon^{-1}(\varepsilon(A)/r) = A/r$, so that $A/r \in K$.

Next we shall define the notion of rational equivalence which strengthens the notion of equivalence in the case of general classes of algebras. We start by recalling the notion of term.

Let $x_1, x_2, x_3, \dots$ be an infinite countable sequence of symbols, called the variables. Let Δ be a type. Δ -terms are formal expressions which can be obtained by a finite number of applications of the following three rules:

(T1) every variable is a Δ -term;

(T2) if $F \in \Delta$ and $n_F = 0$, then F is a Δ -term;

(T3) if $F \in \Delta$, $n_F \geq 1$ and $t_1, \dots, t_{n_F}$ are Δ -terms, then the inscription $F(t_1, \dots, t_{n_F})$ is a Δ -term, too.

We denote by W_Δ the set of all Δ -terms, Moreover, given a non-negative integer n , we denote by $W_{n,\Delta}$ the set of Δ -terms containing no variables different from $x_1, \dots, x_n$. If A is an algebra of type Δ , then for any $t \in W_{n,\Delta}$ we define an n-ary operation $t^{(A)}$ on A , called the algebraic operation of the algebra A , corresponding to t , as follows:

(i) if $t = x_i$, then $t^{(A)}(a_1, \dots, a_n) = a_i$;

(ii) if $t = F$ and $n_F = 0$, then $t^{(A)}(a_1, \dots$ $\dots, a_n) = F_A$;

(iii) if $t = F(t_1, \dots, t_n)$, then

$t^{(A)}(a_1, \dots, a_n) = F_A(t_1^{(A)}(a_1, \dots, a_n), \dots,$
$t_n^{(A)}(a_1, \dots, a_n))$.

Let Δ_1 and Δ_2 be two types. By a translation of Δ_1 into Δ_2 we mean a mapping τ of Δ_1 into W_{Δ_2} such that $\tau(F) \in W_{n_F, \Delta_2}$ for any $F \in \Delta_1$. Given such a translation τ and a Δ_2-algebra A, we define a Δ_1-algebra $T_\tau(A)$ as follows: its underlying set coincides with the underlying set of A; if $F \in \Delta_1$ then $F_{T_\tau(A)}$ is just the n_F-ary algebraic operation of A, corresponding to $\tau(F)$.

Let K_1 and K_2 be two classes of algebras (of types Δ_1 and Δ_2, respectively). A one-to-one mapping ε of K_1 onto K_2 is called a rational equivalence between K_1 and K_2 if there exists a translation τ of Δ_1 into Δ_2 and a translation σ of Δ_2 into Δ_1 such that $\varepsilon(A) = T_\sigma(A)$ for any $A \in K_1$ and $\varepsilon^{-1}(B) = T_\tau(B)$ for any $B \in K_2$. We say that K_1 and K_2 are rationally equivalent under τ, σ.

1.4. Proposition. Any two rationally equivalent classes are equivalent.

Proof is easy.

It is proved e.g. in [3] that two varieties are equivalent iff they are rationally equivalent. However, this will not be used in the following.

1.5. Proposition. Let ε be a rational equivalence between K_1 and K_2.

(i) If $A \in K_1$ and $Z \subseteq A$, then Z is closed in A iff it is closed in $\varepsilon(A)$

(ii) If $A \in K_1$ and r is an equivalence on A, then r is a congruence of A iff it is a congruence of $\varepsilon(A)$.

(iii) K_1 is a variety iff K_2 is a variety.

Proof is easy.

We shall finish this section with several remarks on classes K^* of algebras with fixed points.

Given a class K of Δ-algebras, we define a new type Δ^* and a class K^* of Δ^*-algebras as follows: $\Delta^* = \Delta \cup \{e\}$, where e is a nullary symbol, not belonging to Δ; K^* is the class of all Δ^*-algebras A such that the algebra $A \mid \Delta$ (which results from A by forgetting the nullary operation e) belongs to K.

1.6. Proposition. K^* is a variety iff K is a variety.

Proof is obvious.

1.7. Proposition. Let K be a variety, $A \in K^*$ and $Z \subseteq A$. The algebra A is free in K^*, with K^*-basis Z, iff $A \mid \Delta$ is free in K, with K-basis $Z \cup \{e_A\}$.

Proof is obvious.

Problem. Let K be a variety with only countably many subvarieties. Is it true that K^* has countably many subvarieties, too ?

2. The variety of quasigroups isotopic to a group. We denote by $\mathcal{S}$ the class of all quasigroups which are iso-

topic to a group. We shall show in Theorem 2.1 that the individual correspondence between quasigroups of the class $\mathcal{S}$ and groups has a global character (in certain sense). To this purpose we introduce several definitions.

Let $\mathcal{G}$ denote the variety of algebras of the type $\{+, -, 0, \alpha, \beta, \gamma, \delta\}$ (where $+$ is a binary symbol and $-, \alpha, \beta, \gamma, \delta$ are unary symbols, determined by the following identities:

$$(x + y) + z = x + (y + z),$$
$$x + 0 = 0 + x = x,$$
$$x + (- x) = (- x) + x = 0,$$
$$\alpha(\gamma(x)) = \gamma(\alpha(x)) = x,$$
$$\beta(\delta(x)) = \delta(\beta(x) = x,$$
$$\alpha(0) = \beta(0) = 0.$$

The type of algebras from $\mathcal{S}^*$ will be denoted by $\{\cdot, /, \backslash, u\}$ and the type of algebras from $\mathcal{G}^*$ by $\{+, -, 0, \alpha, \beta, \gamma, \delta, e\}$. We have added two nullary symbols u and e.

Let us define a translation τ of $\{\cdot, /, \backslash, u\}$ into $\{+, -, 0, \alpha, \beta, \gamma, \delta, e\}$ and a translation σ of $\{+, -, 0, \alpha, \beta, \gamma, \delta, e\}$ into $\{\cdot, /, \backslash, u\}$ in the following way:

$\tau(\cdot) = \alpha(x) + e + \beta(y)$ (the distribution of parantheses is inessential),

$$\tau(/) = \gamma(x - \beta(y) - e),$$
$$\tau(\backslash) = \delta(- e - \alpha(x) + y),$$
$$\tau(u) = 0,$$
$$\sigma(+) = (x / u)((u / u) \backslash y),$$

$\sigma(-) = (u / u)((x / u) \backslash u)$,

$\sigma(0) = u$,

$\sigma(\alpha) = x(u \backslash u)$,

$\sigma(\beta) = (u / u)\, x$,

$\sigma(\gamma) = x / (u \backslash u)$,

$\sigma(\delta) = (u / u) \backslash x$,

$\sigma(e) = uu$.

2.1. <u>Theorem</u>. The classes $\mathcal{S}^*$ and $\mathcal{G}^*$ are rationally equivalent under τ, σ .

<u>Proof</u>. Let $Q \in \mathcal{S}^*$. As it is well-known, the algebra $T_\sigma(Q)$ is a loop with respect to $+$ and u is its unit. Albert's theorem, together with $Q \in \mathcal{S}^*$, implies that this loop is a group. Now it can be verified easily that $T_\sigma(Q)$ belongs to $\mathcal{G}^*$. Obviously, $T_\tau(A) \in \mathcal{S}^*$ for any $A \in \mathcal{G}^*$

It remains to prove $T_\sigma(T_\tau(A)) = A$ and $T_\tau(T_\sigma(Q)) = Q$ for any $A \in \mathcal{G}^*$ and $Q \in \mathcal{S}^*$. If $A \in \mathcal{G}^*$, $Q = T_\tau(A)$ and $B = T_\sigma(Q)$, then

$$x +_B y = (x / u)((u / u) \backslash y) = \alpha(\gamma(x - \beta(u) - e)) + e + \beta(\delta(- e - \alpha(\gamma(u - \beta(u) - e)) + y)) = x - u + y = x - 0 + y = x +_A y ,$$

$$\alpha_B(x) = x(u \backslash u) = \alpha(x) + e + \beta(\delta(- e - \alpha(u) + u)) = \alpha(x) - \alpha(0) + 0 = \alpha_A(x) ,$$

$\beta_B(x) = \beta_A(x)$ similarly and

$$e_B = uu = \alpha(u) + e + \beta(u) = \alpha(0) + e_A + \beta(0) = e_A .$$

Now let $Q \in \mathcal{S}^*$, $A = T_\sigma(Q)$ and $H = T_\tau(A)$. There exists a group $Q(+,-,0)$ and its two permutations λ , μ such that $xy = \lambda(x) + \mu(y)$ for all $x, y \in Q$. We have

$x/y = \lambda^{-1}(x - \mu(y))$,

$x \backslash y = \mu^{-1}(-\lambda(x) + y)$,

$x +_A y = (x/u)((u/u)\backslash y) = x - \mu(u) - (u - \mu(u)) + y = x - u + y$,

$\alpha_A(x) = x(u \backslash u) = \lambda(x) - \lambda(u) + u$,

$\beta_A(x) = (u/u)x = u - \mu(u) + \mu(x)$,

$e_A = uu = \lambda(u) + \mu(u)$,

$x \cdot_H y = \alpha_A(x) +_A e_A +_A \beta_A(y) = \lambda(x) - \lambda(u) + u - u + \lambda(u) + \mu(u) - u + u - \mu(u) + \mu(y) = \lambda(x) + \mu(y) = xy$,

$u_H = 0_A = u_Q$,

so that $H = Q$.

2.2. Corollary. The class $\mathcal{S}$ is a variety.

Proof. This follows from 1.6 and 1.5(iii).

Let Q be a quasigroup. A mapping $\lambda : Q \longrightarrow Q$ is called left regular if there exists a mapping $\lambda^* : Q \longrightarrow Q$ such that $\lambda(xy) = \lambda^*(x) \cdot y$ for all $x, y \in Q$. Clearly, the sets L_Q of all left regular mappings and L_Q^* of the corresponding $*$ -mappings are groups with respect to the composition. Similarly we define right regular and middle regular mappings and we obtain groups R_Q , R_Q^* , M_Q , M_Q^* . The following proposition is well-known:

2.3. Proposition. The following conditions are equivalent for a quasigroup Q :

(i) $Q \in \mathcal{S}$.

(ii) At least one of the groups L_Q , L_Q^* , R_Q , R_Q^* , M_Q , M_Q^* operates transitively on Q .

(iii) Every of the six groups operates transitively on Q .

3. Linear quasigroups. A quasigroup Q is called left linear (right linear) if there exists a group $Q(+, -, 0)$, its automorphism λ and a permutation μ of Q such that $\mu(0) = 0$ and

$$xy = \lambda(x) + \mu(y) \qquad (xy = \mu(x) + \lambda(y))$$

for all $x, y \in Q$.

Denote by $\mathcal{L}_\ell$ the class of all left linear quasigroups and by $\mathcal{L}_r$ that of the right ones. Further, let $\mathcal{G}_\ell$ ($\mathcal{G}_r$) denote the subvariety of $\mathcal{G}$ determined by the identity

$$\alpha(x + y) = \alpha(x) + \alpha(y) \quad (\beta(x + y) = \beta(x) + \beta(y)) .$$

3.1. Theorem. The classes $\mathcal{L}_\ell^*$ and $\mathcal{G}_\ell^*$ ($\mathcal{L}_r^*$ and $\mathcal{G}_r^*$) are rationally equivalent under τ, σ .

Proof. It is sufficient to show that $A \in \mathcal{G}_\ell^*$ implies $T_\tau(A) \in \mathcal{L}_\ell^*$ and $Q \in \mathcal{L}_\ell^*$ implies $T_\sigma(Q) \in \mathcal{G}_\ell^*$. If $A \in \mathcal{G}_\ell^*$ then $T_\tau(A) \in \mathcal{L}_\ell^*$ evidently. Let $Q \in \mathcal{L}_\ell^*$. Put $A = T_\sigma(Q)$. There are a group $Q(+, -, 0)$, its automorphism λ and a permutation μ of Q such that $\mu(0) = 0$ and $xy = \lambda(x) + \mu(y)$ for all $x, y \in Q$. As in the proof of 2.1, we have

$$x +_A y = x - u + y ,$$

$$\alpha_A(x) = \lambda(x) - \lambda(u) + u .$$

Since λ is an automorphism, we get

$$\alpha_A(x +_A y) = \lambda(x - u + y) - \lambda(u) + u = \lambda(x) - \lambda(u) +$$
$$+ \lambda(y) - \lambda(u) + u = \lambda(x) - \lambda(u) + u - u + \lambda(y) -$$
$$- \lambda(u) + u = \alpha_A(x) +_A \alpha_A(y) ,$$

so that α_A is an automorphism and $A \in \mathcal{G}_\ell^*$.

3.2. <u>Corollary</u>. The classes $\mathcal{L}_\ell$ and $\mathcal{L}_r$ are varieties.

A quasigroup Q is called linear if it belongs to $\mathcal{L}_\ell \cap \mathcal{L}_r = \mathcal{L}$

3.3. <u>Corollary</u>. The class $\mathcal{L}$ is a variety. The varieties $\mathcal{L}^*$ and $(\mathcal{G}_\ell \cap \mathcal{G}_r)^*$ are rationally equivalent under τ, σ .

3.4. <u>Proposition</u>. Let Q be a quasigroup. The following four conditions are equivalent:

(i) Q is linear;

(ii) there exists a group $Q(+, -, 0)$, its automorphisms λ, μ and an element $g \in Q$ such that $xy = \lambda(x) + g + \mu(y)$ for all $x, y \in Q$;

(iii) there exists a group $Q(+, -, 0)$, its automor-

phisms λ, μ and an element $g \in Q$ such that $xy =$
$= g + \lambda(x) + \mu(y)$;

(iv) there exists a group $Q(+, -, 0)$, its automorphisms λ, μ and an element $g \in Q$ such that $xy =$
$= \lambda(x) + \mu(y) + g$.

Proof. (i) $\Longrightarrow$ (ii): Choose an arbitrary element $u \in$ $\in Q$ and denote $T_\sigma(Q(\cdot, /, \backslash, u))$ by $Q(+, -,$ $0, \alpha, \beta, \gamma, \delta, g)$. By 3.3, α and β are automorphisms of $Q(+, -, 0)$ and $xy = \alpha(x) + g + \beta(y)$.

(ii) $\Longrightarrow$ (i): λ is an automorphism and $y \mapsto g +$ $+ \mu(y)$ is a permutation, so that Q is left linear; similarly, Q is right linear.

(ii) $\Longrightarrow$ (iii): put $\mu' = \mu$ and $\lambda'(x) = - g +$ $+ \lambda(x) + g$.

(iii) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (ii) similarly.

A quasigroup Q is called T-quasigroup if there exists an abelian group $Q(+, -, 0)$, its two automorphisms λ, μ and an element $g \in Q$ such that $xy =$ $= \lambda(x) + g + \mu(y)$ for all $x, y \in Q$.

We denote by $\mathcal{T}$ the class of T-quasigroups and by $\mathcal{A}$ the subvariety of $\mathcal{G}$ determined by the identities

$$x + y = y + x ,$$
$$\alpha(x + y) = \alpha(x) + \alpha(y) ,$$
$$\beta(x + y) = \beta(x) + \beta(y) .$$

3.5. Theorem. The class $\mathcal{T}$ is a variety. The varieties $\mathcal{T}^*$ and $\mathcal{A}^*$ are rationally equivalent under τ, σ .

Proof follows from 3.3 and from Albert's theorem

The study of T-quasigroups in [5] and [4] was founded on several propositions and lemmas which were stated and proved in the introductory section of [5]. All the applications of these propositions and lemmas can be replaced by applications of the present Theorem 3.5.

Theorems 2.1, 3.3 and 3.5 allow us to formulate many problems concerning the varieties $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{T}$ in the more familiar terms of $\mathcal{G}$, $\mathcal{G}_{\ell} \cap \mathcal{G}_{\kappa}$ and $\mathcal{A}$. For example, the description of free algebras in the varieties $\mathcal{G}$, $\mathcal{G}_{\ell} \cap \cap \mathcal{G}_{\kappa}$ and $\mathcal{A}$ is very simple; Propositions 1.2(iv) and 1.7 show how to describe free quasigroups in the varieties $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{T}$. However, we were not able to give answer to the following

Problem. Let V be one of the three varieties $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{T}$. Is it true that subquasigroups of free V-quasigroups are free ?

Let Q be a quasigroup and a, b, c $\in$ Q . Then we define four permutations of Q as follows:

$$\varphi[a, b, c] = L_b^{-1} L_{ac}^{-1} L_{ab} L_c ,$$

$$\psi[a, b, c] = L_a^{-1} R_{bc}^{-1} L_{ab} R_c ,$$

$$\rho[a, b, c] = R_c^{-1} L_{ab}^{-1} R_{bc} L_a ,$$

$$\eta[a, b, c] = R_b^{-1} R_{ac}^{-1} R_{bc} R_a .$$

Clearly,

$$ab \cdot cd = ac \cdot b\varphi[a, b, c](d) ,$$

$ab \cdot dc = a\psi[a, b, c](d) \cdot bc$,

$ad \cdot bc = ab \cdot \rho[a, b, c](d)c$,

$da \cdot bc = \eta[a, b, c](d)b \cdot ac$

for any $a, b, c, d \in Q$.

3.6. Theorem. Let Q be a quasigroup. Then the following are equivalent:

(i) $Q \in \mathcal{L}_\ell$,

(ii) If $u, v, w, x, y, z \in Q$ and $uv \cdot z = uw \cdot y$ then $xv \cdot z = xw \cdot y$;

(iii) $\varphi[a, b, c] = \varphi[d, b, c]$ for all $a, b, c, d \in Q$;

(iv) $\psi[a, b, c] = \psi[d, b, c]$ for all $a, b, c, d \in Q$;

(v) $\rho[a, b, c] = \rho[d, b, c]$ for all $a, b, c, d \in Q$.

Proof. (i) implies (iv) and (i) implies (v). This follows from the simple fact that

$$ab \cdot cd = \lambda^2(a) + \lambda\mu(b) + \mu(\lambda(c) + \mu(d))$$

for all $a, b, c, d \in Q$.

(iv) implies (ii). There are $a, b \in Q$ such that $vb = y$ and $ab = z$. Then $uv \cdot ab = uw \cdot vb$, and therefore $\psi[u, v, b](a) = w$. According to the hypothesis, $\psi[u, v, b] = \psi[x, v, b]$. Thus

$$xv \cdot z = xv \cdot ab = x\psi[x, v, b](a) \cdot vb =$$
$$= x\psi[u, v, b](a) \cdot vb = xw \cdot y .$$

The implication (v) $\Longrightarrow$ (ii) can be proved similarly (it is also obvious from the fact that $\rho[a, b, c] = (\psi[a, b, c])^{-1}$).

(ii) implies (iii), Let $a, b, c, d \in Q$. Then

$ab \cdot cq = ac \cdot b\varphi[a, b, c](q)$ for all $q \in Q$. Now, setting $a = u$, $b = v$, $cq = z$, $c = w$, $x = d$ and $y = b\varphi[a, b, c](q)$ we get $db \cdot cq = dc \cdot b\varphi[a, b, c](q) = dc \cdot b\varphi[d, b, c](q)$. Then $\varphi[a, b, c](q) = \varphi[d, b, c](q)$.

(iii) implies (i). By the hypothesis,

$\varphi[a, b, c] = \varphi[c/c, b, c] = L_b^{-1} L_c^{-1} L_{(c/c)b} L_c$,

and hence $R_b R_c^{-1}(a) \cdot d = a \cdot L_c^{-1} L_{(c/c)b}(d)$. From this we see that the mapping $R_b R_c^{-1}$ is a middle regular permutation of Q. In particular, the group M_Q is transitive on Q, and consequently $Q \in \mathcal{S}$. Now, let $x \in Q$ be arbitrary and $a + b = R_{x\backslash x}^{-1}(a) \cdot L_x^{-1}(b)$ for all $a, b \in Q$. Then $Q(+)$ is a group and $ab = \lambda(a) + \mu(b)$, where $\lambda = R_{x\backslash x}$ and $\mu = L_x$. Moreover, $\lambda(x) = x$ and x is the identity of the group $Q(+)$. Further,

$\varphi[a, b, c](d) = \mu^{-1}(-\lambda(b) + \mu^{-1}(-\lambda(\lambda(a) + \mu(c)) + \lambda(\lambda(a) + \mu(b)) + \mu(\lambda(c) + \mu(d))))$

for all $a, b, c, d \in Q$. Using the equality $\varphi[a, b, c](d) = \varphi[z, b, c](d)$ for every $z \in Q$ we obtain

$$-\lambda(a + c) + \lambda(a + b) = -\lambda(z + c) + \lambda(z + b).$$

In particular,

$$-\lambda(a) + \lambda(a + b) = -\lambda(x) + \lambda(b) = \lambda(b),$$

and consequently $\lambda(a + b) = \lambda(a) + \lambda(b)$

Combining 3.6 with its dual we get:

3.7. Theorem. Let Q be a quasigroup. Then the following are equivalent:

(i) $Q \in \mathcal{L}$;

(ii) $\varphi[a, b, c] = \varphi[d, b, c]$ and $\eta[a, b, c] = \eta[a, b, d]$ for all $a, b, c, d \in Q$;

(iii) $\psi[a, b, c] = \psi[d, b, c] = \psi[a, b, d]$ for all $a, b, c, d \in Q$;

(iv) $\wp[a, b, c] = \wp[d, b, c] = \wp[a, b, d]$ for all $a, b, c, d \in Q$.

3.8. __Theorem__. Let Q be a quasigroup. Then the following are equivalent:

(i) $\varphi[a, b, c] = \varphi[a, b, d]$ for all $a, b, c, d \in Q$;

(ii) $\varphi[a, b, c] = \varphi[a, d, c]$ for all $a, b, c, d \in Q$;

(iii) $\psi[a, b, c] = \psi[a, d, c]$ for all $a, b, c, d \in Q$;

(iv) $\wp[a, b, c] = \wp[a, d, c]$ for all $a, b, c, d \in Q$;

(v) $\eta[a, b, c] = \eta[a, d, c]$ for all $a, b, c, d \in Q$;

(vi) $\eta[a, b, c] = \eta[d, b, c]$ for all $a, b, c, d \in Q$;

(vii) $ab \cdot cd = ac \cdot bd$ for all $a, b, c, d \in Q$.

Proof. As one may check easily,

$$\varphi[a, b, b](d) = d = \eta[b, b, a](d) = \psi[a, d, b](d) =$$
$$= \rho[a, d, b](d) .$$

The rest is clear.

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