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A NOTE ON A LOCAL ERGODIC THEOREM

Ryotaro SATO, Sakado

Abstract: Let $1 \leq p < \infty$ and let $\Gamma = \{T_t: t > 0\}$ be a strongly continuous semigroup of bounded linear operators on L_p of a finite measure space which is assumed to be strongly integrable over every finite interval. In this note we consider the problem of the almost everywhere convergence of the average $\frac{1}{b} \int_0^b T_t f \, dt$ as $b \to 0$.

Key words: Local ergodic theorem, semigroup of bounded linear operators on L_p, strong continuity and integrability.

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1. Introduction and theorems. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $L_p(\mu) = L_p(X, \mu) = L_p(X, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$, the usual Banach spaces. If $A \in \mathcal{F}$ then $L_p(A, \mu)$ is the Banach space of all $L_p(\mu)$-functions that vanish a.e. on $X - A$. Let $\Gamma = \{T_t; t > 0\}$ be a strongly continuous semigroup of bounded linear operators on $L_p(\mu)$, where p is fixed, $1 \leq p < \infty$. This means that T_t is a bounded linear operator on $L_p(\mu)$, $T_t T_s = T_{t+s}$ for all $t, s > 0$, $\lim_{t \to s} \| T_t f - T_s f \|_p = 0$ for all $s > 0$ and $f \in L_p(\mu)$. Throughout this note we shall assume that Γ is strongly integrable over every finite interval, i.e., for each $f \in L_p(\mu)$ the vector-valued function $t \to T_t f$ is

Lebesgue integrable on every finite interval $(a, b) \subset \subset (0, \infty)$. It follows (cf. [2, p. 686]) that for each $f \in \in L_p(\mu)$ there exists a scalar function $T_t f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathcal{F}$, such that for almost all t, $T_t f(x)$ belongs, as a function of x, to the equivalence class of $T_t f$. Moreover there exists a set $N(f) \in \mathcal{F}$ with $\mu(N(f)) = 0$, dependent on f but independent of t, such that if $x \notin N(f)$, then $T_t f(x)$ is Lebesgue integrable over every finite interval (a,b) and the integral $\int_a^b T_f(x)\, dt$ belongs, as a function of x, to the equivalence class of $\int_a^b T_t f\, dt$. Hence, from now on, we shall write $S_a^b f(x)$ for $\int_a^b T_t f(x)\, dt$. The purpose of this note is to investigate the almost everywhere convergence of averages $\frac{1}{b} S_0^b f(x)$ as $b \longrightarrow 0$.

In [1] Akcoglu and Chacon proved that if $\Gamma = \{T_t ; t > 0\}$ is a positive L_1-contraction semigroup, then the limit

$$(1) \qquad \lim_{b \to 0} \frac{1}{b} S_0^b f(x)$$

exists a.e. for any $f \in L_1(\mu)$. See also Krengel [3] and Ornstein [7]. Later in [4] Kubokawa proved that if $\Gamma = \{T_t ; t > 0\}$ is a positive (not necessarily contraction) L_1-operator semigroup and satisfies strong-$\lim_{t \to 0} T_t = I$ (the identity operator), then the limit (1) exists a.e. for any $f \in L_1(\mu)$. Recently Kubokawa [5] proved that if $\Gamma = \{T_t;$

$t > 0\}$ is a (not necessarily positive) L_1-contraction semigroup satisfying $\text{strong-}\lim_{t \to 0} T_t = I$, then the limit (1) exists a.e. for any $f \in L_1(\mu)$. In this note we shall prove the following results.

<u>Theorem 1</u>. Let $1 \leq p < \infty$ and $\Gamma = \{T_t ; t > 0\}$ a strongly continuous semigroup of positive (not necessarily contraction) operators on $L_p(\mu)$. Assume that $\limsup_{t \to 0} \| T_t f \|_p \leq \| f \|_p$ for any $f \in L_p(\mu)$. Then the limit (1) exists a.e. for any $f \in L_p(\mu)$, provided (i) $1 < p < \infty$, or (ii) $p = 1$ and there exists a strictly positive function $h \in L_1(\mu)$ such that the set $\{T_t h ; 0 < t < 1\}$ is weakly sequentially compact in $L_1(\mu)$.

<u>Theorem 2</u>. Let $\Gamma = \{T_t ; t > 0\}$ be a strongly continuous semigroup of (not necessarily positive) contractions on $L_1(\mu)$. Assume that there exists a $p > 1$ such that all the T_t map $L_p(\mu)$ into $L_p(\mu)$ and $\sup_{0<t<1} \| T_t \|_p < \infty$. Then the limit (1) exists a.e. for any $f \in L_1(\mu)$.

2. <u>Lemmas</u>. For the proofs of the above theorems we need the following lemmas.

<u>Lemma 1</u>. Let $\Delta = \{\xi_t ; t > 0\}$ be a strongly continuous semigroup of bounded linear operators on a Banach space B. Assume that the set $\{\xi_t f ; 0 < t < 1\}$ is weakly sequentially compact for any $f \in B$. Then ξ_t converges strongly as $t \to 0$, hence if we let $\xi_0 = \text{strong-}\lim_{t \to 0} \xi_t$

then $\{\xi_t \,;\, t \geqq 0\}$ is a strongly continuous semigroup on $[0,\infty)$.

<u>Proof</u>. By the uniform boundedness principle [2, Theorem II.1.11], $\sup_{0<t<1} \|\xi_t f\| < \infty$ for any $f \in B$. Hence again by the uniform boundedness principle,

$$\sup_{0<t<1} \|\xi_t\| < \infty \quad . \tag{2}$$

It follows that

$$\{f \in B \,;\, \lim_{t\to 0} \|\xi_t f - f\| = 0\} = \overline{\bigcup_{t>0} \xi_t B} \;. \tag{3}$$

Since for any $f \in B$ there exists a closed separable subspace B_f of B containing f such that $\xi_t B_f \subset B_f$ for all $t > 0$, to prove the lemma it may be assumed without loss of generality that B itself is separable. Let $\{f_n \,;\, n \geqq 1\}$ be a dense subset of B . Then, by Cantor's diagonal method, we can find a strictly decreasing sequence t_1 , $t_2,\ldots$ of positive reals with $\lim_m t_n = 0$ such that weak-$\lim_m \xi_{t_m} f_i$ exists for all the f_i . Thus by (2) and an approximation argument, weak-$\lim_m \xi_{t_m} f$ exists for any $f \in B$. Let $\xi_0 =$ = weak-$\lim_m \xi_{t_m}$. It follows that $\xi_t \xi_0 = \xi_t = \xi_0 \xi_t$ for any $t \geqq 0$, and any $f \in B$ can be written as $f = f_1 + f_2$, where $\xi_0 f_1 = f_1$ and $\xi_0 f_2 = 0$. Then, since $f_1 =$ = weak-$\lim_m \xi_{t_m} f_1$, we have by (3) and the Hahn-Banach theorem that $\lim_{t\to 0} \|\xi_t f_1 - f_1\| = 0$. This completes the proof, since $\xi_t f = \xi_t f_1$ for any $t > 0$.

Lemma 2. Let $\Gamma = \{T_t \,;\, t > 0\}$ be a strongly continuous semigroup of positive linear operators on $L_p(\mu)$, where p is fixed, $1 \leqq p < \infty$. Assume that $\text{strong-}\lim_{t \to 0} T_t = I$. Then for any $f \in L_p(\mu)$

$$\int_{A(f)} f^- \, d\mu \leqq \int_X f^+ \, d\mu \ ,$$

where $A(f) = \{x \,;\, \sup_{0 < b < \alpha} S_0^b f(x) > 0 \text{ for any } \alpha > 0\}$.

Proof. Since μ is finite, we may assume without loss of generality that $\mu(X) = 1$. Hence, by Hölder's inequality, $\| f \|_1 \leqq \| f \|_p$ for any $f \in L_p(\mu)$. Let $D = \{1/2^n \,;\, n \geqq 1\}$. Then given an $f \in L_p(\mu)$ and an $\varepsilon > 0$, we can choose a $\delta \in D$ such that $0 < t \leqq \delta$ implies

$$\| T_t(f^- 1_{A(f)}) \|_1 \geqq (1 - \varepsilon) \| f^- 1_{A(f)} \|_1$$

and

$$\| T_t f^+ \|_1 \leqq (1 + \varepsilon) \| f^+ \|_1 \ .$$

Let $K = \sup_{0 < t \leqq \delta} \| T_t \|_p \ (< \infty)$, and choose an $\eta \in D$ such that

$$0 < \frac{2K\eta}{\delta - \eta} \| f^- \|_p < \varepsilon \ .$$

It may be readily seen that there exists a positive integer k and a measurable subset A of $A(f)$ such that $k\eta$ and $k(\delta - \eta)$ are integers.

$$\sup_{0 \leq j \leq k\eta} \sum_{i=0}^{j} (T_{1/k})^i f > 0 \quad \text{a.e. on } A ,$$

and

$$K \| f^- 1_{A(f)} - A \|_p < \varepsilon \ .$$

Therefore a slight modification of the argument in Kubokawa [4 , pp. 463 - 464] shows that

$$(1 - \varepsilon) \int_{A(f)} f^- d\mu \leq (1 + \varepsilon) \int_X f^+ \, d\mu + \frac{2K\eta}{\delta - \eta} \| f^- \|_p$$

$$+ K \| f^- 1_{A(f)-A} \|_p$$

$$< (1 + \varepsilon) \int_X f^+ \, d\mu + 2\varepsilon \ .$$

This completes the proof, since ε is arbitrary.

Lemma 3. Let $\Gamma = \{T_t \ ; \ t > 0\}$ be as in Lemma 2. Then the limit (1) exists a.e. for any $f \in L_p(\mu)$.

Proof. By virtue of Lemma 2 the proof is the same as that of the theorem in [4].

It should be noted that Kubokawa [6] also gave a different proof of the above result. His method of proof is dependent upon the use of another local maximal ergodic lemma which is similar to our Lemma 2.

3. <u>Proof of Theorem 1</u>. By the uniform boundedness principle, $\sup_{0<t<1} \| T_t \|_p < \infty$. If $1 < p < \infty$, then the space $L_p(\mu)$ is reflexive and hence the set $\{ T_t f ; 0 < t < 1 \}$ is weakly sequentially compact for any $f \in L_p(\mu)$. If $p = 1$ and there exists a strictly positive function $h \in L_1(\mu)$ such that the set $\{ T_t h ; 0 < t < 1 \}$ is weakly sequentially compact, then it follows from [2, Theorem IV.8.9] that the set $\{ T_t f ; 0 < t < 1 \}$ is weakly sequentially compact for any $f \in L_1(\mu)$. Thus in any case, $T_0 = \text{strong-}\lim_{t \to 0} T_t$ exists by Lemma 1. Clearly T_0 is a positive contraction on $L_p(\mu)$ and $T_0 T_t = T_t = T_t T_0$ for any $t \geqq 0$. Let us set $h = T_0 1$ and $Q = \operatorname{supp} h$. It then follows that $T_t L_p(Q, \mu) \subset L_p(Q, \mu)$ and $T_t L_p(X - Q, \mu) = \{0\}$ for any $t \geqq 0$. Therefore to prove the theorem we may assume without loss of generality that $X = Q$.

Let λ be the measure on $(X, \mathcal{F})$ defined by $d\lambda = h^p d\mu$, and let S_t , $t \geqq 0$, be defined on $L_p(X, \lambda) = L_p(X, \mathcal{F}, \lambda)$ by

$$S_t f = \frac{1}{h} T_t(fh) , \qquad f \in L_p(X, \lambda) .$$

Since the mapping $f \longrightarrow fh$ is a positive isometry of $L_p(X, \lambda)$ onto $L_p(X, \mu)$, $\{ S_t ; t \geqq 0 \}$ is a strongly continuous semigroup of positive linear operators on $L_p(X, \lambda)$, and hence for the proof of the theorem it suffices to show that for any $f \in L_p(X, \lambda)$ the limit

$$\lim_{b \to 0} \frac{1}{b} \int_0^b S_t f(x) \, dt \tag{4}$$

exists a.e. To see this, however, it suffices to show that the limit (4) exists a.e. for any $f \in L_p(X, \lambda)$ with $S_o f = f$, since $S_o^2 = S_o$. Let $\mathcal{J} = \{A \in \mathcal{F} ; S_o 1_A = 1_A\}$. Since $S_o 1 = (Th)/h = 1$, it follows easily that $\mathcal{J}$ is a σ-field. We shall now prove that

$$(5) \qquad \{f \in L_p(X, \lambda) ; S_o f = f\} = L_p(X, \mathcal{J}, \lambda) .$$

Clearly $f \in L_p(X, \mathcal{J}, \lambda)$ implies $S_o f = f$. Conversely let $S_o f = f$. Since S_o is a positive contraction on $L_p(X, \lambda)$, it then follows that $S_o |f| = |f|$, and hence we may assume without loss of generality that f is nonnegative. If a is any positive real, let $g = \min(f, a)$ and $h = f - g$. Then $\| S_o g \|_\infty \leqq a$ and $S_o h \geqq h$. Hence $S_o h = h$. Thus if we let $A = \{x ; f(x) > a\}$, then $S_o L_p(A, \lambda) \subset L_p(A, \lambda)$ and $S_o 1_{X-A} = 1_{X-A}$. Consequently $f \in L_p(X, \mathcal{J}, \lambda)$.

By (5) and the fact that $S_o S_t = S_t$ for any $t \geqq 0$, each S_t may be considered to be an operator on $L_p(X, \mathcal{J}, \lambda)$ and $S_o = I$ on $L_p(X, \mathcal{J}, \lambda)$. Therefore by Lemma 3 the limit (4) exists a.e. for any $f \in L_p(X, \mathcal{J}, \lambda)$. The proof is complete.

<u>Remark</u>. The argument in the proof of Theorem 1 can be suitably modified to yield a proof of the following result:

If $\Gamma = \{T_t ; t \geqq 0\}$ is a strongly continuous semigroup of positive linear operators on $L_p(\mu)$ with $1 \leqq p < \infty$ and if $0 \leqq f \in L_p(\mu)$ and $\| f \|_p > 0$ imply

$\| T_0 f \|_p > 0$, then the limit (1) exists a.e. for any $f \in$ $\in L_p(\mu)$.

4. <u>Proof of Theorem 2</u>. By the Riesz convexity theorem [2, Theorem VI.10.11] we may assume without loss of generality that $1 < p < \infty$. Hence the set $\{ T_t f \ ; \ 0 < t < 1 \}$ is weakly sequentially compact in $L_p(\mu)$ and hence in $L_1(\mu)$ for any $f \in L_p(\mu)$. Since $L_p(\mu)$ is dense in $L_1(\mu)$, an approximation argument shows that for any $f \in L_1(\mu)$ the set $\{ T_t f \ ; \ 0 < t < 1 \}$ is weakly sequentially compact in $L_1(\mu)$. Lemma 1 now implies that $\text{strong-}\lim_{t \to 0} T_t = T_0$ exists. Clearly T_0 is a contraction on $L_1(\mu)$ and $T_t T_0 = T_t = T_0 T_t$ for any $t \geqq 0$.

Let f_0 be a function in $L_1(\mu)$ with $T_0 f_0 = f_0$ such that if $g \in L_1(\mu)$ satisfies $T_0 g = g$ then $\operatorname{supp} g \subset \operatorname{supp} f_0$ [8]. Let $Q = \operatorname{supp} f_0$ and $h = | f_0 |$. Since $T_t L_1(Q, \mu) \subset L_1(Q, \mu)$ and $T_t L_1(X - Q, \mu) = 0$ for any $t \geqq 0$, for the proof of the theorem we may assume without loss of generality that $X = Q$. As in the proof of Theorem 1, let λ be the measure on $(X, \mathcal{F})$ defined by $d\lambda = h \, d\mu$, and let S_t , $t \geqq 0$, be defined on $L_1(X, \lambda)$ by

$$S_t f = \frac{1}{h} T_t(fh) , \qquad f \in L_1(X, \lambda) .$$

For the proof it suffices to show that the limit (4) exists a.e. for any $f \in L_1(X, \lambda)$. Let e be a function in $L_\infty(X, \lambda)$ with $| e | = 1$ such that $e S_0(\bar{e} f) \geqq 0$ whenever $0 \leqq f \in L_1(X, \lambda)$ [8], and let R_t, $t \geqq 0$, be defined on $L_1(X,\lambda)$ by

$$R_t f = eS_t(\tau f) , \qquad f \in L_1(X, \lambda) .$$

Then clearly $\{R_t \, ; \, t \geq 0\}$ is a strongly continuous semigroup of contractions on $L_1(X, \lambda)$. Since R_0 is positive and satisfies $R_0 1 = 1$, as in the proof of Theorem 1 we have that $\{f \in L_1(X, \lambda) \, ; \, R_0 f = f\} = L_1(X, \mathcal{J}, \lambda)$ where $\mathcal{J} = \{A \in \mathcal{F} \, ; \, R_0 1_A = 1_A\}$, and hence each R_t may be considered to be a contraction on $L_1(X, \mathcal{J}, \lambda)$ and $R_0 = I$ on $L_1(X, \mathcal{J}, \lambda)$. Therefore Kubokawa's local ergodic theorem [5] shows that the limit

$$(6) \qquad \lim_{\delta \to 0} \frac{1}{\delta} \int_0^{\delta} R_t f(x)\, dt$$

exists a.e. for any $f \in L_1(X, \mathcal{J}, \lambda) = \{f \in L_1(X, \lambda) \, ; \, R_0 f = f\}$, and hence the limit (6) exists a.e. for any $f \in L_1(X, \lambda)$, since $R_0^2 = R_0$. This completes the proof.

References

[1] AKCOGLU M.A., CHACON R.V.: A local ratio theorem, Canad. J.Math.12(1970),545-552.

[2] DUNFORD N., SCHWARTZ J.T.: Linear operators,Part I, Interscience,New York 1958.

[3] KRENGEL U.: A local ergodic theorem, Invent.Math.6(1969), 329-333.

[4] KUBOKAWA Y.: A general local ergodic theorem, Proc.Japan

Acad.48(1972),461-465.

[5] KUBOKAWA Y.: Ergodic theorems for contraction semigroups to appear in J.Math.Soc.Japan.

[6] KUBOKAWA Y.: A local ergodic theorem for semi-group on L_p, to appear in Tohoku Math.J.

[7] ORNSTEIN D.S.: The sums of iterates of a positive operator, Advances in Probability and Related Topics, Vol.2,pp. 85-115,Dekker,New York 1970.

[8] SATO R.: On the limit of weighted operator averages, Math.Ann.

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