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Miroslav Hušek; Jan Pelant

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NOTE ABOUT ATOM-CATEGORIES OF TOPOLOGICAL SPACES

M. HUŠEK and J. PELANT, Praha

Abstract: Minimal members of the "lattice" of epireflective subcategories of topological spaces are investigated. They are in a close connection with subspaces of Čech-Stone compactifications of discrete spaces.

Key-words: Epireflective subcategory, Čech-Stone compactification.

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All topological spaces are assumed to be completely regular Hausdorff; the category of all such spaces together with continuous mappings will be denoted by Top_{CR}.

We are going to investigate the ordering given by inclusion between epireflective subcategories of Top_{CR} (by Kennison theorem, [K], between closed-hereditary and productive classes of topological spaces). We shall use without references simple facts about epireflective subcategories (see e.g.: [M_2],[H_1],[H_2]). The epireflective categories $\mathcal{K}(E)$ of E-compact spaces will play a great role in the sequel (E-compact spaces, [M_1], are homeomorphs of closed subspaces of powers E^m). The first fact that is relevant to our consideration is due to Mrówka, [M_2]: Let N be a countable discrete topological space and let $D(2)$ be a two-point discrete space; then there is no epireflective subcategory $\mathcal{K}$ such that $\mathcal{K}(D(2)) \subsetneq \subsetneq \mathcal{K} \subsetneq \mathcal{K}(N)$. We take this property as a foundation for the following definition:

Definition. Let $\mathcal{K}$, $\mathcal{L}$ be epireflective subcategories of Top_{CR}. Then $\mathcal{L}$ is said to be an atom-category above $\mathcal{K}$ if $\mathcal{L} \supsetneq \mathcal{K}$ and there is no epireflective subcategory $\mathcal{M}$ of Top_{CR} such that $\mathcal{K} \subsetneq \mathcal{M} \subsetneq \mathcal{L}$.

Atom-categories above $\mathcal{K}(D(2))$ will be called briefly atom-categories. The Mrówka's result quoted above asserts that $\mathcal{K}(N)$ is an atom-category. It is clear that atom-categories are of the form $\mathcal{K}(E)$ for a suitable space E and that they are minimal in the sense that the only epireflective subcategories of Top_{CR} strictly contained in them are the categories $\mathcal{K}(D(2))$ and $\mathcal{K}(D(1))$. R. Blefko was interested in the question whether $\mathcal{K}(T\omega_\alpha)$ are atom-categories ($T\omega_\alpha$ is the ordered space of all ordinals less than ω_α); the answer was negative [B_1],[B_2], if $cf\,\omega_\alpha \neq \omega_0$, of course. Nevertheless, it is proved in [P] that there is an atom-category $\mathcal{K}(A_\alpha)$ contained in $\mathcal{K}(T\omega_\alpha)$ for any ω_α and, moreover, A_α can be chosen in such a way that $\mathcal{K}(P \times A_\alpha)$ $(card\ P > 2)$ is an atom-category above $\mathcal{K}(P)$ for regular ordinals ω_α provided $comp\ \mathcal{K}(P) > > \omega_\alpha$ (by $comp\ \mathcal{B}$, $\mathcal{B}$ a class of topological spaces, we mean $min\ \{\alpha \mid \exists X \in \mathcal{B},\ \exists A \subset X,\ card\ A = \alpha,\ \bar{A}^X$ is not compact$\}$ if it exists, i.e., if $\mathcal{B}$ contains noncompact spaces).

The aim of this paper is to exhibit other examples of atom-categories and to give properties of a topological space E sufficient for $\mathcal{K}(E)$ to contain an atom-category.

We have mentioned that atom-categories are simple but we can say more about "generators" of such categories (βP is the Čech-Stone compactification of P , ${}^{\alpha}\overline{X} = \bigcup \{ \overline{A} \mid A \subset X , \ \mathrm{card}\, A < \alpha \}$) :

<u>Proposition 1</u>: Let $\mathcal{K}$ be an atom-category containing noncompact spaces. Then there is an object X of $\mathcal{K}$ such that $\mathcal{K}(X) = \mathcal{K}, D \subset X \subsetneq \beta D, {}^{\alpha}\overline{X}^{\beta D} = X$ where D is a discrete space of cardinality $\alpha = \mathrm{comp}\ \mathcal{K}$.

<u>Proof</u>: Put $X = \beta_{\mathcal{K}} D$, the reflection of D in $\mathcal{K}$.

We do not know whether the following converse of Proposition 1 is true: Let D be a discrete space of cardinality α , $D \subset X \subsetneq \beta D$, ${}^{\alpha}\overline{X}^{\beta D} = X$, $\beta_{\mathcal{K}(X)} D = X$, then $\mathcal{K}(X)$ is an atom-category.

We can prove the converse in special cases, e.g. if $\alpha = \omega_0$ or $X = {}^{\alpha}\overline{D}^{\beta D}$, $\mathrm{card}\, D = \alpha$ is regular. (P is a strongly discrete subset of Q if there is a disjoint open family $\{ U_p \mid p \in P \}$ in Q with $p \in U_p$.)

<u>Theorem 1</u>. Suppose that D is a discrete space of cardinality α , $D \subset X \subsetneq \beta D$, ${}^{\alpha}\overline{X}^{\beta D} = X$, $\beta_{\mathcal{K}(X)} D = X$ and

that each subset of $\mathcal{X}$ of cardinality α and with noncompact closure in $\mathcal{X}$ contains a strongly discrete subset of the same cardinality. Then $\mathcal{K}(\mathcal{X})$ is an atom-category.

Proof: Let $E \in \mathcal{K}(X)$, E be noncompact ($\mathcal{K}(D(2))$ is a class of all compact spaces contained in $\mathcal{K}(X)$). We have to prove that $X \in \mathcal{K}(E)$. We may suppose that E is a closed subspace of X^I. There is an $i \in I$ such that $\overline{pr_i[E]}^X$ is not compact and, thus, $card\, pr_i[E] \geq \alpha$. By the assumption, there is a strongly discrete subset A of $pr_i[E]$, $card\, A = \alpha$, with the corresponding disjoint open family $\{U_a\}$. Making use of the equality $\beta_{\mathcal{K}(X)} D = X$ we can prove that $\overline{A}^X$ is homeomorphic to $\mathcal{X}$ (if $\varphi : A \longrightarrow D$ is bijective, there is an $f : D \longrightarrow D$ such that the continuous extension $\tilde{f}$ on βD into βD extends φ; then $\tilde{f}/_{\overline{A}^X}$ is the homeomorphism). Now, let $g : A \longrightarrow E$ be a bijective mapping with the inverse $pr_i/_{g[A]}$. There exists a continuous extension $\tilde{g} : \overline{A}^X \longrightarrow E$ which must be a homeomorphism then. Consequently, $\mathcal{X}$ can be embedded as a closed subspace into E.

As mentioned above, the condition about strongly discrete subsets is clearly fulfilled if $\alpha = \omega_0$ or if $\mathcal{X} = {}^{\alpha}\overline{D}^{\beta D}$, $card\, D = \alpha$ is regular. In the second case we receive atom-categories $\mathcal{K}(X)$ contained in $\mathcal{K}(T\omega_\alpha)$ and described in [P]. The first can give:

Theorem 2. If $\mathcal{K}$ is an epireflective subcategory of Top_{CR} containing an object which is not strongly countably compact (i.e., $comp\,\mathcal{H} = \omega_0$), then there exists an atom-category $\mathcal{L} \subset \mathcal{H}$.

We do not know whether Theorem 2 holds generally without any assumption on $comp\,\mathcal{H}$. To prove a more general version one must remove condition on strongly discrete subsets in Theorem 1 because as Hajnal and Juhász [HJ] proved under generalized continuum hypothesis, for any infinite cardinal α , there exists a set A in βD , $card\,D = \alpha$, such that $card\,A = 2^{2^{\alpha}}$ and no uncountable $B \subset A$ is strongly discrete.

Theorem 1 for $\alpha = \omega_0$ suggests the following construction of spaces X generating atom-categories (we write $\tilde{f}$ for the continuous extension of $f : N \longrightarrow \beta N$ on βN). Let $X_0 \supset N$ and all X_ξ , $\xi < \eta$, be defined; then we put $X_\eta = \cup \{ \tilde{f} [\bigcup_{\xi < \eta} X_\xi] \mid f : N \longrightarrow \bigcup_{\xi < \eta} X_\xi \}$.

It is easy to prove the following properties of $\{ X_\xi \}$: if $\xi \leq \eta$, then $X_\xi \subset X_\eta$; if $f : N \longrightarrow X_\xi$ then $\tilde{f} [X_\xi] \subset X_{\xi+1}$; $X_{\omega_1} = \bigcup_{\xi < \omega_1} X_\xi$. It follows that $X_{\omega_1 + 1} = X_{\omega_1}$ and $\beta_{\mathcal{K}(X_{\omega_1})} N = X_{\omega_1}$, i.e. by Theorem 1 that $\mathcal{K}(X_{\omega_1})$ is an atom-category provided $X_{\omega_1} \neq \beta N$. This last condition is guaranteed by the assumption $card\,X_0 \leq 2^{\omega_0}$ (then $card\,X_{\omega_1} \leq 2^{\omega_0}$). One can deduce that there is exactly $2^{2^{\omega_0}}$ different atom-categories $\mathcal{K}(X)$ generated by the spaces X with properties $N \subset X \subsetneq \beta N$, $\beta_{\mathcal{K}(X)} N = X$, $card\,X \leq 2^{\omega_0}$.

Remark: Proposition 10 of [P] can be generalized: Let $\mathcal{K}(A)$ be an atom-category, $comp\ A = \alpha$, $D(\alpha) \subset A \subset \beta D(\alpha)$. Suppose that each non-compact set of A contains a strongly discrete subset of cardinality α. Let $\mathcal{P}$ be an epireflective subcategory of Top_{CR}, $comp\ \mathcal{P} > \alpha$. Denote by $\mathcal{K}(A) \vee \mathcal{P}$ the least epireflective subcategory of Top_{CR} containing both $\mathcal{K}(A)$ and $\mathcal{P}$. This category is an atom-category above $\mathcal{P}$.

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Matematický ústav
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

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