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ON THE GEOMETRIC CHARACTERIZATION OF DIFFERENTIABILITY II.

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Abstract: In this paper, the geometric characterization of differentiability in Banach spaces is given. It is shown that a mapping $F: X \longrightarrow Y$ possesses the Fréchet derivative $F'(x_0)$ at a point x_0 iff F is continuous at x_0 and certain tangent cone to the graph of F coincides with the graph of some continuous linear mapping $L: X \longrightarrow Y$ (it is $F'(x_0) = L$ in that case).

Key words: Banach space, Fréchet derivative, conic limit, tangent cone.

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The present paper is a free continuation of [11]. Both these papers deal with geometric characterizations of differentiability in Banach spaces.

The problem of geometric characterization, especially in finitely dimensional spaces, was studied by many authors, e.g. [2] - [8],[10],[11]; the characterizations given there were based on two basic notions: tangent plane [6],[11] and tangent cone [4]. The latter notion, in fact generalizing the first one, was then used in various applications, namely to nonlinear programming (see e.g. [1],[4],[5],[9]).

In the first part of our paper [11], the geometric characterization of differentiability of mappings in Banach spaces in terms of tangent flats (planes) was presented. In the second part of [11], we discussed the problem stated by T.M. Flett in [4] (see also [5]): whether the F-differentiability in Banach spaces can be characterized in terms of tangent cones (in the sense of Flett [4]). We showed there in an example that such characterization is not possible even under very strong restrictions (e.g. in case of a Lipschitzian mapping from the real line into a Hilbert space) and we tried to find the cause of it.

Bearing in mind our conclusions made at the end of [11], we shall now modify the notion of a tangent cone in such a manner to obtain the required characterization of differentiability. The relations between this new notion and the similar ones of other authors ([1],[4],[9]) will be stated, too.

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1^o Let Z be a Banach space and $x_0 \in Z$. A set $C \subset Z$ such that $\lambda(C - x_0) \subset C - x_0$ for every $\lambda \geq 0$ is said to be a cone with a vertex x_0; the cone $C = \{x_0\}$ is said to be degenerated. Denote $B_r: \{x \in Z; \|x\| < r\}$ and $S = \{x \in Z : \|x\| = 1\}$.

Definition. A cone $C \subset Z$ with a vertex x_0 is said to be generated by a set $M \subset Z$ iff $C = \bigcup_{\lambda \geq 0} (x_0 + \lambda (M - x_0))$. Let C be a cone with a vertex x_0, let $\varepsilon > 0$; the cone with the vertex x_0 generated by the set $(C \cap (x_0 + S) + B_\varepsilon$

is said to be the conic ε -neigbourhood of C and denoted by $\mathfrak{U}_\varepsilon(C)$.

Let C_m $(m = 1, 2, \dots)$ be cones in Z with a common vertex z_0 . Then two possibilities arise: either such a set $C_0 \subset Z$ can be chosen that there is m_0 for every $\varepsilon > 0$ such that $C_0 \subset \mathfrak{U}_\varepsilon(C_m)$ and $C_m \subset \mathfrak{U}_\varepsilon(C_0)$ whenever $m \geq m_0$, or no $C_0 \subset Z$ has this property. It is easy to see that if C_0, C_0' are two sets having the property above, then $\overline{C}_0 = \overline{C}_0'$ and $\overline{C}_0$ has that property, too; moreover, $\overline{C}_0$ is a closed cone with a vertex at z_0 .

Definition. Let C_m $(m = 1, 2, \dots)$ be cones in Z with a common vertex z_0 . The conic limit of C_m is defined to be the union of the set $\{z_0\}$ with all cones $C \subset Z$ having the property: there is m_0 for every $\varepsilon > 0$ such that $C \subset \mathfrak{U}_\varepsilon(C_m)$ and $C_m \subset \mathfrak{U}_\varepsilon(C)$ whenever $m \geq m_0$. We denote this limit by $C\text{-}\lim_{m \to \infty} C_m$ and call it regular if it contains more than one point. The conic limit of an uncountable system of cones is defined in a similar way.

It follows from the preceding discussion that a conic limit is always a closed cone with a vertex at z_0 (which is degenerated in case of irregular limit). Moreover, the following assertions hold; their proofs are straightforward and so we omit them.

Proposition 1. Let C_m $(m = 0, 1, 2, \dots)$ be closed cones in Z with a common vertex z_0 . Then C_0 is the regular conic limit of C_m $(m = 1, 2, \dots)$ if and only if for every $\varkappa > 0$,

$$C_m \cap (x_0 + \overline{B}_\varkappa) \longrightarrow C_0 \cap (x_0 + \overline{B}_\varkappa)$$

in the sense of Hausdorff metric in the space of closed bounded subsets of Z .

Proposition 2. Let C_m $(m = 1, 2, \dots)$ be cones in Z with a common vertex x_0 , $C_{m+1} \subset C_m$ for all m and suppose that there is the regular conic limit $C_0 = \underset{m \to \infty}{C-\lim}\, C_m$. Then $C_0 = \bigcap_{m=1}^{\infty} \overline{C}_m$.

2^0 Now, we are prepared to define the improved notion of a tangent cone (see the end of (2.2) in [11]). Hereafter, we shall use the term "tangent cone" only in the sense of the following definition.

Definition. Let Z be a Banach space, $M \subset Z$ a non-empty set and $x_0 \in \overline{M}$. Denoting

(1)
$$\mathcal{C}_\varkappa (M, x_0) = \{ \xi : \xi = x_0 + \lambda \frac{x - x_0}{\| x - x_0 \|} , \lambda \geq 0 ,$$
$$x \in M \setminus \{ x_0 \} , \| x - x_0 \| \leq \varkappa \}$$

for $\varkappa > 0$, the set

$$\mathcal{C}_0 (M, x_0) = \underset{\varkappa \to \infty}{C-\lim}\, \mathcal{C}_\varkappa (M, x_0)$$

is said to be the tangent cone to M at the point x_0 .

It is evident that all $\mathcal{C}_\varkappa(M, x_0)$ are cones in Z with the common vertex x_0 , they are generated by the sets $M \cap (x_0 + \overline{B}_\varkappa)$ and $\mathcal{C}_{\varkappa_1}(M, x_0) \subset \mathcal{C}_{\varkappa_2}(M, x_0)$ if $\varkappa_1 \leq \varkappa_2$; we call $\{ \mathcal{C}_\varkappa(M, x_0) : \varkappa > 0 \}$ the quasi-tangent system of cones.

The tangent cone defined in this way is always a nonempty closed cone with a vertex z_0 (that may be degenerated to $\{z_0\}$). It is in close connection with similar cones of some other authors ([9],[4],[1]) as will be shown in Section 3^0 but there is a difference there which makes it possible to characterize the F-differentiability of mappings.

Now, we prove our main theorem.

<u>Theorem 1</u>. Let X, Y be Banach spaces, $D \subset X$, x_0 an interior point of D and let $F: D \longrightarrow Y$ be a mapping. Then F possesses the Fréchet derivative $F'(x_0)$ at x_0 if and only if F is continuous at x_0 and there is a continuous linear mapping $L: X \longrightarrow Y$ so that

$$(2) \quad \mathcal{C}_0(\mathcal{G}(F), (x_0, F(x_0))) = (x_0, F(x_0)) + \mathcal{G}(L) \ ;$$

if it is the case, then $F'(x_0) = L$.

<u>Proof</u>. Denote $Z = X \times Y$ and $z_0 = (x_0, F(x_0))$. We shall consider the maximum norm in $X \times Y$, i.e. $\|(x,y)\|_Z = \max(\|x\|_X, \|y\|_Y)$, but it is not essential - arbitrary equivalent norm in $X \times Y$ (e.g. a sum norm) can be considered. Suppose that any neighbourhoods of x_0 or z_0 will be anywhere dealt with, these will be sufficiently small to be contained in D or $D \times Y$, respectively.

1) Let F be F-differentiable at x_0 and denote $F'(x_0) = L$. Suppose that $\mathcal{C}_0(\mathcal{G}(F), z_0) \neq z_0 + \mathcal{G}(L)$, i.e. that the sequence $\{\mathcal{C}_{\kappa}(\mathcal{G}(F), z_0)\}$ does not converge in the sense of Section 1^0 to $z_0 + \mathcal{G}(L)$. Then there are $\varepsilon > 0$ and $\kappa_n > 0$ $(n = 1, 2, \ldots)$ such that $\kappa_n \longrightarrow 0$ and

that for every $n = 1, 2, \dots$,

(3) $$\mathcal{C}_{\varkappa}(\mathcal{G}(F), x_0) \not\subset x_0 + $$

$$+ \{\xi \in Z : \xi = \mu(w + c), \mu \geq 0, w \in \mathcal{G}(L) \cap S, c \in B_{\varepsilon}\}$$

or

(4) $$x_0 + \mathcal{G}(L) \not\subset \{\xi \in Z : \xi = x_0 + \lambda\left(\frac{x - x_0}{\|x - x_0\|} + c\right),$$

$$\lambda \geq 0, x \in \mathcal{G}(F), \|x - x_0\| \leq \varkappa_n, c \in B_{\varepsilon}\}$$

holds. Denote N_1 and N_2 the sets of those n for which (3) or (4) is true, respectively; at least one of these sets must be infinite.

Suppose N is infinite and denote the set on the right side of the inclusion (3) by $(x_0 + \mathcal{U})$. By (3), there is $x_n \in \mathcal{C}_{\varkappa_n}(\mathcal{G}(F), x_0)$ for every $n \in N_1$ such that $x_n \notin x_0 + \mathcal{U}$ and hence

$$x_0 + \lambda(x_n - x_0) \notin x_0 + \mathcal{U}$$

for all $n \in N_1$ and $\lambda > 0$ because $\mathcal{U}$ is a cone. This means that

$$\left\| \frac{1}{\mu}[\lambda(x_n - x_0) - \mu w] \right\| \geq \varepsilon$$

for all λ, $\mu > 0$ and $w \in \mathcal{G}(L)$ with $\|w\| = 1$; particularly,

(5) $$\|x_n - x_0 - \mu w\| \geq \mu\varepsilon$$

holds for all $n \in N_1$, $\mu > 0$ and $w \in \mathcal{G}(L)$ with $\|w\| = 1$ where

$$\|z_n - z_0\| \le r_m , \ r_m \longrightarrow 0$$

according to the choice of z_m .

By assumption, there is $\delta > 0$ such that

$$\|F(x) - F(x_0) - L(x - x_0)\| < \varepsilon \|x - x_0\|$$

whenever $\|x - x_0\| < \delta \ (x \in X)$. Let $\|z_m - z_0\| < \delta$ for all $m \ge m_0$ and choose $x_m \in X$ such that $z_m = (x_m, F(x_m))$. Then $\|x_m - x_0\| < \delta$ if $m \ge m_0$ and hence

$$\|F(x_m) - F(x_0) - L(x_m - x_0)\| < \varepsilon \|x_m - x_0\|$$

for all such m . In the space $X \times Y$, the relation

$$\|(0, F(x_m) - F(x_0) - L(x_m - x_0))\| < \varepsilon \|x_m - x_0\|$$

follows and therefore

$$(6) \qquad \|z_m - z_0 - (x_m - x_0, L(x_m - x_0))\| <$$
$$< \varepsilon \max (\|x_m - x_0\|, \|L(x_m - x_0)\|)$$

whenever $m \ge m_0$. Put

$\mu_m = \|(x_m - x_0, L(x_m - x_0))\| = \max (\|x_m - x_0\|, L(x_m - x_0)\|)$

and $w_m = \frac{1}{\mu}(x_m - x_0 , L(x_m - x_0))$. Then $\mu_n \ge 0$, $w_m \in \mathcal{G}(L)$, $\|w_m\| = 1$ and (according to (6))

$$\|z_m - z_0 - \mu_m w_m\| < \mu_m \varepsilon$$

for all $m \ge m_0$; but this contradicts (5) and hence, the set N_1 cannot be infinite.

Now, suppose N_2 to be infinite. Denoting $(z_0 + U_{r_m})$ the set on the right side of (4), it follows from (4) that there are $\{w_m\} \subset \mathcal{G}(L)$ such that $w_m \notin U_{r_m}$ for every

$n \notin N_2$. However, $\mathcal{G}(L)$ is linear and $U_{\varkappa_n}$ are cones and so $w \notin U_{\varkappa_n}$ holds for all $w \in \mathcal{G}(L)$ and $n \in N_2$. It means, with respect to the structure of $U_{\varkappa_n}$ and linearity of $\mathcal{G}(L)$ that

$$(7) \qquad \left\| w - \frac{z - z_0}{\|z - z_0\|} \right\| \geq \varepsilon$$

for all $w \in \mathcal{G}(L)$, $z \in \mathcal{G}(F)$ with $\|z - z_0\| \leq \varkappa_n$ and $n \in$ $\in N_2$. Now, in the same way as (6) was proved, we can prove that

$$\|z - z_0 - (x - x_0, L(x - x_0))\| < \varepsilon \|x - x_0\| \leq \varepsilon \|z - z_0\|$$

for all $z \in \mathcal{G}(F)$ sufficiently near to z_0, say $0 < \|z - z_0\| <$ $< \delta$. Choose n_0 to be $\varkappa_n < \delta$ whenever $n \geq n_0$ and choose $z_n \in \mathcal{G}(F)$ such that $0 < \|z_n - z_0\| < \varkappa_n$ for every $n \geq$ $\geq n_0$. Then setting

$$w_n = \frac{(x_n - x_0, L(x_n - x_0))}{\|z_n - z_0\|},$$

we have $w_n \in \mathcal{G}(L)$ and

$$\left\| \frac{z_n - z_0}{\|z_n - z_0\|} - w_n \right\| < \varepsilon$$

for all $n \geq n_0$ which contradicts (7). It proves the first part of our theorem.

2) On the other hand, suppose now that there is a linear continuous mapping $L : X \to Y$ such that (2) holds but that F is not differentiable at x_0. In such case, there are $\varepsilon > 0$ and $x_n \in X$ such that $x_n \to x_0$, $x_n \neq x_0$ and

$$\| F(x_m) - F(x_0) - L(x_m - x_0) \| > \varepsilon \, \| x_m - x_0 \| \tag{8}$$

for all $m = 1, 2, \ldots$; we can assume $\varepsilon < \frac{1}{2}$. Set $\varepsilon' = \varepsilon(1-\varepsilon)(1+\|L\|)^{-1}$ if $\|L\| \le \frac{1}{2}$ and $\varepsilon' = \varepsilon(1-\varepsilon)[2\|L\|(1+\|L\|)]^{-1}$ if $\|L\| > \frac{1}{2}$; it is $0 < \varepsilon' < \varepsilon < \frac{1}{2}$ in both cases. The relation (2) implies that there is $\delta > 0$ such that

$$\mathcal{C}_{\varkappa}(\mathcal{G}(F), z_0) \subset \{ \zeta \in Z : \zeta = z_0 + \mu(w + c), \ \mu \ge 0, \ w \in \mathcal{G}(L) \cap S, \ c \in B_{\varepsilon'} \} \tag{9}$$

whenever $0 < \varkappa \le \delta$.

It follows from $x_m \longrightarrow x_0$ and from continuity of F at x_0 that there is m_0 such that $\|x_m - x_0\| < \delta$ and $\|F(x_m) - F(x_0)\| < \delta$ whenever $m \ge m_0$. Set $z_m = (x_m, F(x_m))$, $m = 1, 2, \ldots$; then $\|z_m - z_0\| < \delta$ and s $z_m \in \mathcal{C}_{\delta}(\mathcal{G}(F), z_0)$ if $m \ge m_0$ .By (9), we can choose $w_m \in \mathcal{G}(L)$ with $\|w_m\| = 1$, $c_m \in Z$ with $\|c_m\| \le \varepsilon'$ and $\mu_m > 0$ (it is $x_m \neq x_0$) so that

$$z_m = z_0 + \mu_m (w_m + c_m) \tag{10}$$

whenever $m \ge m_0$,that is

$$x_m = x_0 + \mu_m (v_m + a_m) , \tag{11}$$

$$F(x_m) = F(x_0) + \mu_m (L(v_m) + b_m) \tag{12}$$

where $(a_n, b_n) = c_n$ and hence $\|a_n\|, \|b_n\| \le \varepsilon'$. Now, (10) implies

$$(13) \qquad \|x_n - x_0\| \ge \mu_n (1 - \varepsilon') > \mu_n (1 - \varepsilon) .$$

It holds

$$L(v_n) = \frac{1}{\mu_n} L(x_n - x_0) - L(a_n)$$

according to (11) and on the other hand, it is

$$L(v_n) = \frac{1}{\mu_n} (F(x_n) - F(x_0)) - b_n$$

by (12). We conclude from these equalities and (13) that

$$(14) \qquad \|F(x_n) - F(x_0) - L(x_n - x_0)\| = \|\mu_n L(a_n) - \mu_n b_n\| \le$$

$$\le \mu_n \|L\| \varepsilon' + \mu_n \varepsilon' < \frac{\varepsilon'(1 + \|L\|)}{1 - \varepsilon} \|x_n - x_0\|$$

for all $n \ge n_0$.

Two cases are to be distinguished now. First, let $\|L\| \le \frac{1}{2}$. Then $\varepsilon' = \varepsilon(1-\varepsilon)\cdot(1 + \|L\|)^{-1}$ and so (14) implies that

$$(15) \qquad \|F(x_n) - F(x_0) - L(x_n - x_0)\| < \varepsilon \|x_n - x_0\|$$

whenever $n \ge n_0$. Moreover, it holds

$$(16) \qquad \|x_n - x_0\| \ge \|F(x_n) - F(x_0)\|$$

in this case; in fact, if the reverse inequality were valid then (9) would imply

$$\|F(x_m)-F(x_0)\| - \|L(x_m-x_0)\| \leq \|F(x_m)-F(x_0)-L(x_m-x_0)\| <$$

$$< \varepsilon \|z_m - z_0\| = \varepsilon \|F(x_m)-F(x_0)\|$$

and hence (it is $\varepsilon < \frac{1}{2}$)

$$\|F(x_m)-F(x_0)\| < \frac{1}{1-\varepsilon} \cdot \|L\| \cdot \|x_m - x_0\| \leq \|x_m - x_0\| ,$$

which is the contradiction to our assumption. It follows now from (15) and (16) that

$$\|F(x_m)-F(x_0)-L(x_m-x_0)\| < \varepsilon \|x_m - x_0\| ;$$

however, it contradicts (8).

Now, consider the case $\|L\| > \frac{1}{2}$; then (14) implies

(17) $\|F(x_m)-F(x_0)-L(x_m-x_0)\| < \frac{\varepsilon}{2\|L\|} \|z_m-z_0\| < \varepsilon \|z_m-z_0\|$

for all $m \geq m_0$ .If

$$\|F(x_m) - F(x_0)\| > \|x_m - x_0\|$$

were valid then (17) would imply (similarly as above)

$$\|F(x_m)-F(x_0)\| \leq \frac{\|L\|}{1-\varepsilon} \|x_m-x_0\| < 2\|L\| \cdot \|x_m-x_0\|$$

and hence by (17),

$$\|F(x_m)-F(x_0)-L(x_m-x_0)\| < \frac{\varepsilon}{2\|L\|} \|F(x_m)-F(x_0)\| \leq \varepsilon \|x_m-x_0\|$$

for $m \geq m_0$. On the other hand, if

$$\|F(x_m)-F(x_0)\| \leq \|x_m-x_0\|$$

were valid then (17) would imply directly

$$\|F(x_n)-F(x_0)-L(x_n-x_0)\|<\varepsilon\|z_n-x_0\| = \varepsilon\|x_n-x_0\|$$

for $n \geq n_0$ .Hence in both last cases, we come to the contradiction to (8), too.

Thus we have proved that F is F -differentiable at x_0 and $F'(x_0)=L$. Moreover, since L is continuous, it is the F -derivative of F at x_0 .

The proof is completed.

Note that in the case of our example (2.2) [11], it is $\mathcal{C}_0(\mathcal{G}(F),(0,0)) = \{(0,0)\}$ and hence F is not differentiable at $(0,0)$ according to Theorem 1.

In the same way as Theorem 1, with evident formal modifications only, the analogical theorem can be proved in the case that x_0 is not an interior point of D but that the intersection of $\operatorname{Int} D$ with every sufficiently small neighbourhood of x_0 is non-empty; such a situation occurs in the case of the differentiability relative to a set. [The F -derivative of $F: X \longrightarrow Y$ at x_0 relative to $M \subset X$ (denoting by $F'_M(x_0)$) is defined to be a linear continuous mapping $L: X \longrightarrow Y$ for which

$$\frac{1}{\|x-x_0\|}\cdot\|F(x)-F(x_0)-L(x-x_0)\| \longrightarrow 0$$

if $x \longrightarrow x_0, x_0 \neq x \in M$.] Hence, the following theorem holds ($F|_M$ denotes the restriction of F to M , $\operatorname{sp} M$ denotes the closed linear span of M):

<u>Theorem 2</u>. Let X, Y be Banach spaces, $D \subset X$, $x_0 \in D$, $F: D \longrightarrow Y$ and let $M \subset D$ be a set with a non-empty inte-

rior. Suppose x_0 lies on the boundary of $Int\, M$. Then F possesses the Fréchet derivative $F'_M(x_0)$ at x_0 relative to M if and only if F is continuous at x_0 relative to M (i.e., $F|_{M \cup \{x_0\}}$ is continuous at x_0) and there is a continuous linear mapping $L : X \to Y$ such that

$$(18) \qquad sp\,[\mathscr{C}_0(\mathcal{G}(F|_M),(x_0,F(x_0)))-(x_0,F(x_0))] = \mathcal{G}(L)\,;$$

if it is the case then $F'_M(x_0) = L$. Moreover, the condition

$$\{(x_0,F(x_0))\} \neq \mathscr{C}_0(\mathcal{G}(F|_M),(x_0,F(x_0))) \subset (x_0,F(x_0)) + \mathcal{G}(L)$$

may be equivalently written instead of (18).

Remark that if x_0 is an interior point of M then $F'_M(x_0)$ is the same as $F'(x_0)$ and our Theorem 1 is applicable.

3^0 At the end of our paper, we look over connections between our notion of a tangent cone and similar notions of other authors. The following theorem is the direct consequence of our Proposition 2.

Theorem 3. Let Z be a Banach space, $M \subset Z$, $z_0 \in \overline{M}$ and let $LC(M, z_0)$ be the local closed cone of M at z_0 in the sense of Varaiya [9]. If $\mathscr{C}_0(M, z_0)$ is non-degenerated (i.e., if it is the regular conic limit of a quasi-tangent system of cones to M at z_0) then

$$\mathcal{C}_0(M, x_0) = x_0 + LC(M, x_0) .$$

Corollary. Let Z be a finite dimensional space, $M \subset Z$, $x_0 \in \overline{M}$ and let $T(M, x_0)$ be the cone of tangents to M at x_0 in the sense of [1]. If $\mathcal{C}_0(M, x_0)$ is non-degenerated then

$$\mathcal{C}_0(M, x_0) = x_0 + T(M, x_0) .$$

This is the immediate consequence of the preceding theorem and Theorem 2.1 of [1]. Remark that

$$T(M, x_0) = \{x : x = \lim \lambda_n \cdot (x_n - x_0), \ \lambda_n > 0, \ x_n \in M, \ x_n \longrightarrow x_0\} .$$

Eventually, we shall discuss a connection with a tangent cone in the sense of Flett [4]; denote this cone by $\mathcal{C}(M, x_0)$.

Theorem 4. Let Z be a Banach space, $M \subset Z$ and $x_0 \in M$. If $\mathcal{C}_0(M, x_0)$ is non-degenerated then

$$\mathcal{C}(M, x_0) \subset \mathcal{C}_0(M, x_0) .$$

Proof. Suppose that $\{x_0\} \neq \mathcal{C}_0(M, x_0) \subsetneq \mathcal{C}(M, x_0)$; then there are $x' \in \mathcal{C}(M, x_0)$, $\|x' - x_0\| = 1$, and $\varepsilon \in (0, 1)$ such that

$$\text{(19)} \qquad \|x' - w\| > \varepsilon$$

for all $w \in \mathcal{C}_0(M, x_0)$. By the definition of $\mathcal{C}(M, x_0)$, there are $\lambda' > 0$ and $\{x_n\} \subset M \setminus \{x_0\}$ such that $x_n \longrightarrow x_0$ and

$$x' = x_0 + \lambda' u' \quad \text{where} \quad u' = \lim_{n \to \infty} \frac{x_n - x_0}{\|x_n - x_0\|} .$$

Choose $\delta > 0$ to be $\mathcal{C}_\varkappa(M, x_0) \subset U_{\frac{1}{2}\varepsilon}(\mathcal{C}_0(M, x_0)) = U$ (see Section 1°) whenever $\varkappa \leq \delta$. It is easy to see that then $\overline{\mathcal{C}_\varkappa(M, x_0)} \subset U$, too. Let n_0 be such a number that $n \geq n_0$ implies $\| x_n - x_0 \| < \delta$; then

$$x_0 + \lambda \cdot \frac{x_n - x_0}{\| x_n - x_0 \|} \in \mathcal{C}_\delta(M, x_0)$$

for all $\lambda \geq 0$ and particularly, setting $\lambda = \lambda'$ we obtain

$$x' \in \overline{\mathcal{C}_\delta(M, x_0)} \subset U \quad .$$

Therefore, there are $\mu' > 0$, $x'' \in \mathcal{C}_0(M, x_0)$ and $c' \in B_{\frac{1}{2}\varepsilon}$ such that

(20) $$x' = x_0 + \mu' \frac{x'' - x_0}{\| x'' - x_0 \|} + \mu' c'$$

and hence,

(21) $$x' = w' + \mu' c'$$

where $w' = x_0 + \dfrac{\mu'}{\| x'' - x_0 \|} \cdot (x'' - x_0) \in \mathcal{C}_0(M, x_0)$ .We have

$$\mu' \leq \frac{\| x' - x_0 \|}{1 - \frac{1}{2}\varepsilon} < 2$$

by (20) and it follows now from (21) that

$$\| x' - w' \| \leq \mu' \| c' \| < \varepsilon \quad ;$$

but it contradicts (19). The theorem is proved.

Let us remark that if $\mathcal{C}_0(M, x_0)$ is degenerated then it may be $\mathcal{C}(M, x_0) \supsetneqq \mathcal{C}_0(M, x_0)$ as our example (2.2)

[11] shows.

__Theorem 5.__ Let Z be a Banach space, $M \subset Z$, $x_0 \in \overline{M}$ and let $\dim(\operatorname{sp} M) < \infty$. Then $\mathcal{C}_0(M, x_0) = \mathcal{C}(M, x_0)$.

__Proof.__ We shall prove that there is $\delta > 0$ for every $\varepsilon > 0$ such that $\mathcal{C}(M, x_0) \subset U_\varepsilon(\mathcal{C}_\varkappa(M, x_0))$ and $\mathcal{C}_\varkappa(M, x_0) \subset U_\varepsilon(\mathcal{C}(M, x_0))$ whenever $\varkappa < \delta$, whence the assertion will follow by the definition of a conic limit because $\mathcal{C}(M, x_0)$ is evidently closed.

The first inclusion above is valid for every $\varepsilon, \varkappa > 0$. In fact, let it be not true for some $\varepsilon_0 > 0$ and $\varkappa_0 > 0$. Then there is $x' \in \mathcal{C}(M, x_0)$ such that $\|x' - x_0\| = 1$ and

$$x' \notin U_{\varepsilon_0}(\mathcal{C}_{\varkappa_0}(M, x_0)) \quad . \tag{22}$$

By definition of $\mathcal{C}(M, x_0)$, x' may be written in the form

$$x' = x_0 + u$$

where $u = \lim\limits_{m \to \infty} \dfrac{x_m - x_0}{\|x_m - x_0\|}$, $x_m \in M \setminus \{x_0\}$ and $x_m \to x_0$.

Choose m_0 so that $\|x_m - x_0\| < \varkappa_0$ for $m \geq m_0$ and set

$$x'_m = x_0 + \frac{x_m - x_0}{\|x_m - x_0\|} \quad ;$$

then $x'_m \to x'$ and $x'_m \in \mathcal{C}_{\varkappa_0}(M, x_0)$ for $m \geq m_0$. Hence, $x' \in \overline{\mathcal{C}_{\varkappa_0}(M, x_0)}$ which contradicts (22).

It remains to prove that giving $\varepsilon > 0$ there is $\delta > 0$ such that $\mathcal{C}_\varkappa(M, x_0) \subset U_\varepsilon(\mathcal{C}(M, x_0))$ for all

$r < \delta$. Suppose to the contrary that there are $\varepsilon > 0$ and $r_m \geq 0$ such that $r_m \longrightarrow 0$ and $\mathcal{C}_{r_m}(M, x_0) \not\subset U_\varepsilon(\mathcal{C}(M, x_0))$ $(m = 1, 2, \ldots)$. Then there are $x_m \in \mathcal{C}_{r_m}(M, x_0)$ such that $\|x_m - x_0\| = 1$ and

$$(23) \qquad x_m \notin U_\varepsilon(\mathcal{C}(M, x_0))$$

for all m. We can choose points $x'_m \in M$ by the definition of $\mathcal{C}_{r_m}(M, x_0)$ in such manner that

$$x_m = x_0 + \frac{x'_m - x_0}{\|x'_m - x_0\|} .$$

It is $\dfrac{x'_m - x_0}{\|x'_m - x_0\|} \in (S \cap sp\, M)$ for all m and so there is a subsequence $\{x'_{m_k}\}$ of $\{x'_m\}$ such that $\left\{\dfrac{x'_{m_k} - x_0}{\|x'_{m_k} - x_0\|}\right\}$ converges. Denoting by w the limit of this sequence we can see that

$$x_{m_k} \longrightarrow x_0 + w .$$

Moreover, $(x_0 + w) \in \mathcal{C}(M, x_0)$ because of $\|x'_{m_k} - x_0\| \leq$ $\leq r_{m_k} \longrightarrow 0$. Since $w \neq 0$, we have obtained the contradiction to (23).

Note that setting $Z = X \times Y$ and $M = \mathcal{G}(F)$ where $F : X \longrightarrow Y$, we can obtain Theorem 1(i) and Theorem 5 of Flett [4] as a direct consequence of our Theorem 1 and two last theorems.

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