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COSTABLE RINGS

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Abstract: Let R be an associative ring with identity. A torsion theory (T, F) for R-*mod* is called costable if the torsion part of any projective module is a direct summand. It will be shown that every (hereditary) torsion theory is costable if and only if the ring R is a finite direct sum of rings with trivial (hereditary) torsion parts.

Key words: Torsion theory, preradical, radical, stability and costability.

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In this paper, all rings R have unit element and all modules are left unitary R-modules. The category of left R-modules will be denoted by R-*mod*.

1. Preliminaries.

1.1. A preradical κ for R-*mod* is any subfunctor of the identity functor. We shall say that κ is

- a radical if $\kappa(M/\kappa(M)) = 0$ for all $M \in R$-*mod*,
- idempotent if $\kappa(\kappa(M)) = \kappa(M)$ for all $M \in R$-*mod*,
- hereditary if $\kappa(N) = \kappa(M) \cap N$ for all submodules N of $M \in R$-*mod*,
- cohereditary if $\kappa(M/N) = (\kappa(M)+N)/N$ for all submodules N of $M \in R$-*mod*,

- cosplitting if $\varkappa$ is hereditary and cohereditary,

- stable if $\varkappa(Q)$ is a direct summand for all injective modules Q,

- costable if $\varkappa(P)$ is a direct summand for all projective modules P,

- splitting if $\varkappa(M)$ is a direct summand for all $M \in R\text{-}mod$,

- centrally splitting if $\varkappa$ is splitting and cosplitting.

1.2. Let $\varkappa$ be a preradical. Then we put $T_\varkappa = \{M \mid \varkappa(M) = M\}$ and $F_\varkappa = \{M \mid \varkappa(M) = 0\}$. If $\varkappa$ is stable (costable) then $T_\varkappa$ is closed under injective hulls (every module from $F_\varkappa$ possesses a projective presentation belonging to $F_\varkappa$). The reverse assertion holds provided $\varkappa$ is idempotent (a radical).

1.3. A preradical $\varkappa$ is hereditary (cohereditary) iff $\varkappa$ is idempotent (a radical) and $T_\varkappa$ ($F_\varkappa$) is closed under submodules (factormodules).

1.4. Let $\varkappa$ be a preradical. Then $\varkappa(P) = \varkappa(R).P$ for every projective module P. In particular, $\varkappa(R)$ is a two-sided ideal. Moreover, if $\varkappa$ is cohereditary then $\varkappa(M) = \varkappa(R).M$ for all $M \in R\text{-}mod$.

1.5. Let I be a left or right ideal. We shall say that I satisfies the condition (a) ((b)) if $x \in I.x$ ($x \in x.I$) for all $x \in I$.

1.6. Let I be a left ideal and $\varkappa(M) = IM$ for all $M \in R\text{-}mod$. Then $\varkappa$ is a cohereditary radical. Furthermore, $\varkappa$ is idempotent iff IR is so and $\varkappa$ is hereditary iff IR satisfies (a).

1.7. There is a one-to-one correspondence between cohereditary radicals and two-sided ideals.

1.8. Let $\varkappa$ be such a hereditary preradical that $T_\varkappa$ is closed under direct products. Then $\varkappa(M) = \{m \mid Im = 0\}$ where $I = \bigcap K$, K is a left ideal and $R/K \in T_\varkappa$. Conversely, if I is a two-sided ideal and $\varkappa(M) = \{m \mid Im = 0\}$ then $\varkappa$ is a hereditary preradical and $T_\varkappa$ is closed under direct products. Moreover, $\varkappa$ is a radical iff I is idempotent and $\varkappa$ is stable iff I satisfies (a).

1.9. For every class A of modules we define $A^* = \{M \mid Hom(N,M) = 0$ for all $N \in A\}$ and $A^+ = \{M \mid Hom(M,N) = 0$ for all $N \in A\}$. A pair of classes (T, F) is said to be a torsion theory if $T^* = F$ and $F^+ = T$.

1.10. If (T, F) is a torsion theory then $\varkappa_T$ is an idempotent radical, $\varkappa_T(M) = \sum N$, N is a submodule of M and $N \in T$. Conversely, if $\varkappa$ is an idempotent radical then $(T_\varkappa, F_\varkappa)$ is a torsion theory. Hence there is a one-to-one correspondence between torsion theories and idempotent radicals.

1.11. By a TTF-theory we mean a pair (A, B) (C, D) of torsion theories such that $B = C$. In this case there exists an idempotent two-sided ideal I such that $A = \{M \mid IM = M\}$ and $B = C = \{M \mid IM = 0\}$. Obviously, $I = \varkappa_A(R) = \bigcap K$, K is a left ideal and $R/K \in C$. For the concept of TTF-theories, the reader is referred to [5].

2. TTF-theories.

The following lemma is obvious.

2.1. Lemma. Let (T, F) be a torsion theory and κ_T be stable (costable). Then $T^+ \subseteq F$ ($F^* \subseteq T$).

For every left ideal I we denote by B_I the class $\{M \mid m \in \in Im$ for all $m \in M\}$.

2.2. Lemma. Let I be a left ideal. Then:

(i) B_I is closed under homomorphic images, submodules, direct sums and extensions.

(ii) I satisfies (a) iff $I \in B_I$.

(iii) If I satisfies (a) then I, $IR \in B_I$ and $B_I = \{M \mid IM = M\}$.

(iv) If I satisfies (a) then IR satisfies (a).

2.3. Proposition. Let (S, T) (T, F) be a TTF-theory and I be the corresponding idempotent ideal. The following conditions are equivalent:

(i) (T, F) is stable (i.e. κ_T is so).

(ii) S is closed under submodules (i.e. (S, T) is hereditary).

(iii) $S \subseteq F$.

(iv) $S = B_I$.

(v) I satisfies (a).

Proof. (i) implies (ii), (ii) implies (v) and (v) implies (i) - see 1.6 and 1.8. (v) implies (iii) by 2.2, (iii) implies (iv) trivially and (iv) implies (v) since I is idempotent.

2.4. Proposition. Let (S, T) (T, F) be a TTF-theory and I be the corresponding idempotent ideal. The following conditions are equivalent:

(i) $(\mathcal{S}, T)$ is costable (i.e. $\kappa_{\mathcal{S}}$ is so).

(ii) F is closed under factormodules (i.e. (T, F) is cohereditary).

(iii) $F \subseteq \mathcal{S}$.

(iv) $F = B_I$.

(v) I is a direct summand as a left ideal.

<u>Proof</u>. The implications (i) implies (v), (v) implies (i), (iv) implies (ii), (ii) implies (iii) and (iii) implies (iv) are obvious.

(iv) implies (v). Let $K = \{x \mid x \in R$ and $Ix = 0\}$. Then $K = \kappa_T(R)$ and hence $R/K \in B_I$. In particular, there is $i \in I$ such that $i - 1 \in K$, and consequently I is a direct summand in R as a left ideal.

(i) implies (iii) by 2.1.

2.5. <u>Proposition</u>. Let (S, T) (T, F) be a TTF-theory and I be the corresponding idempotent ideal. The following conditions are equivalent:

(i) $\mathcal{S}$ is closed under submodules and direct products.

(ii) (T, F) is stable and I is finitely generated as a right ideal.

(iii) I is a direct summand as a right ideal.

<u>Proof</u>. (ii) implies (iii). Since (T, F) is stable, I satisfies (a). Hence the right module R/I is flat, and consequently projective.

(i) implies (iii). There is a two-sided ideal J such that $S = \{M \mid JM = 0\}$. From this we get $JI = 0$ and $J + I = R$.

Thus I is a direct summand as a right ideal.
(iii) implies (i) and (ii) trivially.

2.6. Theorem. Let $(\mathcal{S}, \mathcal{T})$ $(\mathcal{T}, \mathcal{F})$ be a TTF-theory and I be the corresponding idempotent ideal. The following conditions are equivalent:

(i) $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are stable.
(ii) $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are costable.
(iii) $(\mathcal{S}, \mathcal{T})$ is costable and $(\mathcal{T}, \mathcal{F})$ is stable.
(iv) $(\mathcal{S}, \mathcal{T})$ is centrally splitting.
(v) $(\mathcal{T}, \mathcal{F})$ is centrally splitting.
(vi) $\mathcal{S} = \mathcal{F}$.
(vii) I is a ring direct summand.

Proof. (i) implies (iv). Let $X \in \mathcal{T}$ and $Y \in \mathcal{S}$. We have the exact sequence

$$\mathrm{Hom}(X, E(Y)/Y) \to \mathrm{Ext}(X, Y) \to \mathrm{Ext}(X, E(Y)) = 0 .$$

However $E(Y)/Y \in \mathcal{S}$ and since $(\mathcal{S}, \mathcal{T})$ is cosplitting, $\mathrm{Hom}(X, E(Y)/Y) = 0$.

(iii) implies (iv). Let $X \in \mathcal{T}$ and $Y \in \mathcal{S}$. There is a projective presentation $0 \to H \to P \to X \to 0$ such that $P \in \mathcal{T}$. Hence

$$0 = \mathrm{Hom}(H, Y) \to \mathrm{Ext}(X, Y) \to \mathrm{Ext}(P, Y) = 0 .$$

(ii) implies (v). Since $(\mathcal{S}, \mathcal{T})$ is costable, $(\mathcal{T}, \mathcal{F})$ is cohereditary, as it follows from 2.4. Now we may proceed similarly as in the proof of (iii) implies (iv).
(iv) implies (vi) by 2.3 and 2.4.
(v) implies (vi) by 2.3 and 2.4.
(vi) implies (vii) by 2.4 and 2.5.
(vii) implies (i), (ii) and (iii) trivially.

3. Costable rings.

A ring R will be called

- 1-costable if every preradical for R-*mod* is costable,

- 2-costable if every idempotent preradical is costable,

- 3-costable if every radical is costable,

- 4-costable if every idempotent radical is costable,

- 5-costable if every hereditary preradical is costable,

- 6-costable if every cohereditary radical is costable,

- 7-costable if every hereditary radical is costable,

- 8-costable if every idempotent cohereditary radical is costable,

- 9-costable if every cosplitting radical is costable,

- an R_1-ring if $\varkappa(R)=0$ or $\varkappa(R)=R$ for every preradical $\varkappa$.

Similarly we define R_2 -rings, etc.

The following lemma is obvious.

3.1. Lemma. Let every two-sided ideal satisfying both (a) and (b) be finitely generated. Then R is a finite direct sum of directly indecomposable rings.

3.2. Corollary. Any 9-costable ring is a finite direct sum of directly indecomposable rings.

3.3. Lemma. Every two-sided ideal in a 6-costable ring is a ring direct summand.

Proof. Let I be a two-sided ideal. The corresponning cohereditary radical is costable, and therefore I is a direct summand as a left ideal. Hence $I = Re$, $ee = e$. From this we get that $(1-e)R$ is a two-sided ideal, and consequently it is a direct summand as a left ideal. Thus $Re = eR$ and we are through.

3.4. Theorem. The following conditions are equivalent for every ring R:

(i) R is 1-costable.

(ii) R is 3-costable.

(iii) R is 6-costable.

(iv) Any two-sided ideal is a direct summand as a left ideal.

(v) Any two-sided ideal is a direct summand as a right ideal.

(vi) Any two-sided ideal is a ring direct summand.

(vii) R is a finite direct sum of simple rings.

(viii) R is a finite direct sum of R_1-rings.

Proof. (i) implies (ii) and (ii) implies (iii) trivially.

(iii) implies (iv), (v) and (vi) by 3.3.

(iv) implies (vi) and (v) implies (vi). The proof is similar to that of 3.3.

(vi) implies (vii). It is an easy exercise.

(vii) implies (viii). This implication follows from the fact that R_1-rings are just the simple rings.

(viii) implies (i). It is obvious.

3.5. <u>Theorem</u>. The following conditions are equivalent for every ring R :

(i) Any idempotent cohereditary radical is centrally splitting.

(ii) R is 8-costable.

(iii) R is a finite direct sum of rings having only trivial idempotent two-sided ideals.

(iv) R is a finite direct sum of R_8-rings.

<u>Proof</u>. (i) implies (ii) trivially.

(ii) implies (iii). With respect to 3.2 we can suppose that R is directly indecomposable. Let I be an idempotent ideal. Then $I = Re$, $ee = e$ and so $(1-e)R$ is idempotent and two-sided. Hence $I = Re = eR$.

(iii) implies (iv) and (iv) implies (iii). It is obvious.

(iii) implies (i). It is sufficient to use the following simple fact. If $\varkappa$ is a cohereditary radical and $\varkappa(R) = 0$ then $\varkappa(M) = 0$ for all $M \in R\text{-}mod$.

3.6. <u>Proposition</u>. Let R be a left hereditary ring. The following conditions are equivalent:

(i) R is 2-costable.

(ii) R is 4-costable.

(iii) R is 8-costable.

<u>Proof</u>. (i) implies (ii) and (ii) implies (iii) trivially.

(iii) implies (i). Let $\varkappa$ be an idempotent preradical and $I = \varkappa(R)$. Since I is projective, $I = \varkappa(I) = II$. Now we may use 3.5.

3.7. Theorem. The following two conditions are equivalent for each $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$ and every ring R:
(i) R is k-costable.
(ii) R is a finite direct sum of R_k-rings.

Proof. For $k = 1, 3, 6, 8$ by 3.4 and 3.5.
$k = 2$. With respect to 3.2 we can suppose that R is directly indecomposable. Let $\varkappa$ be an idempotent preradical. Then we have $\varkappa(R) = I = Re$, $ee = e$.
Hence $(1 - e)R$ is a two-sided ideal satisfying (a) and the corresponding cohereditary radical is cosplitting. Thus $(1 - e)R$ is a ring direct summand, and consequently I has the same property.
For $k = 4, 5, 9, 7$ similarly.

3.8. Corollary. Any 5-costable ring is a finite direct sum of prime rings. Conversely, any finite direct sum of prime rings with ascending chain condition on right annihilators is a 5-costable ring.

Proof. The first assertion is obvious. Let R be a prime ring with maximal condition on right point annulets and $\varkappa$ be a hereditary preradical. There is a finite subset $S \subseteq I = \varkappa(R)$ such that $I \subseteq (0 : (0 : S)_{\ell})_{\varkappa}$. Then $R/(0 : S)_{\ell} \in T_{\varkappa}$. On the other hand, $(0 : S)_{\ell} . I = 0$, and hence either $\varkappa(R) = 0$ or $\varkappa(R) = R$.

3.9. Corollary. Let R be a commutative ring. Then R is a 7-costable ring iff it is a finite direct sum of rings with T-nilpotent annihilators.

Proof. By 3.7 and Corollary 1.4 [1] .
A ring R is said to be k-stable if every hereditary

radical for R-*mod* is stable.

3.10. Proposition. Any $\mathfrak{h}$-stable ring is 8-costable.

Proof. Let R be an $\mathfrak{h}$-stable ring. In view of 3.5 it is enough to show that every two-sided ideal in R is a ring direct summand. For, let I be such an ideal. If $(S, T)(T, F)$ is the corresponding TTF-theory then (T, F) is stable, and hence (S, T) is cosplitting. Therefore, by the hypothesis, (S, T) is stable and we may use 2.6.

3.11. Proposition. Any prime $\mathfrak{h}$-stable ring is an R_{γ}-ring.

Proof. Let κ be a hereditary radical for R-*mod* and let $I = \kappa(R) \neq R$. Then I cannot be essential in R, and consequently $I \cap K = 0$ for some non-zero left ideal K. Since R is a prime ring and I is two-sided, $I = 0$. Recall that an exact sequence $0 \to A \to B \to C \to 0$ is said to be rational if $\mathrm{Hom}(D, B) = 0$ for any submodule D of C.

3.12. Proposition. The following conditions are equivalent for any ring R:

(i) If $0 \to A \to B \to C \to 0$ is a rational exact sequence then $C = 0$.

(ii) Any hereditary radical is cohereditary.

(iii) Any hereditary radical is centrally splitting.

(iv) R is right perfect and left 7-costable.

(v) R is a finite direct sum of full matrix rings over local right perfect rings.

(vi) R is right perfect and left $\mathfrak{h}$-stable.

Proof. (i) implies (ii) trivially.
(ii) implies (iii). Let κ be a hereditary radical. The corresponding torsion theory (T_κ, F_κ) is cosplitting, and hence there is a class S of modules such that $(T_\kappa, F_\kappa)(F_\kappa, S)$ is a TTF-theory. Thus (F_κ, S) is stable and cosplitting and by the proof of 2.6 it is a centrally splitting torsion theory.
(iii) implies (v). Since every hereditary radical for R-*mod* is centrally splitting, R is 7-costable. We can suppose, without loss of generality, that R is directly indecomposable. Then R possesses only trivial hereditary radicals, and consequently R is isomorphic to a full matrix ring over a local right perfect ring (see e.g. Proposition 4, paragraph 3.7 [3]).
(iv) implies (v). We can assume that R is a right perfect R_γ -ring. Then R has only trivial hereditary radicals - see [1].
(v) implies (iii), (iv) and (vi). It is an easy exercise.
(iii) implies (i) trivially.
(vi) implies (iii). Let (T, F) be a hereditary torsion theory. Then, as it is proved in [2], the class T is closed under direct products, and consequently an application of 3.10 yields the result.

Let us note here that the equivalence of (i) and (v) was already proved before by R. Courter - [4] .

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