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Josef Král

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A NOTE ON THE ROBIN PROBLEM IN POTENTIAL THEORY

Josef KRÁL, Praha

(Preliminary communication)

Abstract: The third boundary value problem in potential theory with a weak characterization of the boundary condition is investigated for a general open set $G \subset R^m$ with a compact boundary B. No a priori restrictions on G (like finite connectivity) and B (like smoothness) are imposed.

Key words: Robin problem, third boundary value problem, Laplace equation, Newtonian potential, Riesz-Schauder theory.

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Let G be an arbitrary open set in R^m, $m > 2$, and suppose that its boundary $B = \bar{G} \setminus G$ is compact. Let us denote by $\mathcal{B}$ the Banach space of all signed Borel measures with support in B (the norm $\| \dots \|$ in $\mathcal{B}$ being given by the total variation). Given $\mu \in \mathcal{B}$ then $\mathcal{U}\mu$ will denote the Newtonian potential of μ corresponding to the kernel $h(x) = |x|^{2-m} / m-2$. Let $\lambda \in \mathcal{B}$ be a fixed measure (≥ 0) with a finite continuous $\mathcal{U}\lambda$ and associate with any $\mu \in \mathcal{B}$ the distribution T defined over the class $\mathcal{D}$ of all infinitely differentiable functions φ with compact support in R^m by

$$\langle \varphi, T\mu \rangle = \int_G \operatorname{grad} \varphi(x) \cdot \operatorname{grad} \mathcal{U}\mu(x) dx + \int_B \varphi \mathcal{U}\mu \, d\lambda \quad .$$

If B is a smooth hypersurface with the exterior normal n and σ denotes the area measure then, under appropriate assumptions on μ and λ, $T\mu$ represents a weak characterization of $\frac{\partial \mathcal{U}\mu}{\partial n} + \frac{d\lambda}{d\sigma} \mathcal{U}\mu$. This fact gives a motive for the following formulation of the Robin problem (= third boundary value problem) for the Laplace equation (cf.[5],[8]):

Given $\nu \in \mathcal{L}$, determine a $\mu \in \mathcal{L}$ such that

(1) $$T\mu = \nu$$

in the sense of distribution theory. (For $\lambda \equiv 0$ this reduces to the Neumann problem as treated in [1],[2] .) Properties of the operator $T: \mu \longmapsto T\mu$ were investigated by I. Netuka (cf. [5],[6]) who obtained (without the simplifying assumption on continuity of $\mathcal{U}\lambda$) necessary and sufficient conditions for applicability of the Riesz-Schauder theory to the equation (1).

In order to describe the relevant results we first recall the following notation introduced in [2] - [4] . Given $\theta \in \Gamma = \{\theta \in R^m; |\theta| = 1\}$, $x \in R^m$ and $r > 0$ let $n_r^G(\theta, x)$ denote the number $(0 \leq n_r^G(\theta, x) \leq +\infty)$ of all points $y \in S = \{x + \varrho\theta ; 0 < \varrho < r\}$ such that every neighborhood of y meets both $S \cap G$ and $S \setminus G$ in a set of positive linear measure. Then the integral

$$v_r^G(x) = \int_\Gamma n_r^G(\theta, x)\, d\sigma(\theta)$$

is meaningful (more precisely: $\theta \mapsto n_r^G(\theta, x)$ is a Baire function whenever G is a Borel set) and we put for $M \subset B$

$$V_0^G(M) = \lim_{r \to 0+} \sup_{x \in M} v_r^G(x) .$$

It appears that

(2) $$V_0^G(B) < +\infty$$

is a necessary and sufficient condition for validity of the inclusion $T\mathcal{B} \subset \mathcal{B}$. In what follows we always assume (2) which guarantees the existence of the density

$$d(x) = \lim_{r \to 0+} \frac{\text{volume}\,\{y \in G;\ |y - x| < r\}}{\text{volume}\,\{y \in R^m;\ |y - x| < r\}}$$

at any $x \in B$. Put $A = \int_\Gamma d\sigma$, $B_k = \{x \in B; d(x) = 2^{-k}\}, k = 0,1$.
It follows from the results of I. Netuka (cf. [6]) that

(3) $$V_0^G(B_k) < 2^{-k} A, \quad k = 0, 1$$

is a necessary and sufficient condition for the existence of continuous functions f_i on B and signed measures $\nu_i \in \mathcal{B}$ $(i = 1, \dots, m)$ such that, for suitable $\alpha \in R^1 \setminus \{0\}$,

$$\| T - \alpha I - \sum_{i=1}^{m} \langle f_i, \cdot \rangle \nu_i \| < |\alpha| ,$$

where I stands for the identity operator on $\mathcal{B}$. Accordingly, under the assumption (3) the Riesz-Schauder theory applies to the equation (1) rewritten in the form $[I + \alpha^{-1}(T - \alpha I)]\mu = \alpha^{-1}\nu$.

Our main objective in this note is to describe the range of T under the conditions (2),(3) solely (which was done in [7] for a connected G) without à priori assumptions concerning connectivity or finite connectivity of G . (It should be noted here that G may have infinitely many components even if (2),(3) hold.)

Theorem. If G fulfils (2),(3), then $T\mathcal{B}$ consists precisely of those $\nu \in \mathcal{B}$ such that $\nu(K \cap B) = 0$ for every bounded component K of $\bar{G}$ satisfying $\lambda(K \cap B) = 0$.

The proof of this theorem rests on the following

Proposition. Let C be a Borel set with a compact boundary Q and suppose that every open $U \subset R^m$ with $U \cap Q \neq \emptyset$ meets both C and $R^m \setminus C$ in a set of positive volume. If $V_0^C(Q) < \frac{1}{2} A$, then C has only a finite number of components and their closures are mutually disjoint.

A detailed proof of this result will be presented in a paper to be published in Czech.Math.Journal where further comments and references will be given.

R e f e r e n c e s

[1] Ju.D. BURAGO - V.G. MAZ'JA: Nekotorye voprosy teorii potenciala i teorii funkcij dlja oblastej s nereguljarnymi granicami, Zapiski nauč.sem. LOMI AN 3(1967).

[2] J. KRÁL: The Fredholm method in potential theory, Trans.Amer.Math.Soc.125(1966),511-547.

[3] J. KRÁL: Limits of double layer potentials, Atti Accad. Naz.Lincei,Rend.Cl.Sc.Fis.,mat.e natur.48 (1970),39-42.

[4] J. KRÁL: Flows of heat and the Fourier problem, Czech. Math.J.20(1970),556-598.

[5] I. NETUKA: Generalized Robin problem in potential theory, Czech.Math.J.22(1972),312-324,

[6] I. NETUKA: An operator connected with the third boundary value problem in potential theory, ibid. 462-489.

[7] I. NETUKA: The third boundary value problem in potential theory, ibid. 554-580.

[8] V.D. SAPOŽNIKOVA: Rešenie tret´ej kraevoj zadači metodom teorii potenciala dlja oblastej s neregljarnymi granicami. Sb."Problemy mat.analiza, Kraevye zadači, Integral´nye uravnenija", Izdat.Leningrad.Un-ta,Leningrad 1966,35-44.

Matematický ústav ČSAV

Žitná 25

11567 Praha 1

Československo

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