Commentationes Mathematicae Universitatis Carolinae

Pavel Doktor

On the density of smooth functions in certain subspaces of Sobolev space

Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 4, 609--622

Persistent URL: http://dml.cz/dmlcz/105513

Commentationes Mathematicae Universitatis Carolinae

14,4 (1973)

ON THE DENSITY OF SMOOTH FUNCTIONS IN CERTAIN SUBSPACES OF SOBOLEV SPACE

P. DOKTOR, Praha

Abstract: In the present paper, some results concerning density of smooth functions in certain classes of functions $f \in W_2^{(1)}(\Omega)$ are established. As a consequence, some conditions are given under which the class of spaces $W_t \subset W_2^{(1)}$ tends to some limit space W_0 .

Key words: Sobolev spaces, density of smooth functions.

AMS: 46E35 Ref. Ž. 7.972.272

---

§ 1. Introduction, notations. In this paper, we use the notation of the book [1] . Let $\Omega \subset E_N$ $(N \geq 2)$ be a bounded domain in the Euclidean space E_N . We say that Ω has a lipschitzian boundary $\partial \Omega$ (or $\Omega \in \mathfrak{N}^{(0),1}$) iff the boundary of Ω is locally representable as a graph of a lipschitzian function which divides a sufficiently small neighbourhood of the point in question into two parts belonging to the interior and exterior of Ω , respectively. (For details, see [1, p.15] .)

Let $\Gamma \subset \partial \Omega$ be a relatively open set (i.e. open in the metric space $\partial \Omega$). We say that Γ has a lipschitzian relative boundary $\partial^* \Omega$ (i.e. the boundary in the metric space $\partial\Omega$) iff it has the following property:

Let x_0 be an arbitrary point of $\partial^*\Omega$ and let $\mathcal{U}(x_0)$ be a neighbourhood of x_0 such that $\mathcal{U}(x_0)\cap\cap\,\partial\Omega$ is expressed as a graph: $x_N = a(x_1,\dots,x_{N-1})$. Let further G be the image of $\Gamma\cap\mathcal{U}(x_0)$ in the projection on the hyperplane $x_1, x_2,\dots, x_{N-1}$ with the boundary ∂G . Then G has the same property as Ω , i.e. ∂G is locally representable as a graph of the lipschitzian function of $N-2$ variables (obviously this definition is independent on the description of $\partial\Omega$).

By $W_2^{(1)}(\Omega)$ we denote the Sobolev space of all square integrable functions u such that their first derivatives (in the sense of distributions)

$\dfrac{\partial u}{\partial x_1}$, $\dfrac{\partial u}{\partial x_2}$, ... $\dfrac{\partial u}{\partial x_N}$ are also L_2 -functions. Introduce the norm in $W_2^{(1)}(\Omega)$ by

$$(1)\qquad \|u\|_1 = \|u\|_{1,\Omega} = \{\|u\|_0^2 + \sum_{i=1}^{N}\|\frac{\partial u}{\partial x_i}\|_0^2\}^{\frac{1}{2}}$$

where

$$(2)\qquad \|u\|_0 = \{\int_\Omega |u|^2 dx\}^{\frac{1}{2}} .$$

By $W_2^{(\frac{1}{2})}(\Omega)$ we denote the space of all square integrable functions for which

$$(3)\qquad |||u|||_{\frac{1}{2}} = \{\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{N-1}}\,dx\,dy\}^{\frac{1}{2}} < \infty$$

with the norm

(4) $$\|u\|_{\frac{1}{2}} = \|u\|_{\frac{1}{2},\Omega} = \{\|u\|_0^2 + |||u|||_{\frac{1}{2}}^2\}^{\frac{1}{2}} .$$

Throughout the whole paper we suppose all functions to be defined on E_N and equal zero outside their natural domain of definition.

Let ω_h be a mollifier:

(5) $$\omega_h = \begin{cases} \varkappa h^{-N} \exp \dfrac{|x|^2}{|x|^2 - h^2} & |x| < h \\ 0 & |x| \geq h \end{cases}$$

$$\int_{E_N} \omega_1(x)dx = \int_{E_N} \omega_h(x)dx = 1 .$$

Then the convolution

(6) $$(\omega_h * u)(x) = \int_{E_N} \omega_h(x-y)u(y)dy = \int_{E_N} \omega_h(y)u(x-y)dy$$

is well-defined for $u \in L_2(\Omega)$ and its restriction onto $\bar{\Omega}$ belongs to the space $C^\infty(\bar{\Omega})$ i.e. the space of all infinitely differentiable functions continuous on $\bar{\Omega}$ together with their derivatives of arbitrary order. (For this and for other properties of ω_h see [1, p.58, p. 60].)

Let $x \in E_N$ be an arbitrary point, $x = (x_1, x_2, \dots x_{N-1}, x_N)$. We write for brevity $x = (x', x_N)$, where $x' = (x_1, \dots x_{N-1}) \in E_{N-1}$.

By $supp\, u$ we denote the closure of the set $\{x \in E_N \mid u(x) \neq 0\}$. Using this notation, we denote $\mathcal{D}(\Omega) = \{u \in C^\infty(\bar{\Omega}) \mid supp\, u \subset \Omega\}$.

§ 2. An auxiliary lemma.

Lemma 1. Let Ξ be N-dimensional parallelepiped

$$(7) \qquad \Xi = \{x \in E_N \mid x' \in \Delta,\ x_N \in (-\beta, 0)\} ,$$

where $\Delta = (-\alpha, \alpha)^{N-1}$ and α, β are positive numbers.

Let $G \subset \overline{G} \subset \Delta$ be a domain with a lipschitzian boundary; let G_m be a sequence of sets with the following property: for any open set $\mathcal{U} \subset \Delta$, $\overline{G} \subset \mathcal{U}$ there exists m_0 such that for $m > m_0$: $\overline{G}_m \subset \mathcal{U}$.

Let us denote $\Xi_1 = (-\alpha, \alpha)^{N-1} \times (-\beta, \beta)$. Further, let us denote $\Gamma = G \times \{0\}$ and let $K \subset \Xi_1$ be a compact set.

Then there exists a compact set $K_1 \in \Xi_1$ (which depends only on K) with the following property:

Let $u \in W_2^{(1)}(\Xi)$ be an arbitrary function which equals zero on Γ (in the sense of traces) and $\operatorname{supp} u \subset K$.

Then there exists a sequence $u_m, u_m \in C^\infty(\overline{\Xi_1})$ such that $\operatorname{supp} u \subset K_1 \setminus \overline{\Gamma}_m$, where $\overline{\Gamma}_m = G_m \times \{0\}$ and $u_m \longrightarrow u$ in the space $W_2^{(1)}(\Omega)$.

Proof: According to the assumptions $\overline{G} \subset \Delta$, $K \subset \Xi_1$ we have

$$(8) \quad \min(\operatorname{dist}(\overline{G}, E_{N-1} \setminus \Delta), \operatorname{dist}(K, E_N \setminus \Xi_1)) = \nu > 0 .$$

Denote successively

$$U_\lambda(G) = \{x' \in \Delta \mid \operatorname{dist}(x', G) < \lambda\} ,$$

$$(9) \qquad V_\lambda(G) = \{x' \in \Delta \mid \operatorname{dist}(x', G) < \tfrac{3}{4}\lambda\} ,$$

$$W_\lambda(G) = \{x' \in \Delta \mid dist(x', G) < \frac{1}{2}\lambda\} ,$$
$$Z_\lambda(G) = \{x' \in \Delta \mid dist(x', G) < \frac{1}{4}\lambda\} ,$$

and, correspondently,

$$U_\lambda(\Gamma) = U_\lambda(G) \times (-\lambda, \frac{1}{4}\lambda) ,$$
$$(10) \quad V_\lambda(\Gamma) = V_\lambda(G) \times (-\frac{3}{4}\lambda, \frac{1}{4}\lambda) ,$$
$$W_\lambda(\Gamma) = W_\lambda(G) \times (-\frac{1}{2}\lambda, \frac{1}{4}\lambda) ,$$
$$Z_\lambda(\Gamma) = Z_\lambda(G) \times (-\frac{1}{4}\lambda, \frac{1}{4}\lambda) ,$$

where λ is supposed to be sufficiently small:

$$(11) \qquad \lambda < \frac{1}{2}\nu \quad .$$

Let us put $h = \frac{1}{4}\lambda$ and

$$u_\lambda(x', x_N) = u(x', x_N - h) \qquad (x', x_N) \in E_N ,$$
$$(12) \quad v_\lambda(x) = \begin{cases} 0 & x \in W_\lambda(\Gamma) , \\ u_\lambda(x) & x \in \Xi \setminus \overline{W_\lambda(\Gamma)} , \end{cases}$$
$$w_\lambda(x) = (\omega_h * v_\lambda)(x) \quad .$$

We see immediately that $u_\lambda \in W_2^{(1)}(\Xi)$. It follows from (8),(11) that $supp\, w_\lambda \subset K_1$, where $K_1 = \{x \in E_N \mid dist(x,K) < \frac{\nu}{2}\} \subset \Xi_1$ depends only on K , and that $w_\lambda(x) = 0$ for $x \in Z_\lambda(\Gamma)$ so that there exists $m(\lambda)$ such that for any $m > m(\lambda)$: $(supp\, w_\lambda) \cap \cap \overline{\Gamma_m} = \emptyset$. In the following we show $\| u - w_\lambda \|_{1,\Xi} \longrightarrow 0$ for $\lambda \longrightarrow 0$ which proves the lemma.

Let us denote $\sigma = \Xi \setminus \overline{V_\lambda(\Gamma)}$, $P = V_\lambda(\Gamma) \cap \Xi_1$ and $\Xi_2 = \Delta \times (-\beta, h)$. The proof proceeds as follows: we write $\| w_\lambda - u \|_{1,\Xi} \le \| u - u_\lambda \|_{1,\Xi} + \| u_\lambda - w_\lambda \|_{1,\sigma} +$
$+ \| u_\lambda \|_{1,P} + \| w_\lambda \|_{1,P}$
and prove successively that all the right hand terms tend to zero. The main difficulty is to prove that $\| w_\lambda \|_{1,P} \to 0$ particularly to prove $\| \frac{\partial w_\lambda}{\partial x_1} \|_{0,P} \to 0$.

We obtain as an immediate consequence of the mean continuity of L_2-functions (see [1, p.57]) that
$\| u - u_\lambda \|_{1,\Xi} \longrightarrow 0$ and, because of the absolute continuity of integral, $\| u - u_\lambda \|_{1,\Xi} \to 0$. Further, obviously $v_\lambda \in W_2^{(1)}(\Xi_2 \setminus \overline{W_\lambda(\Gamma)})$ and so $x \in \sigma \Longrightarrow \frac{\partial}{\partial x_i}(\omega_h * v_\lambda) = \omega_h * \frac{\partial}{\partial x_i} v_\lambda$.
It follows from this implication that $\| u_\lambda - w_\lambda \|_{1,\sigma} \longrightarrow 0$ and hence our task is now to estimate the function w_λ as an element of $W_2^{(1)}(P)$. To this end, let us denote $Q = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}$. Because of the choice of $h^\bullet$ we obtain for $x \in P$

$$(13) \quad w_\lambda(x) = \int_{E_N} \omega_h(y-x) v_\lambda(y)\,dy = \int_Q \omega_h(y-x) u_\lambda(y)\,dy$$

and, similarly,

$$(14) \quad \begin{aligned} \psi_i(x) &= \frac{\partial}{\partial x_i} w_\lambda(x) = \int \frac{\partial}{\partial x_i} \omega_h(x-y) u_\lambda(y)\,dy = \\ &= -\int_Q \left(\frac{\partial}{\partial y_i} \omega_h(x-y)\right) u_\lambda(y)\,dy \ . \end{aligned}$$

Using the standard technique of mollifiers, see e.g.

[1, p.58] we obtain $\|w_\lambda - u_\lambda\|_{0,E_N} \to 0$ and hence $\|w_\lambda\|_{0,P} \to 0$ in virtue of the absolute continuity of the integral. Let us now consider the L_2 -norm of ψ_i.

1) Let $i = N$. Without loss of generality, we can suppose that O has a lipschitzian boundary (in the case of necessity, we can replace the boundary of $Z_\lambda(G)$ by the infinite differentiable hypersurface which uniformly approximates ∂G (see [2]) and construct the domains $\mathcal{U}_\lambda(G)$ etc. with respect to this regularization). In that case, we can use the Green formula (see [1, p.121]) and we obtain

$$\psi_N(x) = \int_Q \omega_h(x-y)\frac{\partial}{\partial y_N} u_\lambda(y)dy + \int_{U_\lambda(G)} \omega_h(x-(y',-\lambda))u_\lambda((y',-\lambda))dy' -$$
$$- \int_{W_\lambda(G)} \omega_h(x-(y',-\tfrac{1}{2}\lambda))u_\lambda((y',-\tfrac{1}{2}\lambda))dy' -$$
$$- \int_{U_\lambda(G)\setminus W_\lambda(G)} \omega_h(x-(y',\tfrac{1}{4}\lambda))u_\lambda((y',\tfrac{1}{4}\lambda))dy'.$$

The first integral tends to zero in $L_2(P)$ the second and the fourth equal zero for $x \in P$. Let us consider the third integral

$$\psi_{N,3}(x) = \int_{W_\lambda(G)} (x-(y',-\tfrac{1}{2}\lambda))u_\lambda((y',-\tfrac{1}{2}\lambda))dy' =$$
$$= \int_{W_\lambda(G)} \omega_h(x-(y',-\tfrac{1}{2}\lambda))u_\lambda((y',-\tfrac{3}{4}\lambda))dy'.$$

We obtain

$$|\psi_{N,3}(x)|^2 \le \Big(\int_{E_{N-1}} [\omega_h(x-(y',-\tfrac{1}{2}\lambda))]^2 dy'\Big)\Big(\int_{|y'-x'|<h} u^2((y',-\tfrac{3}{4}\lambda))dy'\Big) =$$

(15)

$$= \Big(\int_{E_{N-1}} \omega_h^2((x',x_N+\tfrac{1}{2}\lambda))dx'\Big)\Big(\int_{|x'-y'|<h} u^2((y',-\tfrac{3}{4}\lambda))dy',$$

and, integrating (15) over P we obtain

$$\|\psi_{N,3}\|^2_{0,P} \leq \left(\int_{E_N} \omega_h^2(x)\,dx\right)\left(\int_{V_\lambda(G)} dx' \int_{|x'-y'|<h} u^2((y', -\tfrac{3}{4}\lambda))\,dy'\right) .$$

We have

$$\int_{E_N} \omega_h^2(x)\,dx = c\,h^{-N} \quad . \tag{16}$$

The Fubini theorem yields

$$\int_{V_\lambda(G)} dx' \int_{|x'-y'|<h} u^2\,dy' \leq \int_{U_\lambda(G)} dy' \int_{|y'-x'|<h} u^2\,dx' =$$
$$= c_1 h^{N-1} \int_{U_\lambda(G)} u^2((y', -\tfrac{3}{4}\lambda))\,dy'$$

and thus we have

$$\|\psi_{N,3}\|^2_{0,P} \leq c_2 h^{-1} \int_{U_\lambda(G)} u^2((y', -\tfrac{3}{4}\lambda))\,dy' .$$

Let us consider the last integral. We obtain for $y' \in U_\lambda(G)$ (supposing at first u to be small enough and extending the result by continuity):

$$u((y', -\tfrac{3}{4}\lambda)) = u((y', 0)) + \int_0^{-\frac{3}{4}\lambda} u((y', \xi))\,d\xi ,$$

$$u^2((y', -\tfrac{3}{4}\lambda)) \leq 2\,[u^2((y', 0)) + \tfrac{3}{4}\lambda \int_{-\lambda}^{\frac{1}{4}\lambda} u^2((y', \xi))\,d\xi]$$

(Hölders inequality), and hence

$$h^{-1} \int_{U_\lambda(G)} u^2((y', -\tfrac{3}{4}\lambda))\,dy' \leq 2\,[h^{-1} \int_{U_\lambda(G)} u^2((y', 0))\,dy' +$$
$$+ 3 \int_{U_\lambda(\Gamma)} u^2((y', \xi))\,dy'\,d\xi) .$$

The absolute continuity of integral implies

$$\int_{U_\lambda(\Gamma)} u^2((y', \xi))\,dy'\,d\xi \longrightarrow 0 \qquad (\lambda \longrightarrow 0)$$

and further we have

$$\int_{U_\lambda(G)} u^2((y',0))\,dy' = \int_{U_\lambda(G)\setminus G} u^2((y',0))\,dy' \quad .$$

It follows from the imbedding theorems (see [3]) that $u((y',0)) \in W_2^{(\frac{1}{2})}(\Delta)$ and again from these theorems $u \in L_2(\partial G)$ in the sense of traces. Then the convergence

$$\int_{U_\lambda(G)\setminus G} u^2((y',0))\,dy' \longmapsto 0$$

follows from the properties of traces. Of course, locally we have

$$\int_R u^2((y',0))\,dy' \le \int_{R''} \Big(\int_{a(x'')}^{a(x'')+2\lambda} u^2((x'', x_{N-1}, 0))\,dx_{N-1} \Big)\,dx'' \ ,$$

where R is the intersection of $U_\lambda(G)\setminus G$ with some suitable neighbourhood $R_1 = \{(x'', x_{N-1}) \mid x'' \in R'', x_{N-1} \in (a(x'')-\gamma, a(x'')+\gamma)\}$ of any fixed point of ∂G , and a is the function which represents ∂G with respect to the local system of axes $(x'', x_{N-1}) = (x_1, \dots x_{N-2}, x_{N-1})$. The function

$$\Phi(\eta) = \int_{R''} u^2((x'', a(x'')+\eta, 0))\,dx''$$

is the continuous function of η (see [3]) and so making the change of variables $\eta = x_{N-1} - a(x'')$ and applying Fubini's theorem we obtain

$$\lim_{\lambda\to 0+} \frac{1}{h} \int_R u^2((y',0))\,dy' \le 8 \lim_{\lambda\to 0+} \frac{1}{2\lambda} \int_0^{2\lambda} \Phi(\eta)\,d\eta = 8\,\Phi(0) = 0 \ ,$$

q.e.d.

2) Let $i = N-1$. Similarly as above, by means of the Green formula we reduce our problem to the consideration of the surface integral

$$(17) \qquad I = h^{-1} \int_S u^2 d\mu \quad ,$$

where $S = \partial W_\lambda(G) \times (-\frac{\lambda}{2}, \frac{\lambda}{4})$. Passing to local systems of axes $(x_1, \dots x_{N-2}, x_{N-1}, y_N)$ $((x_1, \dots x_{N-2}, x_{N-1}) = (x'', x_{N-1})$ as above) we estimate the integral (17) by a sum of terms

$$(18) \quad h^{-1} \int_{-\frac{\lambda}{2}}^{\frac{1}{4}\lambda} dy_N \int_{R''} u^2((x'', a_s(x''), y_N)) \sqrt{1 + |\nabla a_s(x'')|^2}\, dx''$$

where a_s is the local representation of S . Easy calculation yields $|\nabla a_s|^2 < C$ where C depends only on the Lipschitz constant of the function a which locally represents ∂G . Using this fact we obtain as above

$$h^{-1} \int_S u^2 d\mu \longrightarrow 0 \Longleftrightarrow h^{-1} \int_{\partial G \times (-\frac{\lambda}{2}, \frac{\lambda}{4})} u^2 d\mu \rightarrow 0 \quad .$$

However, the last integral tends to zero thanks to the properties of traces (remember $u = 0$ on ∂G), q.e.d.

§ 3. Density theorems.

Theorem 1. Let $\Omega \subset E_N$ be a bounded set, $\Omega \in \mathfrak{N}^{(0),1}$. Let $\Gamma \subset \partial\Omega$ be a relatively open set with a lipschitzian relative boundary; let $\Gamma_m \subset \partial\Omega$ be a sequence of sets with the following property: for any neighbourhood $\mathcal{U}$ of Γ there exists m_0 such that for $m > m_0$ $\overline{\Gamma}_m \subset \mathcal{U}$.

Let $u \in W_2^{(1)}(\Omega)$ be a function which equals zero on Γ (in the sense of traces). Then there exists a sequence $u_m \in C^\infty(\overline{\Omega})$ such that

(i) $u_m = 0$ on the neighbourhood of Γ_m

and

(ii) $u_m \longrightarrow u$ in the space $W_2^{(1)}(\Omega)$.

Proof: The domain Ω has a lipschitzian boundary and hence for any $x \in \partial\Omega$ there exists a cartesian system $(x_1, \dots x_N) = (x', x_N)$ and a lipschitzian function a with the domain of definition $\Delta = (-\alpha, \alpha)^{N-1} \subset E_N$ such that:

(i) $U = \{(x', x_N) | x' \in \Delta, a(x') - \beta < x_N < a(x')\} \subset \Omega$

and

(ii) $V = \{(x', x_N) | x' \in \Delta, a(x') < x_N < a(x') + \beta\} \subset E_N \setminus \overline{\Omega}$

($\alpha > 0$, $\beta > 0$ are suitable constants).

Let us denote $Z = U \cup V \cup \{(x', x_N) | x' \in \Delta, x_N = a(x')\}$. Because of the compactness of $\partial\Omega$ we can cover $\partial\Omega$ by a finite number of such domains Z_κ, $\kappa = 1, 2, \dots m$. We can find a domain $Z_0 : \overline{Z}_0 \subset \Omega$ and $\overline{\Omega} \subset \bigcup_{\kappa=0}^{m} Z_\kappa$. Because of the compactness of $\overline{\Omega}$ we can construct a partition of unit to this covering, i.e. a system of functions $\varphi_\kappa \in \mathcal{D}(Z_\kappa)$ ($\kappa = 0, 1, \dots m$), $0 \le \varphi_\kappa \le 1$, $\sum_{\kappa=0}^{m} \varphi_\kappa(x) = 1$ for $x \in \overline{\Omega}$.

We can now transform U_κ ($\kappa = 1, 2, \dots m$) to the parallelepiped $\Xi = (-\alpha, \alpha)^{N-1} \times (-\beta, \beta)$ by means of the lipschitzian mapping

$T_k: \xi_i = x_i \ (i = 1, 2, \dots N-1), \quad \xi_N = x_N - a_k(x')$.

This mapping transforms continuously $W_2^{(1)}(U_k)$ to $W_2^{(1)}(\Xi)$, see [1, p.66], and $\operatorname{supp} \varphi_k$ to a compact set $K_k \subset \Xi_1 = T_k(Z_k)$. Let $K_{k,1} \subset \Xi_1$ be the compact set from Lemma 1, and let $G_k \subset \Delta$, $G_{k,m} \subset \Delta$ be the images of $T_k(\Gamma)$, $T_k(\Gamma_k)$, respectively, in the projection along x_N. Obviously we can find domains G'_k, $G'_{k,m}$: $\overline{G'_k} \subset \Delta$, $\overline{G'_{k,m}} \subset \Delta$, $G'_k \cap K_1 = G_k \cap K_1$, $G'_{k,m} \cap K_1 = G_{k,m} \cap K_1$ and G'_k, $G'_{k,m}$ satisfy the assumptions of Lemma 1. Hence we can approach $T_k(\varphi_k u)$ by the sequence $v_{m,k} \in W_2^{(1)}(\Xi)$, $\operatorname{supp} v_{m,k} \subset K_{1,k}$. The functions $\tilde{u}_{m,k} = T_k^{-1}(v_{m,k})$ belong to $W_2^{(1)}(C)$ ($C \subset E_N$ is an N-dimensional cube which contains $\overline{\Omega}$), $\tilde{u}_{m,k} = 0$ in a neighbourhood of $\overline{\Gamma}_m$ and $\tilde{u}_{m,k} = u\varphi_k$ in $W_2^{(1)}(\Omega)$. Applying the mollifier ω_h we can replace $\tilde{u}_{m,k}$ by $u_{m,k} \subset C^\infty(\overline{\Omega})$ with the same properties. Finally, we approach $u\varphi_0$ by the sequence $u_{m,0} \in \mathcal{D}(\Omega)$ and write $u_m = \sum_{k=0}^{m} u_{m,k}$, which proves the theorem.

__Remark__. Higher smoothness of $\partial\Omega$ guarantees higher smoothness of the mappings T_k and hence gives analogous results for the space $W_2^{(k)}(\Omega)$, $k > 1$. For example, the following theorem holds:

__Theorem 2__. Let $\Omega \subset E_N$ be a bounded domain with an infinitely differentiable boundary; let $\Gamma \subset \partial\Omega$ be a relatively open set with an infinitely differentiable relative boundary and let $\Gamma_m \subset \partial\Omega$ be a sequence of sets such that the conditions of Theorem 1 are fulfilled.

Let $u \in W_2^{(k)}(\Omega)$ $(k \geq 2)$ be a function which equals zero on Γ (in the sense of traces) and

$$\frac{\partial^i u}{\partial \nu^i} = 0 \qquad \text{(the normal derivative) for } i = 1, 2, \dots k-1.$$

Then there exists a sequence $u_n \in C^{(\infty)}(\bar{\Omega})$, $u_n = 0$ on a neighbourhood of Γ_n and u_n tends to u in the space $W_2^{(k)}(\Omega)$.

Proof is completely analogous to that of Theorem 1.

§ 4. Relations to the convergence of spaces.

Certain assumptions are introduced in [1] (see p. 169 and following) under which the weak solution of the linear boundary value problem $Au = f$, $u - u_0 \in V_n \subset W_p^{(k)}(\Omega)$ depends continuously on V_n. One of these conditions is that $V_n \longrightarrow V$ in the following sense:

(i) $$\forall u \in V \exists u_n \in V_n : u_n \longrightarrow u$$

and

(ii) $$V = \bigcap_{n=1}^{\infty} \overline{[\bigcup_{i=n}^{\infty} V_i]}$$

(by $[M]$ we denote the minimal linear space which contains M).

Let $k = 1$, $V_n = \{u \in W_2^{(1)}(\Omega) \mid u = 0 \text{ on } \Gamma_n\}$,

$$V = \{u \in W_2^{(1)}(\Omega) \mid u = 0 \text{ on } \Gamma\}$$

($\Gamma_n, \Gamma \subset \partial\Omega$ are relatively open sets). Theorem 1 gives conditions on Γ_n, Γ under which (i) holds. Condi-

tion (ii) is satisfied if in addition $\Gamma = \lim \Gamma_n$, i.e.
$\Gamma = \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} \Gamma_n = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \Gamma_n$.

The proof is the same as in [1, p.173] (see example 6.4).

References

[1] NEČAS Jindřich: Les méthodes directes en théorie des équations elliptiques, Praha-Paris 1967.

[2] NEČAS Jindřich: Sur les domaines du type $\mathfrak{N}$. Czech. Mat.Journ. 12(1962),274-287.

[3] BESOV O.V., ILJIN V.P., KUDRJAVCEC L.D., LIZORKIN G.I., NIKOLSKIJ S.M.: Teorija vloženij klassov differenciruemych funkcij mnogich peremennych, Differencial'nye uravnenija s častnymi proizvodnymi (Trudy simpoziuma, posvjaščennogo 60-letiju akademika S.L. Soboleva), Moskva 1967, 38-70.

Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, Praha 8
Československo

(Oblatum 9.8.1973)