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FACTORIZATION AND NON-ALGEBRAIC CATEGORIES

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Abstract: The current paper studies the relation between the algebraicity of a category and its factorcategories. We show that a factorcategory of an algebraic category need not be algebraic, and we bring some sufficient conditions for a category, not to be a factorization of an algebraic one.

Key-words: algebraic category, left adequate subcategory, factorization, retraction, congruence.

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The notion of an algebraic category was introduced by J.R. Isbell. It is a category which can be fully embedded in some category of algebras. Algebraic categories were studied intensively (see [3],[4],[5] and other papers by Z. Hedrlín, L. Kučera, A. Pultr and V. Trnková). It was later shown that under the set-theoretical assumption $\mathcal{M}$ assuming that there is not "too many" measurable cardinals, every concrete category is algebraic. Furthermore, the assumption $\mathcal{M}$ was shown to be critical ([5]). The current paper studies the relation between the algebraicity of a category and its factorcategories. We show that a factorcategory of an algebraic category need not be algebraic, and we bring some sufficient conditions for a category, not to be a factorization of an algebraic one.

We emphasize here that a factorization of a category is for us any functor from this category onto another one which is one-to-one on the class of objects. The category dual to the category of all sets and mappings ($\mathcal{Set}^*$), and several other common categories (e.g. that of complete boolean algebras ($\mathcal{B}$) , compact topological spaces ($\mathcal{Comp}$) , and the dual to the category of abelian groups ($\mathcal{Ab}^*$) fail to be algebraic under $\mathit{mon}\ \mathbb{M}$, as it was proved in [5].
It is to be noted that the non-algebraicity of $\mathcal{Set}^*$ is essential there; the non-algebraicity of the others follows from this fact. Throughout this paper we shall use the word "algebraic" exclusively in the sense "algebraic under $\mathit{mon}\ \mathbb{M}$ ", similarly "non-algebraic".
We obtain that e.g. $\mathcal{Set}^*$, $\mathcal{B}$, $\mathcal{Comp}$, and $\mathcal{Ab}^*$ fail to be factorizations of algebraic categories. It is proved in [6] that every category is a factorization of a concrete one. This result, then, cannot be strentghened to the assertion that every category is a factorization of a concrete algebraic one. A generalization of the main theorem from [5] and our above mentioned conditions imply the non-algebraicity of other categories, the non-algebraicity of which could not be proved by the methods introduced in [5].

The present paper has four sections. In the first one we introduce conventions and recall some definitions and theorems which will be of use later. In the second one the notion of an $\mathbb{P}$-measure and the main theorem from [5] are generalized. Further, there are conditions sufficient for a category, not to be a factorization of an algebraic one, here. Some corollaries are given. In the third section we study the category of relations. It is proved here that it has no small left adequate subcategory, and it fails to be algebraic. In the fourth section there is an example of factorization of an algebraic category onto a non-algebraic one and further similar examples.

I am much indebted to A. Pultr, who introduced me to the topics of factorizations, and helped me throughout the whole work on this paper. I had also a valuable conversation on these questions with Z. Hedrlín.

I

I.1. We work in the Gödel-Bernays set-theory sometimes assuming the axiom $non\ \mathfrak{M}$ (see I.18). The class of all cardinal numbers will be denoted by Cn, that of all ordinal numbers by On, and the cardinality of a set X by $|X|$.

I.2. We use the language of the theory of categories which is in current use. The class of objects of a category $\mathfrak{K}$ will be denoted by $\mathfrak{K}^o$, the class of morphisms by $\mathfrak{K}^m$. If $X, Y \in \mathfrak{K}^o$, then $\mathfrak{K}(X,Y)$ denotes the set of morphisms of $\mathfrak{K}$ from X to Y. By $\mathfrak{K}^*$ we denote the category dual to $\mathfrak{K}$ identifying $\mathfrak{K}^{*o}$ with $\mathfrak{K}^o$ and $\mathfrak{K}^*(X,Y)$ with $\mathfrak{K}(X,Y)$ for every $X, Y \in \mathfrak{K}^o$. The composition is written from left to right, i.e. the composition of $f: X \longrightarrow Y$ with $g: Y \longrightarrow Z$ is fg. Subcategory is denoted by $\subseteq$. If $\mathfrak{A} \subseteq \mathfrak{K}$, $F: \mathfrak{K} \longrightarrow \mathfrak{L}$ is a functor, then its restriction to $\mathfrak{A}$ is denoted by $F/\mathfrak{A}$. Shirking of $\mathfrak{K}$ (notation $Sh(\mathfrak{K})$) is such a thin category that $Sh(\mathfrak{K})^0 = \mathfrak{K}^0$, and $Sh(\mathfrak{K})(X,Y) \neq 0$ iff $\mathfrak{K}(X,Y) \neq 0$, for every $X, Y \in \mathfrak{K}^0$. A functor $F: \mathfrak{K} \longrightarrow \mathfrak{L}$ is a retraction if there is a functor $J: \mathfrak{L} \to \longrightarrow \mathfrak{K}$ with $JF = 1_{\mathfrak{L}}$.

I.3. The category of sets and mappings is denoted by $\mathcal{Set}$. Let us introduce notation for some mappings:

$i_j : 1 = \{0\} \longrightarrow 2 = \{0,1\}$ is defined by $0 i_j = j$ $(j = 0,1)$

$k_x : 1 \longrightarrow X$ denotes the constant mapping to x $(x \in X,\ X \in \mathcal{Set}^0)$.

$\chi_A : X \longrightarrow 2$ is the characteristic function of A $(A \subseteq X,\ X \in \mathcal{Set}^0)$.

For $x \in X$ we write χ_x instead of $\chi_{\{x\}}$. By the same symbols we denote the corresponding morphisms in the category $\mathcal{Set}^*$.

I.4. __Definition__. Let $\Phi : \mathfrak{K} \longrightarrow \mathfrak{L}$ be a functor. Φ is said to be a factorization if Φ is onto $\mathfrak{L}$, and one-to-one on the class of objects of $\mathfrak{K}$.

I.5. __Definition__. Given a category $\mathfrak{K}$ and an equivalence $\sim_{X,Y}$ on $\mathfrak{K}(X,Y)$ for every $X, Y \in \mathfrak{K}^0$, we call the collection $\{\sim_{X,Y}, X, Y \in \mathfrak{K}^0\}$ a congruence on $\mathfrak{K}$ if $(\forall X,Y,U,V \in \mathfrak{K}^0)(\forall f,g \in \mathfrak{K}(X,Y))(\forall k \in \mathfrak{K}(V,X))(\forall h \in \mathfrak{K}(Y,U))(f \sim_{X,Y} g \Longrightarrow fh \sim_{X,U} gh \ \&\ kf \sim_{V,Y} kg)$.

Notation: We shall write only $\sim$ instead of

$\{\sim_{X,Y}, X, Y \in \mathfrak{K}^o\}$, and $\mathfrak{K}/\sim$ will denote such a category that $(\mathfrak{K}/\sim)^o = \mathfrak{K}^o$ and $\mathfrak{K}/\sim(X,Y) = \mathfrak{K}(X,Y)/\sim_{X,Y}$ for every $X, Y \in \mathfrak{K}^o$.

I.6. Remark. Apparently every congruence on a category $\mathfrak{K}$ determines a factorization $\mathfrak{K} \longrightarrow \mathfrak{K}/\sim$, and conversely we can assign to every factorization Φ a congruence $\sim$ such that $f \sim g$ iff $f\Phi = g\Phi$. Throughout this paper we shall interchange the notion of factorization and that of congruence.

I.7. Definition. A collection of ordinals $\Delta = \{\delta_i, i \in I\}$, where I is a set, is said to be a type.

I.8. Definition. Let $\Delta = \{\delta_i, i \in I\}$ be a type. Denote by $R(\Delta)$ the category the objects of which are all pairs $\langle X, \{R_i, i \in I\}\rangle$ ($X \in Set^o$, $R_i \subseteq X^{\delta_i}$ for every $i \in I$) and morphisms from $\langle X, \{R_i, i \in I\}\rangle$ to $\langle Y, \{S_i, i \in I\}\rangle$ are all mappings $f : X \longrightarrow Y$ with $R_i f^{\delta_i} \subseteq S_i$ for every $i \in I$. Denote by $\mathcal{G} = R(\{2\})$ the category of graphs.

I.9. Definition. Let $\Delta = \{\delta_i, i \in I\}$ be a type. The category of algebras of the type Δ is a category $A(\Delta)$, the objects of which are all pairs $\langle X, \{w_i, i \in I\}\rangle$ where X is a set and $w_i : X^{\delta_i} \longrightarrow X$ are mappings. Morphisms from $\langle X, \{w_i, i \in I\}\rangle$ to $\langle Y, \{v_i, i \in I\}\rangle$ are all mappings $f : X \longrightarrow Y$ with $w_i f = f^{\delta_i} v_i$ for every $i \in I$.

1.10. Definition. A category $\mathfrak{K}$ is said to be algebraic if there is a full embedding $J : \mathfrak{K} \longrightarrow \mathcal{G}$.

<u>Remark</u>. The original definition of an algebraic category (Isbell) is the following: $\mathfrak{K}$ is algebraic if there is a type Δ and a full embedding $J : \mathfrak{K} \longrightarrow A(\Delta)$ (hence the word "algebraic").

It was proved in [3] that both definitions are equivalent.

I.11. <u>Notation</u>. Let $\mathfrak{K}$ be a category, $\{A_i, i \in I\}$ a collection of its objects, where I is a set. Then by $\mathfrak{K}(\{A_i, i \in I\})$ we denote a category, the objects of which are all pairs $\langle X, \{R_i, i \in I\}\rangle$ ($X \in \mathfrak{K}^0, R_i \subseteq \mathfrak{K}(A_i, X)$ for every $i \in I$). Morphisms from $\langle X, \{R_i, i \in I\}\rangle$ to $\langle Y, \{S_i, i \in I\}\rangle$ are such morphisms $f \in \mathfrak{K}(X, Y)$ that $gf \in S_i$ for every $f \in R_i$ and $i \in I$.

I.12. <u>Theorem</u>. If $\mathfrak{K}$ is algebraic then every $\mathfrak{K}(\{A_i, i \in I\})$ is algebraic, too.

<u>Proof</u>: See [5].

I.13. <u>Definition</u>. Let $\mathfrak{K}$ be a category, $\mathfrak{A} \subseteq \mathfrak{K}$. $\mathfrak{A}$ is said to be a left adequate of $\mathfrak{K}$, if the only transformations from $\mathfrak{K}(-, X)/(\mathfrak{A} \longrightarrow \mathfrak{K}(-, Y))/\mathfrak{A}$ are τ_f with $f \in \mathfrak{K}(X, Y)$ ($g\tau_f^A = gf$ for every $g \in \mathfrak{K}(A, X)$ and arbitrary $A \in \mathfrak{A}^0$), and if distinct f and g induce distinct τ_f and τ_g . The notion of a right adequate subcategory is defined dually. If $\mathfrak{A}$ is both a left and a right adequate of $\mathfrak{K}$, then it is said to be an adequate of $\mathfrak{K}$. If there is a small category $\mathfrak{A} \subseteq \mathfrak{K}$ such that $\mathfrak{A}$ is a left adequate of $\mathfrak{K}$, then $\mathfrak{K}$ is said to have a small left adequate subcategory.

I.14. Theorem. Let $\mathfrak{K}$ have a small left adequate subcategory. Then $\mathfrak{K}$ is algebraic.

Proof: See [2].

I.15. Theorem. Let $\mathfrak{A}$, $\mathfrak{L}$, $\mathfrak{K}$ be categories, and $\mathfrak{A} \subseteq \subseteq \mathfrak{L} \subseteq \mathfrak{K}$. Let $\mathfrak{L}$ be an adequate of $\mathfrak{K}$, and $\mathfrak{A}$ a left adequate of $\mathfrak{L}$. Then $\mathfrak{A}$ is a left adequate of $\mathfrak{K}$.

Proof: See [1].

I.16. Notation. Denote by P^- a functor from Set^* to Set defined by $XP^- = \exp X$ for $X \in Set^{*0}$, and $(A)fP^- = \{x \in X, xf \in A\}$ for $f \in Set^*(Y,X)$, $A \subseteq Y$.

I.17. Definition. Let $\alpha \in Cn$. A mapping $\mu : XP^- \longrightarrow 1P^-$ is said to be an α-additive measure, if for every set A with $|A| \leq \alpha$ and for every mapping $f : A \longrightarrow XP^-$ there is $x \in X$ with $f\mu = f k_x P^-$. If $\mu = k_\xi P^-$ for some $\xi \in X$, then it is said to be trivial. If contrary, non-trivial.

Remark. A classical definition of an α-additive measure is the following: A mapping μ from $\exp X$ to $\{0,1\}$ is an α-additive measure if for every collection $\{A_i, i \in I\}$ of disjoint subsets of X with $|I| \leq \alpha$ we have $(\bigcup_{i \in I} A_i)\mu = \sum_{i \in I} A_i \mu$. The equivalence with our definition is obvious (see [5]).

I.21. Axiom $mon\ M$: For every $\alpha \in Cn$ there exists a non-trivial α-additive measure $\mu : XP^- \longrightarrow 1P^-$.

II

II.1. Definition. Denote by $\mathcal{S}$ the following subcategory of $\mathcal{S}et^*$:

$\mathcal{S}^0 = \mathcal{S}et^{*0} - \{0\}$,

$\mathcal{S}(2,1) = \{i_0, i_1\}$, $\mathcal{S}(1,2) = 0$,

$\mathcal{S}(X,1) = \{k, x \in X\}$, $\mathcal{S}(1,X) = 0$ for $X \neq 1,2$,

$\mathcal{S}(2,X) = \{\chi, x \in X\}$, $\mathcal{S}(X,2) = 0$ for $X \neq 1,2$,

$\mathcal{S}(X,Y) = 0$ if $X \neq Y$ and $X, Y \neq 1, 2$,

$\mathcal{S}(X,X) = \{1_X\}$ for every $X \in \mathcal{S}^0$.

For $X \in \mathcal{S}^0 - \{1,2\}$ denote by $\mathcal{S}_X$ the full subcategory of $\mathcal{S}$ generated by the objects $1, 2, X$.

<u>Remark</u>. $\mathcal{S}$ is isomorphic to $\mathcal{S}^*$, and $\mathcal{S}$ has no small left adequate subcategory.

II.2. <u>Definition</u>. Let $\mathcal{U}, V$ be sets, $\mathcal{M} = \{f_x : \mathcal{U} \longrightarrow V$, $x \in X\}$ be a collection of mappings, $\alpha \in Cn$. A mapping $\mu : \mathcal{U} \longrightarrow V$ is called an α-additive $\mathcal{M}$-measure if for every mapping $f: A \longrightarrow \mathcal{U}$ with $|A| \leq \alpha$ there is some $x \in X$ with $f\mu = ff_x$.

An $\mathcal{M}$-measure is said trivial if there is some $x_0 \in X$ with $\mu = f_{x_0}$. Otherwise it is called non-trivial.

II.3. <u>Lemma</u>. Let $\mathcal{U}, V$ be sets, $\alpha \in Cn$, $\alpha \geq \max(\aleph_0, |V|)$. Let $\mu : XP^- \longrightarrow 1P^-$ be an α-additive measure and $\mathcal{M} = \{f_x : \mathcal{U} \longrightarrow V, x \in X\}$ be a collection of mappings. For $u \in \mathcal{U}$, $v \in V$ put $\mathcal{M}(u,v) = \{x \in X, uf_x = v\}$. Then $\bar{\mu} : \mathcal{U} \longrightarrow V$, defined by $u\bar{\mu} = v$ iff $(\mathcal{M}(u,v))\mu = 1$, is an α-additive $\mathcal{M}$-measure.

<u>Proof</u>. For every $u \in \mathcal{U}$ we have $\bigvee_{v \in V} \mathcal{M}(u,v) = X$. Since $|V| \leq \alpha$, it follows that there is the unique $v \in V$

with $(\mathbb{M}(u,v))\mu = 1$. Thus $\bar{\mu}$ is a correctly defined mapping. To show that $\bar{\mu}$ is an α-additive $\mathbb{M}$-measure consider some mapping $f : A \longrightarrow \mathcal{U}$ with $|A| \leq \alpha$. Define $\bar{f} : A \times V \longrightarrow XP^-$ by $(\langle a, v\rangle)\bar{f} = \mathbb{M}(af, v)$ for every $a \in A$, $v \in V$. Since $|A \times V| \leq \alpha$, it follows that $\bar{f}\mu = f\mathfrak{K}_x P^-$ for some $x \in X$. As
$(\forall u \in \mathcal{U})(uf\bar{\mu} = v \equiv (\mathbb{M}(uf,v))\mu = 1 \equiv x \in \mathbb{M}(uf,v) \equiv uff_x = v)$,
we have $f\bar{\mu} = ff$ and thus $\bar{\mu}$ is an α-additive $\mathbb{M}$-measure.

<u>Definition</u>. $\bar{\mu}$ is said to be induced by μ.

The following generalizes a theorem from [5]; the proof is a simple modification of that given in [5].

II.4.Theorem (*mon* $\mathbb{M}$). Let $\mathfrak{K}$ be such a category that $\mathcal{S} \subseteq \mathfrak{K}$, and $\mathcal{S}(X,1) = \mathfrak{K}(X,1)$ for every $X \in \mathcal{S}^0 - \{1,2\}$. Then $\mathfrak{K}$ is not algebraic.

II.5. <u>Corollary</u> (*mon* $\mathbb{M}$). $\mathcal{S}et^*$ is not algebraic.

II.6. <u>Theorem</u> (*mon* $\mathbb{M}$). Let $\mathfrak{K}$ be a category, $\mathcal{S} \subseteq \mathfrak{K}$ and $\mathfrak{K}(X,1) = \mathcal{S}(X,1)$ for every $X \in \mathcal{S}^0 - \{1,2\}$. Then $\mathfrak{K}$ is not a factorization of any algebraic category.

<u>Proof</u>. Let $\mathcal{O}\!\mathcal{L}$ be algebraic, $\Phi : \mathcal{O}\!\mathcal{L} \longrightarrow \mathfrak{K}$ be a factorization. Suppose that $\mathfrak{K}^0 = \mathcal{O}\!\mathcal{L}^0$ and $X\Phi = X$ for every $X \in \mathcal{O}\!\mathcal{L}^0$. Let $J : \mathcal{O}\!\mathcal{L} \longrightarrow \mathcal{G}$ be a full embedding, denote by $\square$ the natural forgetful functor $\square : \mathcal{G} \longrightarrow \mathcal{S}et$, and put $F = J\square : \mathcal{O}\!\mathcal{L} \longrightarrow \mathcal{S}et$. Put $\alpha = max(\aleph_0 |1F|, |2F|, |\mathcal{O}\!\mathcal{L}(2,1)|)$. Let $\mu : XP^- \longrightarrow 1P^-$ be a non-trivial α-additive measure. For every $x \in X$ choose arbitra-

rily some $f_x \in \mathcal{O}\!\mathcal{L}(X,1)$ with $f_x \Phi = k$. Let $M = \{f_x F : XF \longrightarrow 1F, x \in X\}$. Denote by $\overline{\mu}$ the α-additive M-measure induced by μ (see II.3). Since J is a full embedding, it follows from the additivity of $\overline{\mu}$ that there is some $\xi \in X$ and $q_\xi \in \mathcal{O}\!\mathcal{L}(X,1)$ with $q_\xi \Phi = k_\xi$ and $q_\xi F = \overline{\mu}$. Choose some $h_\xi \in \mathcal{O}\!\mathcal{L}(2,X)$ with $h_\xi \Phi = \chi_\xi$. Let us show that $h_\xi F \overline{\mu} = h_\xi F f_\xi F$: As $|2F| \leq \alpha$, there is some $f_x F \in M$ with $h_\xi F f_x F = h_\xi F \overline{\mu} = h_\xi F q_\xi F$, it follows that $h_\xi f_x = h_\xi q_\xi$, and $(h_\xi f)\Phi = \chi_\xi k = (h_\xi q_\xi)\Phi = i_1$. Therefore necessarily $x = \xi$, and $h_\xi F \overline{\mu} = h_\xi F f_\xi F$. Put $j_1 = h_\xi f_\xi$, and let $\{j_0^\gamma, \gamma \in \beta\}$ be the set of all Φ-preimages of i_0. Then we have $\beta \leq |\mathcal{O}\!\mathcal{L}(2,1)| \leq \alpha$. For every $\gamma \in \beta$ put $X_\gamma = \{x \in X, x \neq \xi, h_\xi f_x = j_0^\gamma\}$. Apparently $X - \{\xi\} = \bigvee_{\gamma \in \beta} X_\gamma$.

F is faithful, hence for every $\gamma \in \beta$ there is $b_\gamma \in 2F$ with $b_\gamma j_0^\gamma F \neq b_\gamma j_1 F = a_\gamma$. Then $M(b_\gamma h_\xi F, a_\gamma) \subseteq X - X_\gamma$ for every $\gamma \in \beta$, because $x \in M(b_\gamma h_\xi F, a_\gamma)$ iff $b_\gamma h_\xi F f_x F = a_\gamma$, i.e. $x \notin X_\gamma$. As $b_\gamma h_\xi F \overline{\mu} = b_\gamma h_\xi F f_\xi F = b_\gamma j_1 F = a_\gamma$, we have $(M(b_\gamma h_\xi F, a_\gamma))\mu = 1$. Therefore $(X_\gamma)\mu = 0$ for every $\gamma \in \beta$ and as $\beta \leq \alpha$ also $(\bigvee_{\gamma \in \beta} X_\gamma)\mu = 0$. Thus necessarily $(\{\xi\})\mu = 1$, which is a contradiction.

II.7. Corollary (*non* M). *Set** is not a factorization of any algebraic category.

II.8. Lemma. If $\mathfrak{K}$ is a factorization of an algebraic category, then every category $\mathfrak{K}(\{A_i, i \in I\})$ is also a factorization of an algebraic category.

Proof: Let $\Phi : \mathfrak{A} \longrightarrow \mathfrak{K}$ be a factorization, and let $\mathfrak{A}$ be algebraic. Suppose that $\mathfrak{A}^o = \mathfrak{K}^o$ and $X\Phi = X$ for every $X \in \mathfrak{A}^o$. For every object $\langle X, \{R_i, i \in I\}\rangle$ of $\mathfrak{K}(\{A_i, i \in I\})$ (see I.11) let $\langle X, \{\overline{R}_i, i \in I\}\rangle$ be an object of the category $\mathfrak{A}(\{A_i, i \in I\})$ such that $(\forall i \in I)(\forall g \in \mathfrak{A}(A_i, X))(g \in \overline{R}_i \equiv g\Phi \in R_i)$. Denote by $\mathfrak{A}'$ the full subcategory of $\mathfrak{A}(\{A_i, i \in I\})$ generated by these objects. We can suppose that if $\langle X, \{R_i, i \in I\}\rangle \neq \langle Y, \{S_i, i \in I\}\rangle$ then $\langle X, \{\overline{R}_i, i \in I\}\rangle \neq \langle Y, \{\overline{S}_i, i \in I\}\rangle$, for we can add more isomorphic copies of any object to the category $\mathfrak{A}'$. By I.12 $\mathfrak{A}'$ is algebraic. To define a factorization $\Phi' : \mathfrak{A}' \longrightarrow \mathfrak{K}(\{A_i, i \in I\})$, put $\langle X, \{\overline{R}_i, i \in I\}\rangle\Phi' = \langle X, \{\overline{R}_i, i \in I\}\rangle\Phi$, $f\Phi' = f\Phi$. It suffices to verify that $f\Phi \in \mathfrak{K}(\{A_i, i \in I\})^m$ for every $f \in \mathfrak{A}'^m$. Let $f : \langle X, \{\overline{R}_i, i \in I\}\rangle \to \langle Y, \{\overline{S}_i, i \in I\}\rangle$ be a morphism of $\mathfrak{A}'$, and let $g \in R_i$ for some $i \in I$. Let $h \in \mathfrak{A}(A_i, X)$ with $h\Phi = g$, i.e. $h \in \overline{R}_i$. Then $hf \in \overline{S}_i$ and $(hf)\Phi = g f \Phi \in S_i$. Therefore $f\Phi$ is a morphism of the category $\mathfrak{K}(\{A_i, i \in I\})$.

II.9. Corollary (*non* $\mathbb{M}$). The following categories are not factorizations of any algebraic category:
the category of complete boolean algebras ($\mathfrak{B}$),
the category of compact Hausdorff topological spaces (Comp),
the category dual to that of abelian groups ($\mathcal{Ab}^*$).

Proof: In [5] it is proved that Set^* can be fully embedded into $\mathfrak{B}$, $\mathcal{Ab}^*(\{X_i, i \in I\})$ and $\mathit{Comp}(\{Y_j, j \in J\})$ for suitable $\{X_i, i \in I\}$ and $\{Y_j, j \in J\}$.

The following two theorems will be needed in the following. The proof of the first one is a modification of the proof of the theorem from [5] quoted above. The proof of the second one is analogous to that of II.8.

II.10. Theorem (*mon* $\mathcal{M}$). Let $\mathfrak{K}$ be a category, $S \subseteq \mathcal{S}^o - \{1,2\}$ be a proper class. Let for every $X \in S$ be $J_X : \mathcal{S}_X \longrightarrow \mathfrak{K}$ an embedding with $(\mathcal{S}_X(X,1))J_X = \mathfrak{K}(XJ_X, 1J_X)$, and let $1J_X = 1J_{X'}$ for every $X, X' \in S$. Then $\mathfrak{K}$ is not algebraic.

II.11. Notation. Let $\mathfrak{A}$ be a category, $\mathcal{S}^o \subseteq \mathfrak{A}^o$. For every $X \in \mathcal{S}^o - \{1,2\}$ denote by $\mathfrak{A}_X$ the following subcategory of $\mathfrak{A}$:

$\mathfrak{A}_X^o = \{1,2,X\}$,

$\mathfrak{A}_X(2,1) = \mathfrak{A}(2,1)$, $\qquad \mathfrak{A}_X(1,2) = 0$,

$\mathfrak{A}_X(X,1) = \mathfrak{A}(X,1)$, $\qquad \mathfrak{A}_X(1,X) = 0$,

$\mathfrak{A}_X(2,X) = \mathfrak{A}(2,X)$, $\qquad \mathfrak{A}_X(X,2) = 0$,

$\mathfrak{A}_X(1,1) = \{1_1\}$, $\mathfrak{A}_X(2,2) = \{1_2\}$, $\mathfrak{A}_X(X,X) = \{1_X\}$.

II.12. Theorem (*mon* $\mathcal{M}$). Let $\mathfrak{A}$ be a category, $\mathcal{S}^o \subseteq \mathfrak{A}^o$. For every $X \in \mathcal{S}^o - \{1,2\}$ given a functor $\Phi_X : \mathfrak{A}_X \longrightarrow \mathcal{S}_X$ with $X\Phi_X = X, 2\Phi_X = 2$ and $1\Phi_X = 1$, such that all Φ_X are factorizations, then $\mathfrak{A}$ is not algebraic.

III

III.1. Definition. Denote by $\mathbf{Rel}$ the category of sets and relations, i.e. $\mathbf{Rel}^o = \mathbf{Set}^o - \{0\}$, and $\mathbf{Rel}(X,Y) = \{R, R \subseteq X \times Y\}$ for every $X, Y \in \mathbf{Rel}^o$. The composition is defined for $R \in \mathcal{Rel}(X,Y)$ and $T \in \mathcal{Rel}(Y,Z)$ by $RT = \{\langle x,z\rangle \in X \times Z, (\exists y \in Y)(\langle x,y\rangle \in R \;\&\; \langle y,z\rangle \in T)\}$.

Notation. Let $R \subseteq X \times Y$, $x \in X$, $y \in Y$. Put $xR = \{y \in Y, \langle x,y\rangle \in R\}$, $Ry = \{x \in X, \langle x,y\rangle \in R\}$.

III.2. Lemma. If $\mathcal{Rel}$ has a small left adequate subcategory, then it has a one-object left adequate subcategory.

Proof: Let $\mathcal{A}$ be a small left adequate of $\mathcal{Rel}$. We can suppose that $\mathcal{A}$ is a full subcategory of $\mathcal{Rel}$, and that there is some $\alpha \in Cn$ with $\alpha \in \mathcal{A}^0$ and $|X| \leq \alpha$ for every $X \in \mathcal{A}^0$. It is easily seen that $\mathcal{A}$ is also a right adequate of $\mathcal{Rel}$. By I.15 it suffices to show that the full subcategory of $\mathcal{Rel}$ with one object α is a left adequate of $\mathcal{A}$. Let $X, Y \in \mathcal{A}^0$, and $\tau : \mathcal{Rel}(-,X) \longrightarrow \mathcal{Rel}(-,Y)$ be a transformation of the covariant hom-functors on it. Putting $|X| = \beta \leq \alpha$, we get $X = \{x_\gamma, \gamma \in \beta\}$. Let $J \in \mathcal{Rel}(\alpha, X)$ and $\tilde{J} \in \mathcal{Rel}(X,\alpha)$ are defined by $J = \{\langle \gamma, x_\gamma\rangle, \gamma \in \beta\}$, $\tilde{J} = \{\langle x_\gamma, \gamma\rangle, \gamma \in \beta\}$. Obviously $\tilde{J}J = 1_X$. Put $R = (J)\tau$ and $\tilde{R} = \tilde{J}R$, τ is induced by $\tilde{R}$, for $(T)\tau = (T\tilde{J}J)\tau = T\tilde{J}(J)\tau = T\tilde{J}R = T\tilde{R}$ for every $T \in \mathcal{Rel}(\alpha, X)$.

III.3. Theorem. $\mathcal{Rel}$ has no small left adequate subcategory.

Proof. By III.2 it suffices to prove that $\mathcal{Rel}$ has no one-object left adequate subcategory $\mathcal{R}$, say, the object of which is α , $\alpha \in Cn$. Consider an arbitrary set X with $\alpha < |X|$. We have $T = \bigcup_{\gamma \in \alpha} \{\gamma\} \times \gamma T$ for every $T \in \mathcal{Rel}(\alpha, X)$. To define $\tau : \mathcal{Rel}(\alpha, X) \longrightarrow \mathcal{Rel}(\alpha, X)$ put $(T)\tau = \bigcup_{\gamma \in \alpha} \{\gamma\} \times \overline{\gamma T}$ where $\overline{\gamma T} = \gamma T$ if $|\gamma T| < |X|$, $\overline{\gamma T} = X$

if $|\gamma T| = |X|$. Verify that τ is a transformation of functors on $\mathcal{R}$. For arbitrary $R \in \mathit{Rel}(\alpha, \alpha)$ and $T \in \mathit{Rel}(\alpha, X)$ we have $(RT)\tau = \bigcup_{\gamma \in \alpha} \{\gamma\} \times \overline{\gamma RT}$, and $(R)T\tau = R(\bigcup_{\beta\in\alpha}\{\beta\} \times \overline{\beta T}) = \bigcup_{\gamma\in\alpha}(\{\gamma\} \times \bigcup_{\beta\in\gamma R} \overline{\beta T})$. It suffices to verify that $\overline{\bigcup_{\beta\in\gamma R} \beta T} = \bigcup_{\beta\in\gamma R} \overline{\beta T}$. $|\gamma R| \leq \alpha$, hence if $|\bigcup_{\beta\in\gamma R} \beta T| = |X|$ then there is some $\beta \in \gamma R$ with $|\beta T| = |X|$, i.e. $\overline{\beta T} = X$. If $|\bigcup_{\beta\in\gamma R} \beta T| < |X|$ then $|\beta T| < |X|$ for every $\beta \in \gamma R$. Thus τ is a transformation. If τ were induced by some $S \in \mathit{Rel}(X, X)$ then we would have $yS = \{y\}$ for every $y \in Y$, for $\{\langle 0, y\rangle\}\tau = \{\langle 0, y\rangle\} = \{\langle 0, y\rangle\}S = \{0\} \times yS$. Hence $(\{0\} \times Y)S = \{0\} \times Y$ for every $Y \subseteq X$. But by definition of τ, we have $(\{0\} \times Y)\tau = \{0\} \times X$ for any $Y \subsetneq X$ with $|Y| = |X|$. Thus τ cannot be induced by S, and $\mathcal{R}$ fails to be a left adequate of Rel.

III.4. <u>Lemma</u>. Let $F : \mathit{Set}^* \longrightarrow \mathit{Set}$ be a faithful functor, $\alpha \in Cn$, $\alpha \geq \aleph_0$, $\mu : XP^- \longrightarrow 1P^-$ be a non-trivial α-additive measure. Let $M = \{k_x F : XF \longrightarrow 1F, x \in X\}$. Then the α-additive M-measure $\overline{\mu}$ induced by μ (see II.3) is non-trivial.

<u>Proof</u>: See [5].

III.5. <u>Theorem</u> ($mon\ M$). Rel is not algebraic.

<u>Proof</u>: Denote by $J : \mathit{Set}^* \longrightarrow \mathit{Rel}$ the full embedding defined by $XJ = X$, $fJ = \{\langle xf, x\rangle, x \in X\}$ for every $f \in \mathit{Set}^*(Y, X)$. Let $P' : \mathit{Rel} \longrightarrow \mathcal{G}$ be a full embedding, $\square : \mathcal{G} \longrightarrow \mathit{Set}$ the natural forgetful functor, put $F = P'\square$.

Let $\alpha = max(\aleph_0, |1F|, |2F|)$ and $\mu : XP^- \longrightarrow 1P^-$ be a non-trivial α-additive measure. Putting $\mathcal{M} = \{k_x JF : XF \longrightarrow 1F, x \in X\}$, denote by $\bar{\mu}$ the α-additive $\mathcal{M}$-measure induced by μ. By III.4 $\bar{\mu}$ is non-trivial, too. The additivity of $\bar{\mu}$, in virtue of the fact that F' is a full embedding, guarantees the existence of some $T \in Rel(X,1)$ with $\bar{\mu} = TF$. We have $T \neq 0$. Let us suppose the contrary: Considering $\chi_x \in Set^*(2,X)$, we obtain $\chi_x JF\bar{\mu} = \chi_x JF\, 0F = 0F$. Since $|2F| \leq \alpha$ there exists $\xi \in X$ with $\chi_x JF\bar{\mu} = \chi_x JF k_\xi JF = i_1 JF$. F is faithful, hence $i_1 JF \neq 0F$. Thus $T \neq 0$ holds. So we can write $T = A \times 1$, where $0 \neq A \subseteq X$. To get a contradiction with the non-triviality of $\bar{\mu}$, it suffices to show that $|A| = 1$, i.e. $\bar{\mu} = (\{x_0\} \times 1)F = k_{x_0} JF$ for some $x_0 \in X$. Suppose that there are $x, y \in A$ with $x \neq y$. We have $\chi_x J(A \times 1) = 2 \times 1 = \chi_y J(A \times 1)$. As F is faithful, there is some $b \in 2F$ with $(b)(2 \times 1)F \neq b\, i_0 JF$. Denote by f the mapping from $\{x, y\}$ to XF defined by $xf = (b)\chi_x JF$, $yf = (b)\chi_y JF$. Now, the additivity of $\bar{\mu}$ guarantees the existence of some $\xi \in X$ with $f\bar{\mu} = f k_\xi JF$. $x \neq y$ implies $x \neq \xi$ or $y \neq \xi$. Let e.g. $x \neq \xi$. Then

$$(b)\chi_x JF\bar{\mu} = (b)\chi_x JF k_\xi JF = (b) i_0 JF \neq (b)(2 \times 1)F =$$
$$= (b)\chi_x JF(A \times 1)F = (b)\chi_x JF\bar{\mu} .$$

Therefore $\bar{\mu} \neq (A \times 1)F$ for any A with $|A| > 1$.

Remark. By I.14 the preceding theorem implies that under *non M* **Rel** has no small left adequate subcategory. Theorem III.3 states more: **Rel** has no small left adequate subcategory independently on set-theoretical assumptions.

IV

IV.1. Notation. Denote by D the following assumption: There exists a proper class $C \subseteq Cn$ such that for every $\alpha \in C$ there is $R_\alpha \subseteq \alpha^2$ such that the full subcategory of $\mathcal{G}$ consisting of objects $\langle \alpha, R_\alpha \rangle$ for every $\alpha \in C$ is discrete (i.e. besides of identities, it has no other morphisms). The negation of this assumption will be denoted by *non* D .

Remark. It is easy to see that D is equivalent to the statement that no big discrete category is algebraic. It is not known, whether D is independent on M or not.

IV.2. Theorem (*non* M). There exists a non-algebraic category which can be obtained as a factorization of an algebraic one.

Proof: We shall construct two examples of algebraic categories which can be factorized on a non-algebraic category; one assuming *non* D , the other assuming D .

(1) Denote by $\mathfrak{M}$ the subcategory of $\mathcal{S}et$, the objects of which are all cardinals and the morphisms are all one-to-one mappings between them. This category is algebraic; e.g. the mapping $\alpha \longrightarrow \langle \alpha, \{\langle \gamma, \delta \rangle, \gamma \neq \delta, \gamma, \delta \in \alpha \} \rangle$ defines a full embedding of $\mathfrak{M}$ into $\mathcal{G}$. Denote by $\mathcal{C}ard$ the thin category such that $\mathcal{C}ard^o = Cn^o$ and $\alpha \longrightarrow \beta$ iff $\alpha \leq \beta$ for every $\alpha, \beta \in Cn$. Let Φ : $: \mathfrak{M} \longrightarrow \mathcal{C}ard$ be the unique functor with $\alpha\Phi = \alpha$ for every $\alpha \in Cn$. Φ is a factorization. Moreover Φ is

a retraction, for we can define a functor $J : \mathcal{C}ard \longrightarrow \mathfrak{M}$ by $\alpha J = \alpha$, $(\alpha \to \beta)J = j_{\alpha,\beta}$, where $j_{\alpha,\beta}$ is the inclusion of α to β. It is known that under $non\ \mathrm{D}$ $\mathcal{C}ard$ is non-algebraic. (It is easily seen that if there were a full embedding $F : \mathcal{C}ard \longrightarrow \mathcal{G}$ then there would exist a proper class $C \subseteq Cn$ such that $\bigcup_{\beta < \alpha} \beta F(\beta \to \alpha)F \subsetneqq \alpha F$ for every $\alpha \in C$. Then we can choose some $x_\alpha \in \alpha F - \bigcup_{\beta < \alpha} \beta F(\beta \to \alpha)F$ for every $\alpha \in C$. Consider the full subcategory of $R(\{2,1\})$ consisting of objects $\langle \alpha, \{R_\alpha, \{x_\alpha\}\}\rangle$, where $\alpha \in C$ and $\langle \alpha, R_\alpha \rangle = \alpha F$. Obviously it is a big and discrete category, which contradicts to $non\ \mathrm{D}$.)

(2) Let $\alpha \in Cn$, put $\bar{\alpha} = \alpha \times \{0\}$ and for $A \subseteq \alpha$ put $\bar{A} = A \times \{0\}$. Given $\alpha \in Cn$, $\beta \in \alpha$, then denote by $f_\beta : \alpha \longrightarrow 2$ the mapping such that $\gamma f_\beta = 1$ iff $\gamma \geqq \beta$, $\gamma \in \alpha$. Further denote by $g_\beta : \bar{\alpha} \longrightarrow \alpha$ the mapping defined by $\bar{\gamma} g_\beta = \gamma$ if $\gamma \in \alpha$, $\gamma \geqq \beta$, $\bar{\gamma} g_\beta = 0$ if $\gamma \in \alpha$, $\gamma < \beta$. Assuming D, denote by $R_{\bar{\alpha}} \subseteq \bar{\alpha}^2$ for every $\alpha \in C$ the copies of R_α (see IV.1) on $\bar{\alpha}$ under the bijection $\gamma \longrightarrow \langle \gamma, 0 \rangle$. Let $\mathfrak{K}$ be the full subcategory of $R(\{1,1,2,2\})$ consisting of the following objects:

$\langle 2, \{\{0\}, \{0\}, 2^2, \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}\}\rangle$,

$\langle \alpha, \{\{0\}, 0, \alpha^2, \{\langle \gamma, \delta\rangle, \gamma < \delta, \gamma, \delta \in \alpha\}\}\rangle$ for every $\alpha \in C$, $\alpha > 2$,

$\langle \bar{\alpha}, \{0, 0, R_{\bar{\alpha}}, 0\}\rangle$ for every $\alpha \in C$, $\alpha > 2$.

It is easy to verify that we have $\mathfrak{K}(2, \alpha) = 0$,

$\mathfrak{K}(2,\bar{\alpha})=0,\ \mathfrak{K}(\alpha',\bar{\alpha})=0,\ \mathfrak{K}(\alpha,\bar{\alpha})=0,\ \mathfrak{K}(\bar{\alpha},\bar{\alpha}')=0,\ \mathfrak{K}(\bar{\alpha},2)=$
$=Set(\bar{\alpha},2),\ \mathfrak{K}(\bar{\alpha},\alpha)=Set(\bar{\alpha},\alpha),\ \mathfrak{K}(\alpha,2)=\{f_\beta,\beta\in\alpha,\ \beta>0\}$

for every $\alpha, \alpha' \in C$ with $\alpha \neq \alpha', \alpha, \alpha' > 2$. Apparently the following is true for every $\beta, \gamma \in \alpha, \gamma > 0$: $g_\beta f_\gamma = \chi_0$ if $\beta < \gamma$, $g_\beta f_\gamma = \chi_{\langle\gamma,\beta\rangle}$ if $\gamma \leq \beta$. Define a congruence $\sim$ on $\mathfrak{K}$: Let $\varphi, \psi \in \mathfrak{K}^m$, then $\varphi \sim \psi$ as soon as one of the following conditions is fulfilled:

(a) $\varphi = \psi$,

(b) There is some $\alpha \in C, \alpha > 2$ such that $\varphi, \psi \in \mathfrak{K}(\bar{\alpha}, 2)$, $\varphi = \chi_A, \psi = \chi_B$ where $A, B \subseteq \bar{\alpha}$ with $|A| = 1 = |B|$,

(c) There is $\alpha \in C, \alpha > 2$ such that $\varphi, \psi \in \mathfrak{K}(\bar{\alpha}, 2)$, $\varphi = \chi_A$, $\psi = \chi_B$, where $A, B \subseteq \bar{\alpha}$ with $|A| > 1$ or $A = 0$, and $|B| > 1$ or $B = 0$.

This defines correctly a congruence on $\mathfrak{K}$, as we have $\mathfrak{K}(2, X) = 0$ for every $X \in \mathfrak{K}^o$, $X \neq 2$ and $\mathfrak{K}(X, \bar{\alpha}) = 0$ for every $X \in \mathfrak{K}^o, X \neq \bar{\alpha}$. Denote by i_1^α the class in $\mathfrak{K}/\sim$ containing the characteristic functions of all one-point subsets of $\bar{\alpha}$, and by i_0^α the class in $\mathfrak{K}/\sim$ containing $\chi_0 : \bar{\alpha} \longrightarrow 2$. As other classes in $\mathfrak{K}/\sim$ consist of one point only, we shall denote them by the same symbols as their representatives. To show that $\mathfrak{K}/\sim$ is not algebraic, construct functors $J_\alpha : \mathcal{S}_{\alpha'} \longrightarrow \mathfrak{K}/\sim$ ($\alpha \in C, \alpha > 2, \alpha' = \alpha - \{0\}$) fulfilling the assumptions of II.10. Put

$1J_\alpha = 2$, $2J_\alpha = \bar{\alpha}$, $\alpha' J_\alpha = \alpha$, $i_0 J_\alpha = i_0^\alpha$, $i_1 J_\alpha = i_1^\alpha$,

$\chi_\beta J_\alpha = g_\beta$, $k_\beta J_\alpha = f_\beta$ for every $\beta \in \alpha'$. Obviously all J_α are embeddings, and

$\mathfrak{K}/\sim(\alpha, 2) = \mathfrak{K}(\alpha, 2) = \{f_\beta, \beta \in \alpha, \beta > 0\} = \{k_\beta J_\alpha, \beta \in \alpha\}$.

Therefore $\mathfrak{K}/\sim$ is not algebraic, though it is a factorization of an algebraic category $\mathfrak{K}$.

IV.3. Proposition (*non* M). There exists an algebraic category which can be obtained as a retract of an algebraic one.

Proof. As a consequence of III.5 the category $\mathcal{Rel}$ (see III.1) fails to be algebraic. We shall show that $\mathcal{Sh}(\mathcal{Rel})$ (see I.2) is algebraic and likewise it is a retract of $\mathcal{Rel}$. To define functors $J: \mathcal{Sh}(\mathcal{Rel}) \longrightarrow \mathcal{Rel}$, $\Phi: \mathcal{Rel} \longrightarrow \mathcal{Sh}(\mathcal{Rel})$ put $XJ = X, (X \rightarrow Y)J = X \times Y$ and $X\Phi = X, R\Phi = X \longrightarrow Y$ for every $R \in \mathcal{Rel}(X, Y)$. Obviously J and Φ are functors and J is a coretraction of Φ. $\mathcal{Sh}(\mathcal{Rel})$ is algebraic since it is a thin category whose all objects are mutually isomorphic. So it is isomorphic to a full subcategory of $\mathcal{G}$ consisting of a proper class of copies of the same rigid graph.

Remark. The preceding example can also serve as an example of a category without a small left adequate subcategory, the retract of which has a small left adequate. By III.3 $\mathcal{Rel}$ has no small left adequate subcategory, while apparently an arbitrary object of $\mathcal{Sh}(\mathcal{Rel})$ generates its left adequate subcategory.

IV.4. Proposition. A retract of a category with a small left adequate subcategory need not have a small left adequate.

Proof. Let $\mathcal{M}$ and $\mathcal{Card}$ be categories considered in IV.2 (1). Obviously $Card$ has no small left adequate subcategory. $Card$ is simultaneously a factorization and a retract of $\mathfrak{M}$. To show that $\mathfrak{M}$ has a small left adequate, consider the following subcategory $\mathcal{L} \subseteq \mathfrak{M} : \mathcal{L}^o = \{1, 2\}$, $\mathcal{L}(1,2) = \{i_0, i_1\}$, $\mathcal{L}(2,1) = 0$, $\mathcal{L}(1,1) = \{1_1\}$, $\mathcal{L}(2,2) = \{1_2\}$. To verify that $\mathcal{L}$ is a left adequate subcategory, consider a transformation $\tau : \mathfrak{M}(-,\alpha)/\mathcal{L} \to \mathfrak{M}(-,\beta)/\mathcal{L}$ for some $\alpha, \beta \in \mathfrak{M}^o$. Define $f: \alpha \longrightarrow \beta$ by $\gamma f = \delta$ iff $(k_\gamma)\tau^1 = k_\delta$ for every $\gamma \in \alpha$, $\delta \in \beta$. It is easily seen that f is one-to-one, and induces τ . Obviously two distinct $f, g : \alpha \longrightarrow \beta$ induce distinct transformations from $\mathfrak{M}(-,\alpha)/\mathcal{L}$ to $\mathfrak{M}(-,\beta)/\mathcal{L}$.

IV.5. <u>Remark</u> ($mon\,\mathrm{M}$). Under $\mathbb{D}$ the category $\mathcal{S}$ is a minimal non-algebraic category, i.e. every proper full subcategory of it is algebraic or isomorphic to $\mathcal{S}$.

<u>Proof</u>: It suffices to show that the full subcategories $\mathcal{S}_{1,2}$ (generated by the class $\mathcal{S}^o - \{1, 2\}$), $\mathcal{S}_2$ (generated by $\mathcal{S}^o - \{2\}$), and $\mathcal{S}_1$) (generated by $\mathcal{S}^o - \{1\}$) are algebraic.

$\mathcal{S}_{1,2}$ is a big discrete category. To define $J_1 : \mathcal{S}_1 \longrightarrow \mathcal{G}$ put $2J_1 = \langle 1, 0 \rangle, \alpha J_1 = \langle \alpha, \mathbb{R}_\alpha \rangle$ for $\alpha \in C, \alpha > 1$ (see IV.1) (the functor J_1 is not defined on the whole category $\mathcal{S}_1$, but on its full subcategory which is isomorphic to $\mathcal{S}_1$). Obviously J_1 is a full embedding. To define $J_2 : \mathcal{S}_2 \to \mathbb{R}(\{1,1,2,2\})$ put $1J_1 = \langle 2, \{\{0\}, \{0\}, 2^2, \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}\}\rangle$, $\alpha J_2 = \langle \alpha, \{\{0\}, 0, \mathbb{R}_\alpha, \{\langle \gamma, \delta\rangle, \gamma < \delta, \gamma, \delta \in \alpha\}\} \rangle$.
It is easy to verify that J_2 is a full embedding.

(We identify again $\mathcal{S}_2$ with its full subcategory which is isomorphic to it.)

IV.6. Remark. In the introduction we formulated the problem, whether algebraicity is invariant under factorization. It is solved negatively by IV.2. So far, a similar problem remains open: Is algebraicity invariant under factorization which is simultaneously a retraction? Under *non* $\mathcal{D}$ it is also negatively solved by IV.2 (1), but our example given in IV.2 (2) under $\mathcal{D}$ deals only with a factorization, and not a retraction.

Let us recall another problem which seems to be of interest with regard to the examples constructed in this paper: Under which assumption can Theorem I.14 be reversed? In particular: Does there exist a complete and cocomplete algebraic category which is not thin, and has no small left adequate subcategory?

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