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SET FUNCTOR II - CONTRAVARIANT CASE

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Abstract:

The paper gives a description of contravariant set functors F from the point of view of the powers FX for various sets X. The methods are analogous to those used for covariant functors, in the author's paper "Set functor". In contravariant case the situation proves to be clearer: where for covariant functors we gave estimations of the powers, here we give the precise equalities. The paper also brings some generalizations of the results for covariant functors and some constructions of contravariant functors.

Key words: set-functor, smallness of functors, cardinal numbers of images

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In his paper [4] the author studied the covariant set functors (i.e. functors F from the category of sets and mappings into itself). He defined a class of cardinals (called the unattainable cardinals of the functor F) on which, roughly speaking, F increases, and he showed, for a given set X , an estimation of the power of FX from the powers of $F\alpha$ for all unattainable α . In the finite case (X finite) these estimations change into precise equalities.

The aim of the present paper is to solve analogous problems for contravariant set functors. The situation here proves to be clearer in the sense that it is possible to form precise equalities even in the infinite case. The author also gives some better estimations for covariant set functors.

Contents

I

Convention. Denote $\mathcal{S}$ the category of sets and mappings. In what follows, a functor means a set functor i.e. a functor from $\mathcal{S}$ into $\mathcal{S}$ (covariant, or contravariant).

Conventions. 1) As usual in the set theory, a cardinal α is the set of all ordinals less than the type of α. Denote X^+ the follower of the cardinal of X, $X \leq Y$ denotes that $card\, X \leq card\, Y$ (analogously $X < Y$), while $X \subset Y$ has the usual meaning that X is a subset of Y, $X \simeq Y$ denotes that $card\, X = card\, Y$.

2) Given a mapping (or a functor) $f : A \longrightarrow B$ and C a subset (subcategory) of A, f/C denotes the domain restriction of f to C. Let $f : X \longrightarrow Y$ be a mapping, denote $Im\, f = \{ f(x) ; x \in X \}$.

3) Let $X \subset Y$, then i_Y^X denotes the inclusion of X to Y, $i_Y^X(x) = x$ for all $x \in X$. Denote id_X the identity of X.

4) X^Y denotes the set of all mappings from Y to X.

5) Denote

Q_M - the covariant homfunctor $Q_M = Hom(M, -)$,

P_M - the contravariant homfunctor $P_M = Hom(-, M)$.

6) Let F be a functor, α a cardinal. Then F^α denotes the subfunctor of F with

$$F^\alpha X = \bigcup_{Y < \alpha} \bigcup_{f : Y \to X} Im\, Ff \quad \text{if } F \text{ is covariant,}$$

$$F^\alpha X = \bigcup_{Y < \alpha} \bigcup_{f : X \to Y} Im\, Ff \quad \text{if } F \text{ is contravariant}$$

and such that $F^\alpha h$ is the domain-range restriction of Fh, for every mapping h.

We shall make use of the following well-known facts:

Lemma 1.1. Let F be a functor, let $f : X \longrightarrow Y$ be a monomorphism, $X \neq \emptyset$. Then Ff is a monomorphism, if F is covariant and Ff is an epimorphism, if F is contravariant. Analogously, if f is an epimorphism.

We recall the special case of the Yoneda lemma.

Lemma 1.2. For every functor F and every $x \in FX$ there exists just one transformation τ_x from the homfunctor to X (covariant if F is covariant, contravariant if F is contravariant) such that $\tau_x^X(id_X) = x$.

Note. For every contravariant functor F and every set $A \neq \emptyset$ there exists a contravariant functor F', such that $F'\emptyset = A$ and $F = F'$ on the category of non-void sets and mappings.

II

We recall the notion of α-semidisjoint systems of subsets.

Definition. Let X be a set, $\mathcal{A}$ a system of subsets of X, α a cardinal. We say that $\mathcal{A}$ is an α-semidisjoint system on X, if

$$Z \in \mathcal{A} \Longrightarrow Z \simeq \alpha \ ,$$

$$Z_1 , Z_2 \in \mathcal{A} \Longrightarrow (Z_1 \cap Z_2) < \alpha \ .$$

Lemma 2.1. Assume the generalized continuum hypothesis. Let X be an infinite set, α a cardinal. There exists an α-semidisjoint system $\mathcal{A}$ on X with $\mathcal{A} \simeq 2^X$ iff $X > \alpha$ and $\mathrm{conf}\, X = \mathrm{conf}\, \alpha$.

Proof: see [7].

Definition. A couple of mappings $f, g : X \longrightarrow Y$ is called diverse if f and g are epimorphisms and there exists $Z \subset X$, with $f(Z) = Y$, $g(Z) < Y$ or $f(Z) < Y$, $g(Z) = Y$. A subset $\mathcal{A} \subset Y^X$ is called diverse if every couple from it is diverse.

Lemma 2.2. Let α, β be cardinals, α infinite, $\alpha \geq \beta > 1$. There exists a diverse system $\mathcal{A} \subset \beta^\alpha$ with $\mathcal{A} \simeq 2^\alpha$.

Proof: Let $V = \beta \times \alpha$, given $A \subset \alpha$ put $f_A : V \rightarrow \beta$, $f_A(i, j) = i$ if $j \in A$, $f_A(i,j) = 0$ if $j \notin A$ (remember $0 \in \beta$). Then $\{ f_A ; \emptyset \neq A \subset \alpha \}$ is diverse, as for $A, B \subset \alpha$, $A - B \neq \emptyset$. Choose $x \in A - B$ and put $Z = \beta \times \{x\} \subset V$. Then $f_A(Z) = \beta$, $f_B(Z) = \{0\} < \beta$. So we have a diverse system in β^V

with power 2^{α}. As clearly $\gamma \simeq \alpha$, this completes the proof.

Convention. Let k be a finite cardinal, then $p_k(m)$ denotes the number of partitions of a set of cardinality m into exactly k non-empty sets.

Lemma 2.3. For every finite cardinal $\beta > 1$ and every cardinal α, $\alpha \geq \beta$, there exists a diverse system $\mathcal{D}$ in β^{α} with power $p_{\beta}(\alpha)$.

Proof: Let $P_{\beta}(\alpha)$ be the set of all partitions of the set α into β non-empty sets. For every $\mathcal{A} \in P_{\beta}(\alpha)$, $\mathcal{A} = \{A_1, A_2, \dots, A_{\beta}\}$ choose an epimorphism $f_{\mathcal{A}} : \alpha \longrightarrow \beta$ such that $x, y \in \alpha$, $f_{\mathcal{A}}(x) = f_{\mathcal{A}}(y)$ iff there exists i with $x, y \in A_i$. Denote $\mathcal{D} = \{f_{\mathcal{A}}, \mathcal{A} \in P_{\beta}(\alpha)\}$. Prove that $\mathcal{D}$ is diverse: if $f, g \in \mathcal{D}$, $f \neq g$, then clearly there exists $x, y \in \alpha$ with $f(x) \neq f(y)$, $g(x) = g(y)$. Clearly there exists a set $Z \subset \alpha$ with power β such that $x, y \in Z$ and f/Z is a monomorphism. As g/Z is not a monomorphism, we have $g(Z) < < \beta = f(Z)$, therefore f, g are diverse.

Note. It is well-known that

$$p_k(m) = \begin{cases} 2^m & \text{if } m \text{ is infinite,} \\ \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^m & \text{if } m \text{ is finite.} \end{cases}$$

III

In [4] we defined an unattainable cardinal for a covariant functor. An analogous definition is possible without the consideration of variances.

Definition. Cardinal $\alpha > 1$ is an unattainable cardinal of a functor F if $F\alpha - F^{\alpha}\alpha \neq \emptyset$. Denote $\mathcal{A}_F$ the class of all unattainable cardinals of F.
The cardinal of the set $F\alpha - F^{\alpha}\alpha$ is called the increase of F on α.

In [4] the following results concerning the cardinalities of the images of an arbitrary set X through a covariant functor F are proved.

Theorem 3.1. Let F be a covariant functor. Let $\beta = \sup \mathcal{A}_{FX}$. Then

1) if $X \geq \min(\aleph_0, \min \mathcal{A}_F)$ then $\max(F\beta, X) \leq FX \leq \max(F\beta, X^{\beta})$,

2) if $\aleph_0 > X \geq \min \mathcal{A}_F$ then $FX \simeq F^{\beta}X + [(F\beta - F^{\beta}\beta)\binom{X}{\beta}]$,

3) there exist γ, δ such that, if $\emptyset < X < \min \mathcal{A}_F$ then $FX \simeq \gamma +$

Proof: see [4].

Theorem 3.2. Assume the generalized continuum hypothesis. Let F be a covariant functor. Then $FX \geq 2^{X}$ if X is infinite and $X \in \mathcal{A}_F$.

Proof: see [4].

Lemma 3.1. Let $\mathcal{B}$ be an α-semidisjoint system on X, $\alpha \in \mathcal{A}_F$. Then $FX \geq \mathcal{B}$, where F is a covariant functor.

Proof: see [4].

Now, using Lemma 2.1, we are able to give a better estimation.

Corollary 3.3 . Assume the generalized continuum hypothesis. Let F be a covariant functor, X an infinite set, $\beta = \sup \mathcal{A}_F X$. If either for every $\alpha \in \mathcal{A}_F X$ it holds $\operatorname{conf} X > \operatorname{conf} \alpha$ or there exists $\alpha \in \mathcal{A}_F X$ such that $\operatorname{conf} X = \operatorname{conf} \alpha$, then $FX = \max (F\beta, X^\beta)$.

Proof: Use Theorem 3.1 , and Lemmas 2.1 and 3.1 .

Now, we present analogous results concerning contravariant functors. In what follows F is a contravariant functor.

Lemma 3.2 . Let $f : X \to Y$ be an epimorphism. Then for every cardinal $\beta \leq Y$,

$Ff(F^{\beta^+}Y - F^\beta Y) \subset (F^{\beta^+}X - F^\beta X)$.

Proof: Clearly $Ff(F^{\beta^+}Y\ F^\beta Y) \subset F^{\beta^+}X$. Assume that there exists $x \in F^{\beta^+}Y - F^\beta Y$ with $Ff(x) \in$ $\in F^\beta X$. Let $g : Y \to X$ with $fg = id_Y$. Clearly $Fg(Ff(x) \in F^\beta X$ (F^β is a functor) but then $x =$ $= F(fg)(x) \in F^\beta X$ which is a contradiction.

Lemma 3.3 . Let $f : X \to Y$, we have $\operatorname{Im} Ff \subset$ $\subset F^{(\operatorname{Im} f)^+} X$.

Proof: Denote by $\operatorname{Im} f = A$. Let $\tilde{f} : X \to A$ with $f = i_Y^A \cdot \tilde{f}$, then $Ff = F\tilde{f} \cdot Fi_Y^A$. As F^{A^+} is a functor and $\operatorname{Im} Fi_Y^A \subset F^{A^+}A$, we have $\operatorname{Im} Ff \subset$ $\subset F^{A^+}X$ which concludes the proof.

Lemma 3.4 . Let $f, g : X \to Y$ be diverse, then
$Ff(FY - F^Y Y) \cap Fg(FY - F^Y Y) = \emptyset$.

Proof: Let $Z \subset X$ be the set from the definition of diverse couple. Assume the existence of

$x \in Ff(FY - F^YY) \cap Fg(FY - F^YY)$. As $f \circ i_X^Z$ is an epimorphism we have, due to 3.2 , $Fi_X^Z(x) \notin \notin F^YZ$ (let $u \in FY - F^YY$ with $Ff(u) = x$, we have $F(f \circ i_X^Z)(u) \notin F^YZ$). As $g . i_X^Z(Z) < Y$ we get from 3.3 $Fi_X^Z(x) \in F^XZ$ (let $v \in FY$ with $Fg(v) = x$, we have $F(g \circ i_X^Z)(v) \in F^XZ$). This is a contradiction.

Lemma 3.5 . Let $\mathcal{B}$ be a diverse system from X to Y , where $Y \in \mathcal{A}_F$. Then $(FX - F^YX) \geq \mathcal{B} \cdot (FY - F^YY)$.

Proof: Let $f \in \mathcal{B}$. It follows from 3.2 that $Ff(FY - F^YY) \subset (FX - F^XX) \neq \emptyset$. From 3.4 we get $f_1, f_2 \in \mathcal{B}, f_1 \neq f_2 \Rightarrow Ff_1(FY - F^YY) \cap Ff_2(FY - F^YY) = \emptyset$. Thus $(FX - F^YX) \geq \mathcal{B} . (FY - F^YY)$, because Ff_1 is a monomorphism.

Lemma 3.6 . If $X \leq Y$ then $FX \leq FY$.

Proof: Let $f : Y \longrightarrow X$ be an epimorphism, then Ff is a monomorphism from FX to FY , hence $FX \leq FY$.

Lemma 3.7. Let X, Y be sets, $Y \neq \emptyset$ and if $Y < < Z \leq X$ then $Z \notin \mathcal{A}_F$. Then $FX \leq FY \cdot Y^X$.

Proof: It follows from the presumption that $F^{X+} = F^{Y+}$ and so $FX = F^{Y+}X = \bigcup_{f : Y \to X} Im\, Ff$. Then clearly $FX \leq FY \cdot Y^X$.

Lemma 3.8 . Let X, Y be finite sets, $Y \neq \emptyset$ and if $Y < Z \leq X$ then $Z \notin \mathcal{A}_F$. Then

$$FX \simeq F^YX + [(FY - F^YY) \cdot p_Y(X)] .$$

Proof: Let $P_Y(X)$, $\mathcal{D}$ be as in Lemma 2.3 . The proof will be concluded by showing that for every $x \in FY - F^XY$ there exists $f \in \mathcal{D}$ with $x \in Im\, Ff$. Due to the assumptions there exists $g: Y \longrightarrow X$ an epimorphism, with $x \in Im\, Fg$. Let $f \in \mathcal{D}$ such that $\{f^{-1}(x)\}_{x \in X} = \{g^{-1}(x)\}_{x \in X}$. Then $f = hg$, where h is an epimorphism and $Ff = Fg\ \ Fh$ and therefore $Im\, Fg = Im\, Ff$. Therefore $x \in Im\, Ff$. Hence

$$FX \leq F^YX + [(FY - F^YY) \cdot p_Y(X)] .$$

The other inequality

$$FX \geq F^YX + [(FY - F^YY) \cdot p_Y(X)]$$

follows from 2.3, 3.2 and 3.4.

Lemma 3.9. If $0 \neq X < min\, \mathcal{A}_F$, then $FX \simeq 1$.

Proof: Due to 3.6 $FX \geq F1$. Now, $FX = \bigcup_{f \in 1^X} Im\, Ff$ and as $1^X \simeq 1$, we have $FX \leq F1$.

Theorem 3.4. Let F be a contravariant functor. Let X be an arbitrary non-empty set, $\beta = sup\, \mathcal{A}_{FX}$. If $X \geq min(\aleph_0, min\, \mathcal{A}_F)$ then $FX \simeq max(F\beta, 2^X)$. If $\aleph_0 > X \geq min\, \mathcal{A}_F$ then

$$FX \simeq F1 + \sum_{\alpha \in \mathcal{A}_{FX}} [(F\alpha - F^{\alpha}\alpha)\frac{1}{\alpha!}(\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha}{i}(\alpha - i)^X)] .$$

If $0 \neq X < min\, \mathcal{A}_F$ then $FX \simeq F1$.

Proof: It is a consequence of the preceding lemmas.

Corollary 3.5. Let F be a contravariant functor. Let $X \in \mathcal{A}_F$ be an infinite set. Then

$$(FX - F^XX) \geq 2^X .$$

IV

<u>Definition</u>. Let α, β be cardinals. Define a contravariant functor M_α^β like this:

Let X be a set, then $M_\alpha^\beta X$ is $\{0\}$ joined with the set of all couples $\langle N, i\rangle$ where $i \in \beta$ and N is a partition of X and $N \simeq \alpha$; let $f : X \to Y$, $M_\alpha^\beta f(0) = 0$, let $\langle N, i\rangle \in M_\alpha^\beta Y$, if $\{f^{-1}(V); V \in N\} \simeq \alpha$ then $M_\alpha^\beta f(\langle N, i\rangle) = \langle \{f^{-1}(V); V \in N\}, i\rangle$, if not $M_\alpha^\beta f(\langle N, i\rangle) = 0$.

<u>Lemma</u> 4.1 . For every α, β, $\mathcal{A}_{M_\alpha^\beta} = \{\alpha\}$ and if $\alpha \geq \aleph_0$ then $(M_\alpha^\beta \alpha - (M_\alpha^\beta)^\alpha \alpha) \simeq \beta \cdot 2^\alpha$ if $\alpha < \aleph_0$, $(M_\alpha^\beta \alpha - (M_\alpha^\beta)^\alpha \alpha) \simeq \beta$.

Proof: Let $\gamma < \alpha$. Then we have $M_\alpha^\beta \gamma \simeq M_\alpha^\beta 1 \simeq 1$ and Theorem 3.4 implies $\gamma \notin \mathcal{A}_{M_\alpha^\beta}$. Clearly $\alpha \in \mathcal{A}_{M_\alpha^\beta}$. Let $\langle N, i\rangle \in M_\alpha^\beta X$, $X \geq \alpha$. Then $M_\alpha^\beta f(\langle I, i\rangle) = \langle N, i\rangle$ where $f : X \to \alpha$ and $f(x) = f(y) \Longleftrightarrow x, y \in V \in N$ and I is a disjoint system of an one-point subset of α . Therefore $X > \alpha$, $X \notin \mathcal{A}_{M_\alpha^\beta}$. The second proposition is clear.

<u>Note</u>. <u>*Card*</u> denotes the class of all cardinals.

<u>Proposition</u> 4.1 . For a given class of cardinals γ and a given $f : \mathcal{J} \to \mathit{Card}$. Then there exists a contravariant functor F with $\mathcal{J} = \mathcal{A}_F$, $(F\alpha - F^\alpha\alpha) \simeq f(\alpha)$ for all $\alpha \in \mathcal{J}$ if and only if $f(\alpha) \geq 2^\alpha$ for all $\alpha \in \mathcal{J}$, α infinite and $f(\alpha) \geq 1$ for all $\alpha \in \mathcal{J}$, α finite.

Proof: Put $FX = \bigcup_{\alpha \in \mathcal{J}} M_\alpha^{f(\alpha)} X$, if $\varphi : X \to Y$,

$$Fg/_{M_\alpha^{f(\alpha)}X} = M_\alpha^{f(\alpha)}g .$$

Due to Theorem 3. 4 and Lemma 4. 1 this is the functor we were looking for.

<u>Proposition</u> 4. 2 . Given a class of infinite cardinals $\mathcal{J}$ and $f : \mathcal{J} \longrightarrow \mathcal{C}ard$ there exists a contravariant functor F with $\mathcal{J} = \mathcal{A}_F$ $F\alpha \simeq f(\alpha)$ for all $\alpha \in \mathcal{A}_F$ if and only if $\alpha_1\ \alpha_2 \in \mathcal{J}, \alpha_1 \leq \alpha_2 \Rightarrow 2^{\alpha_1} \leq f(\alpha_1) \leq f(\alpha_2)$.

Proof: Put $FX = \bigcup_{\alpha \in \mathcal{J}} M_\alpha^{f(\alpha)}X$, if $g : X \longrightarrow Y$,
$Fg/_{M_\alpha^{f(\alpha)}X} = M_\alpha^{f(\alpha)}g$.

Clearly F fulfils the conditions.

V

Now, analogously as in [4] for a covariant functor we shall show the relation between $\mathcal{A}_F$ and the property "to be small".

<u>Definition</u>. A functor F is small if it is a colimit of a diagram with homfunctors as objects (the variance of the homfunctors agreeing with that of F).

F is petty, if it is a factorfunctor of a disjoint union of a set of homfunctors (of the same variance as F).

<u>Proposition</u> 5. 1 . A set functor (covariant, or contravariant) is small if and only if it is petty.

Proof: It is proved in [3] that for a category **K** in which no homfunctor has a proper class of factorfunctor, a functor from **K** into $\mathcal{S}$ is small iff it is petty. Our proposition follows from the fact that the above condition is fulfilled both for $\mathcal{S}$ and the category dual to $\mathcal{S}$.

<u>Theorem</u> 5. 2 . A covariant set functor F is small if and only if $\mathcal{A}_F$ is a set.

Proof: see [4].

Lemma 5.1 . $\mathcal{A}_{P_M} = \{\alpha, 1 < \alpha \leq M\}$, P_M is a contravariant homfunctor.

Proof: 1) Let $1 < \alpha \leq M$, let $f : \alpha \longrightarrow M$ be a monomorphism. We shall show that $f \notin (P_M)^{\alpha}\alpha$ (and so $\alpha \in \mathcal{A}_{P_M}$). If $f \in (P_M)^{\alpha}\alpha$ i.e. $f = (P_M)^{\alpha} g(h) = hg$ where $g : \alpha \longrightarrow \gamma$ with $\gamma < \alpha$ then $Im\, f < \alpha$, which is impossible.

2) Let $\alpha > M$. For every $f : \alpha \longrightarrow M$, $(P_M)^{\alpha} f(id_M) = f$ and so $f \in (P_M)^{\alpha}\alpha$.

Lemma 5.2 . Let $\{F_i\}_{i \in I}$ be a collection of arbitrary contravariant functors. Then $\mathcal{A}_{\bigvee_{i \in I} F_i} = \bigcup_{i \in I} \mathcal{A}_{F_i}$.

Proof: It is elementary.

Lemma 5.3 . If F is a factorfunctor of G , both F, G contravariant, then $\mathcal{A}_F \subset \mathcal{A}_G$.

Proof is easy.

Theorem 5.3 . A contravariant functor F is small if and only if $\mathcal{A}_F$ is a set.

Proof: If F is small then $\mathcal{A}_F$ is a set due to Proposition 5.1 and Lemmas 4.2, 4.3 and 4.4 Let $\mathcal{A}_F$ be a set, let $X > \sup \mathcal{A}_F$. Let $\varepsilon : \bigvee_{x \in FX} P_{N_x} \longrightarrow F$ where $N_x = X$ and $\varepsilon(id_{N_x}) = x$ (this defines a transformation). As $F^X = F$, ε is an epitransformation and so F is petty. It follows from the proposition 5.1 that F is small, which concludes the proof.

Corollary 5.4. A set functor F , covariant or contravariant, is small if and only if $\mathcal{A}_F$ is a set.

In the time when I prepared this paper for publication, I got acquainted with a preprint [1] which solves a similar problem only for finite sets.

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