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ON THE ITERATIVE CONSTRUCTION OF A SOLUTION OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Introduction. General existence theorems for nonlinear elliptic boundary value problems for operators of the form

$$A(u) := \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, Du, \ldots, D^m u)$$

are considered in several papers (see e.g. [1-5, 8, 9, 11-13]). The operator $A(u)$ is studied on a closed subspace V of the Sobolev space $W_{m,p}(\Omega)$, where Ω is a bounded open subset of R^n, $n \ge 1$. The existence theorems are based upon different methods and different assumptions. In [1, 2, 5] the theory of monotone operators on reflexive Banach spaces is used, while in [3, 4, 8, 9, 12, 13] the monotonicity condition is replaced by a weaker assumption. Browder considers in [3, 4] noncoercive elliptic boundary value problems, while all other cited papers assume that the operator A satisfies a coercivity condition. Furthermore there are different growth conditions on the functions $A_{\alpha}(x, u, Du, \ldots, D^m u)$ with respect to $u, \ldots, D^m u$.
All these existence theorems are not constructive.

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For operators A of order $2m = n$ an iteration process is given by Koselev [6] and in [7] Kratochvíl studies a similar iteration method for operators A being a potential operator. In both cases it is proved that the iteration sequence converges to the unique solution of the elliptic boundary value problem. It is assumed that the operator A is monotone and coercive and the functions A_α satisfy restrictive growth conditions.

It is the purpose of the present note to apply the general iteration process studied in [10] to nonlinear elliptic boundary value problems on the space $\overset{\circ}{W}_{m,2}$. It is assumed that A is a potential operator which must not be monotone or coercive. The functions $A_\alpha(x, u, \dots, D^m u)$ satisfy less restrictive growth conditions. It is shown that the iteration sequence converges to a solution of the not necessarily uniquely solvable nonlinear elliptic boundary value problem.

2. In this section we will state the assumptions, some known results on nonlinear elliptic operators and the general iteration process studied in [10].

In the following we shall use the usual notations (see e.g. [3]). We introduce the notations: Let ξ, η and φ be the vectors $\{\xi_\alpha : |\alpha| \le m\}$, $\{\eta_\alpha : |\alpha| \le m-1\}$ and $\{\varphi_\alpha : |\alpha| = m\}$, respectively, from the spaces R^{s_m}, $R^{s_{m-1}}$ and $R^{s_m - s_{m-1}}$, respectively.

Furthermore we assume that Ω is a bounded open subset of R^n with sufficiently smooth boundary $\partial\Omega$

such that the Imbedding Theorems of Sobolev hold.
For functions $u(x)$, $v(x)$, defined on Ω a.e., we set $(u,v) = \int_\Omega u(x)\, v(x)\, dx$, where $\int_\Omega$ denotes the Lebesgue integral.

We will study solutions of nonlinear elliptic boundary value problems on the Sobolev space $\mathring{W}_{m,2}$. By $\langle w, u \rangle$ we shall denote the value of $w \in \mathring{W}^*_{m,2}$ at $u \in \mathring{W}_{m,2}$.
For each α, A_α is assumed to be a function from $\Omega \times \times R^{s_m}$ to R^1 satisfying the following conditions (s.[3]):

<u>Assumption A</u>: (1) A_α is measurable in x for fixed $\xi \in R^{s_m}$ and continuous in ξ on R^{s_m} for almost all $x \in \Omega$. Let b be the greatest integer less than $m - n/2$, and ξ_b^+ denote the vector $\{|\xi_\alpha| : |\alpha| \leq b\}$ from $R_+^{s_n}$. There exist continuous functions c_α and c_1 from $R_+^{s_b}$ to $L^{p_\alpha}(\Omega)$ and R_+^1, respectively, such that

$$|A_\alpha(x,\xi)| \leq c_\alpha(\xi_b^+) + c_1(\xi_b^+) \sum_{m-n/2 \leq |\beta| \leq m} |\xi_\beta|^{p_{\alpha\beta}}$$

with the exponents p_α and $p_{\alpha\beta}$ satisfying

$$p_\alpha = 2 \text{ for } |\alpha| = m ,$$

$$p_\alpha > s'_\alpha \text{ for } |\alpha| \in [m - n/2, m[,$$

$$1/s_\alpha = 1/2 - (m - |\alpha|)/n ,$$

$$1/s_\alpha + 1/s'_\alpha = 1 ,$$

$$p_\alpha = 1 \text{ for } |\alpha| \in [0, b] ,$$

and

$$p_{\alpha\beta} \leq 1 \text{ for } |\alpha| = |\beta| = m ,$$

$$p_{\alpha\beta} < s_\beta / s'_\alpha \text{ for } |\alpha|, |\beta| \in [m - n/2, m],$$

$$|\alpha| + |\beta| < 2m \ ,$$
$$r_{\alpha\beta} \le s_\beta \ \text{for} \ |\alpha| \in [0, b] \ ,$$
$$|\beta| \in [m - n/2, m] \ .$$

(2) For almost all $x \in \Omega$ and each $\eta \in \mathbb{R}^{s_{m-1}}$ let $\sum_{|\alpha| = m} (A_\alpha(x, \eta, \varphi) - A_\alpha(x, \eta, \varphi'))(\varphi_\alpha - \varphi'_\alpha) > 0$ for $\varphi \neq \varphi'$.

(3) There exist continuous functions c_0 and c from $\mathbb{R}^{s_b}$ to $\mathbb{R}^1_+$ with $c_0(\xi_b) \ge \tilde{c}_0 > 0$ for all $\xi_b := \{\xi_\alpha : |\alpha| \le b\}$, such that for almost all $x \in \Omega$, all φ and η we have

$$\sum_{|\alpha| = m} A_\alpha(x, \eta, \varphi)\varphi_\alpha \ge c_0(\xi_b)|\varphi|^2 - c(\xi_b^+) \sum_{m-n/2 \le |\beta| \le m-1} |\eta_\beta|^{t_\beta}$$

with $t_\beta < s_\beta$.

(4) $F : \Omega \times \mathbb{R}^{s_m} \longrightarrow \mathbb{R}^1$. For each fixed $\xi \in \mathbb{R}^{s_m}$, $F(\cdot, \xi)$ is measurable on Ω. For almost all $x \in \Omega$, $F(x, \cdot)$ is once continuously differentiable with $\dfrac{\partial F}{\partial \xi_\alpha} = A_\alpha$. Furthermore let

$$|F(x, \xi)| \le c_2(\xi_b^+)(x) + c_3(\xi_b^+) \sum_{m-n/2 \le |\beta| \le m} |\xi_\beta|^{s_\beta} \ ,$$

where c_2 and c_3 are continuous functions from $\mathbb{R}^{s_b}_+$ to $L^1(\Omega)$ and $\mathbb{R}^1_+$, respectively.

(5) For each α and almost all $x \in \Omega$, $A_\alpha(x, \cdot)$ is once continuously differentiable such that

$$\left| \frac{\partial A_\alpha(x, \xi)}{\partial \xi_\beta} \right| \le c_{\alpha\beta}(\xi_b^+)(x) + d(\xi_b^+) \sum_{m-n/2 \le |\gamma| \le m} |\xi_\gamma|^{r_{\alpha\beta\gamma}} \ ,$$

where the exponents satisfy with $q_{\alpha\beta} := s_\beta / s'_\alpha$, $1/q_{\alpha\beta} + 1/q'_{\alpha\beta} = 1$ the following inequalities

$$p_{\alpha\beta\gamma} = s_\gamma / (s'_\alpha q'_{\alpha\beta}) \quad \text{for } |\alpha|, |\beta|, |\gamma| \in [m - n/2, m],$$

$$p_{\alpha\beta\gamma} = q_{\beta\gamma} \quad \text{for } |\beta|, |\gamma| \in [m - n/2, m], \ |\alpha| < m - n/2,$$

$$p_{\alpha\beta\gamma} = q_{\alpha\gamma} \quad \text{for } |\alpha|, |\gamma| \in [m - n/2, m], \ |\beta| < m - n/2,$$

$$p_{\alpha\beta\gamma} = s_\gamma \quad \text{for } |\gamma| \in [m - n/2, m], |\alpha|, |\beta| \in [0, b]$$

and $c_{\alpha\beta}$ (d, respectively) is a continuous function from $\mathbb{R}_+^{s_b}$ to $L^{s'_\alpha q'_{\alpha\beta}}(\Omega)$ for $|\alpha|, |\beta| \in [m - n/2, m]$, to $L^{s'_\alpha}(\Omega)$ for $|\alpha| \in [m - n/2, m]$, $|\beta| \in [0, b]$, to $L^{s'_\beta}(\Omega)$ for $|\alpha| \in [0, b]$, $|\beta| \in [m - n/2, m]$ and to $L^1(\Omega)$ for $|\alpha|, |\beta| \in [0, b]$ ($\mathbb{R}_+^1$, respectively).

Assumption A (1) - (4) is the assumption of Browder [3] ($p = 2$), where the following Lemma is proved:

<u>Lemma 1</u>: Let Assumption A hold. Set

$$a(u, v) := \sum_{|\alpha| \le m} (A_\alpha(\cdot, \xi(u)), D^\alpha v),$$

$$g(u) := \int_\Omega F(x, \xi(u)(x))\, dx .$$

Then it follows:

(a) There exists a bounded continuous mapping $T: \mathring{W}_{m,2} \to \mathring{W}^*_{m,2}$, such that for all $u, v \in \mathring{W}_{m,2}$

$$a(u,v) = \langle Tu, v \rangle \ .$$

Furthermore T satisfies the condition (S^+): If for any sequence $\{u_\nu\}$ in $\mathring{W}_{m,2}$ converging weakly to u in $\mathring{W}_{m,2}$ such that $\limsup_\nu \langle Tu_\nu - Tu, u_\nu - u \rangle \leq 0$, it follows that $\{u_\nu\}$ converges strongly to u in $\mathring{W}_{m,2}$.

(b) g is a once differentiable functional in $\mathring{W}_{m,2}$, and its derivative g' satisfies

$$g'(u) = T(u) \ .$$

(c) For each $R > 0$ there exists a constant $L_R > 0$ such that for all $u, v \in \overline{K}_R = \{u : \|u\| \leq R\}$

$$\|Tu - Tv\| \leq L_R \|u - v\| \ .$$

Proof: The assertions (a) and (b) are proved in [3] using assumption A (1) - (4), while assertion (c) follows also as in [3] by assumption A (5) using the Imbedding Theorem of Sobolev.

The usual norm in $\mathring{W}_{m,2}$ is equivalent to the norm $\|u\|_{m,2} := (\sum_{|\alpha|=m} (D^\alpha u, D^\alpha u))^{1/2}$ (see e.g.[7]), which shall be used in the following.

We need further

<u>Assumption B</u>: There exist $v_0 \in \mathring{W}_{m,2}$, $r > 0$, $\rho > 0$, such that for all

$u \in \overline{K}_{r,\rho} := \{u \in \mathring{W}_{m,2} : r \leq \|u - v_0\| \leq r + \rho\}$

it follows

$$\sum_{|\alpha| \leq m} (A_\alpha(\cdot, \xi(u)), \ D^\alpha(u - v_0)) > 0 \ .$$

<u>Remark</u>: Under the assumptions A and B the mapping T defined by Lemma 1 is not coercive.

If we assume the coercivity condition

Assumption B': Let exist a constant $a_1 > 0$ and a function $a_2 \in L^1(\Omega)$ such that for almost all $x \in \Omega$ and all $\xi \in R^{s_m}$ we have

$$\sum_{|\alpha| \le m} A_\alpha(x, \xi)\xi_\alpha \ge a_1 |\varphi|^2 - a_2(x) ,$$

then we have the following

Proposition: Suppose that the assumptions A and B' hold. Then assumption B is also true with $v_0 = 0$, $\rho > 0$ arbitrary and r sufficiently large.

For our iteration process we need the following Lemma (see e.g. [5], § 8):

Lemma 2: Consider the differential operator

$$B(u) := (-1)^m \sum_{|\alpha| = m} D^{2\alpha} u$$

and the bilinear form

$$b(u, v) := \sum_{|\alpha| = m} (D^\alpha u, D^\alpha v) .$$

Then it follows:

(a) There exists a linear bounded operator $S : \mathring{W}_{m,2} \to \mathring{W}^*_{m,2}$ such that for all $u, v \in \mathring{W}_{m,2}$

$$\langle Su, v \rangle = b(u, v) ,$$

$$\langle Su, u \rangle = \| Su \|^*_{m,2} \| u \|_{m,2} = \| u \|^2_{m,2} .$$

(b) For each $w_0 \in \mathring{W}^*_{m,2}$ there exists a unique $u_0 \in \mathring{W}_{m,2}$ such that $Su_0 = w_0$.

We will now state a special case of the general iteration process studied in [10].

Let B be a real Banach space with dual B^*, and denote by $\langle w, u\rangle$ the value of $w \in B^*$ at $u \in B$. Let $v_0 \in B$ and set

$$K_r := \{u \in B \,;\, \|u - v_0\| < r\} ,$$

$$\overline{K}_{r,\rho} := \{u \in B \,;\, r \leq \|u - v_0\| \leq r + \rho\} ,$$

$$K_{r+\rho} := \{u \in B \,;\, \|u - v_0\| < r + \rho\} .$$

By $\overline{K}_r$ and $\overline{K}_{r+\rho}$ we denote the closure of K_r and $K_{r+\rho}$, respectively.

We assume

<u>Condition I</u>: (a) Let $f : \overline{K}_{r+\rho} \to B^*$ satisfying

$$\sup_{u \in \overline{K}_{r+\rho}} \|f(u)\| \leq M$$

with suitable constant $M > 0$. Let there exist a constant $L_0 > 0$ such that for all $u, v \in \overline{K}_{r+\rho}$ it follows

$$\|f(u) - f(v)\| \leq L_0 \|u - v\| .$$

(b) Suppose that φ is a linear mapping from B to B^* possessing an inverse on B^* such that for all $w \in B^*$

$$\|\varphi^{-1}(w)\| \leq L \|w\|$$

with suitable constant $L > 0$. Set

$h^* := \inf\left(\frac{\rho}{LM}, \frac{1}{LL_0}\right)$.

<u>Condition II</u>: (a) There exists a functional g on $K_{r+\rho}$, possessing a linear Gâteaux-differential $\delta g(u,v) = \langle g'(u), v\rangle$ such that $g'(u) = -f(u)$.

(b) There exists a constant $c_0 > 0$ such that for each $w \in B^*$

$$\langle w, \varphi^{-1}(w) \rangle \geq c_0 \|w\|^2 .$$

Further let $0 < h < \tilde{h} := \inf \left(h^*, \frac{c_0}{L_0 L^2}\right)$.

Condition III: Let $h \in [0, \tilde{h}]$, $v \in K_r$ and suppose that $u \in \overline{K}_{r+\rho}$ satisfies

$$\varphi(u) = \varphi(v) + h f(u) ,$$

then it follows $u \notin \overline{K}_{r,\rho}$.

Condition IV: Let $\{u_\nu\} \subset \overline{K}_r$ such that $f(u_\nu) \to 0$. Then there exists a subsequence $\{u_{\nu'}\}$ and an element $u \in \overline{K}_r$ such that $u_{\nu'}$ converges strongly to u and $f(u) = 0$.

We may now formulate

Lemma 3 (see [10]): Suppose that the conditions I - IV hold and let $u_0 \in K_r$. Then we have:

(a) The nonlinear problem

$$\varphi(u_{\nu+1}) = \varphi(u_\nu) + h f(u_{\nu+1})$$

has in the ball $\{u \in B : \|u - u_\nu\| \leq \rho\}$ a unique solution $u_{\nu+1}$ which can be obtained by the following iteration process

$$\varphi(u_{\nu,i+1}) = \varphi(u_\nu) + h f(u_{\nu,i}) \quad (i = 0, 1, 2, \ldots ; \ \nu \text{ fixed}),$$

$$u_{\nu,0} = u_\nu .$$

It follows $u_{\nu+1} \in K_r$.

(b) The sequence $\{u_\nu\}$ - uniquely defined by (a) - possesses a subsequence $\{u_{\nu'}\}$ which converges to some $u^* \in K_r$ satisfying $f(u^*) = 0$. The sequences $\{u_{\nu'+k}\}$ ($k = \pm 1, \pm 2, \dots$) then also converge to u^* . Each limiting point of $\{u_\nu\}$ is a solution of $f(u) = 0$.

(c) If $f(u) = 0$ has in $\overline{K}_r$ only isolated solutions, then the whole sequence $\{u_\nu\}$ converges to u^*

3. In this section we will now state our theorem on the iterative construction of a solution of nonlinear elliptic boundary value problems.

We apply Lemma 3 to the nonlinear elliptic boundary value problem. Thus we set

$$B := \overset{\circ}{W}_{m,2}\ ,\ B^* := \overset{\circ}{W}{}^{*}_{m,2}\ ,\ f(u) := -T(u),\ \varphi(u) := S(u)$$

and consider the following iteration process

(1) $$S(u_{\nu+1}) = S(u_\nu) - h\,T(u_{\nu+1})$$

and

(2) $$S(u_{\nu,i+1}) = S(u_\nu) - h\,T(u_{\nu,i}) \quad (i = 0,1,2,\dots;\ \nu \text{ fixed}),$$

$$u_{\nu,0} = u_\nu$$

to construct a solution of

(3) $$Tu = 0 \ .$$

By Lemma 1 it follows that (3) is equivalent to

(4) $$a(u,v) := \sum_{|\alpha| \le m} (A_\alpha(\cdot, \xi(u)), D^\alpha v) = 0$$

for all $v \in \overset{\circ}{W}_{m,2}$, i.e. (3) is equivalent to a weak solution of the nonlinear elliptic boundary value problem

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u(x), \ldots, (D^{m}u)(x)) = 0 \quad \text{for}$$

$x \in \Omega$,

$$(D^{\alpha}u)(x) = 0 \text{ for } x \in \partial\Omega, \; |\alpha| \leq m-1 \; .$$

We can now formulate

<u>Theorem</u>: Suppose that assumptions A and B hold. Let $u_0 \in K_r := \{u \in \overset{\circ}{W}_{m,2} : \|u - v_0\|_{m,2} < r\}$. Then there exists a constant $\tilde{h} > 0$ such that for $0 < h < \tilde{h}$ the following holds:

(a) The nonlinear problem (1) has in the ball $\{u \in \overset{\circ}{W}_{m,2} : \|u - u_\nu\|_{m,2} \leq \rho\}$ a unique solution $u_{\nu+1}$ satisfying $u_{\nu+1} \in K_r$, which can be obtained by the iteration process (2).

(b) The sequence $\{u_\nu\}$ - uniquely defined by (a) - possesses a subsequence $\{u_{\nu'}\}$ which converges to a solution u^* of (3) (i.e. (4)) satisfying $u^* \in K_r$. The sequences $\{u_{\nu'+k}\}$ ($k = \pm 1, \pm 2, \ldots$) then also converge to u^* . Each limiting point is a solution of (3) (i.e. (4)).

(c) If (3) has in $\overline{K}_r$ only isolated solutions, then the whole sequence $\{u_\nu\}$ converges to u^*

Proof: We apply Lemma 3. The conditions I(a) and II(a) follow by Lemma 1, while conditions I(b) and II(b) follow by Lemma 2. Let $h > 0, v \in K_r$ and suppose that

$u \in \overline{K}_{\kappa+\rho} := \{u \in \overset{\circ}{W}_{m,2} : \| u - v_0 \|_{m,2} \leq \kappa + \rho \}$

satisfies

$$S(u) = S(v) - h\,T(u)$$

then because S is linear we obtain by Lemma 2 and assumption B

$$\| u - v_0 \|^2_{m,2} = \langle S(u - v_0), u - v_0 \rangle = \langle S(v - v_0), u - v_0 \rangle$$
$$- h \langle T(u), u - v_0 \rangle$$
$$< \| v - v_0 \|_{m,2} \, \| u - v_0 \|_{m,2} < \kappa \, \| u - v_0 \|_{m,2} \quad ,$$

proving $u \in K_\kappa$, i.e. $u \notin \overline{K}_{\kappa,\rho}$. Condition IV follows by Lemma 1(a). Let $\{u_\nu\} \subset \overline{K}_\kappa$ such that $T(u_\nu) \to$ $\to 0$ then by the reflexivity of $\overset{\circ}{W}_{m,2}$ there exists a subsequence $\{u_{\nu'}\}$ converging weakly to some u . Thus we have

$$\lim_{\nu'} \sup \langle T(u_{\nu'}) - T(u), u_{\nu'} - u \rangle = - \langle T(u), u - u \rangle = 0 \, .$$

Hence by Lemma 1(a) $\{u_{\nu'}\}$ converges strongly to u , i.e. $u \in \overline{K}_\kappa$. By the continuity of T it follows $T(u) = 0$ proving Condition IV.

Remark: (a) The iteration process (1),(2) is rather complicated because it consists of a recursive sequence of iteration processes, but the assumptions are very mild. Furthermore one has only to solve a linear elliptic differential equations with constant coefficients of order $2m$.

(b) The application of a slightly more general form of

Lemma 3 to nonlinear ordinary differential equations with boundary conditions is given in [10] without the assumption that the differential operator is a potential operator.

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