Miroslav Katětov

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Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 2, 373--396

Persistent URL: `http://dml.cz/dmlcz/105424`

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13,2 (1972)

BAIRE CLASSIFICATION AND INFINITE PERCEPTRONS

Miroslav KATĚTOV, Praha

In this note, some connections are established between descriptive set theory (Baire classification) and mathematical theory of perceptrons.

Results applicable outside mathematics have been neither aimed at nor achieved. Most theorems contained in the note are rather easy, however, the investigation of possible links between both fields, which are seemingly remote, may be of some interest.

Some approaches of the theory of perceptrons can be of use in the descriptive theory of functions and sets; on the other hand, some ideas of the descriptive theory might suggest, though only indirectly, new viewpoints in the mathematical theory of perceptrons.

For the theory of perceptrons we refer to the books by F. Rosenblatt [5] and M. Minsky, S. Papert [4]. The neurophysiological background is sketched in F. Rosenblatt's monograph. A number of references concerning the connections between the theory of perceptrons and the problems of pattern recognition are contained in the book by Minsky and Papert.

AMS, Primary: 26A21, 54 C50 Secondary: 94 A20

Ref. Ž. 7.517, 3.969.5 51:155.001.57:612.82

As for the descriptive set theory, we refer to K. Kuratowski's book [3].

No definition of a perceptron (in the current sense) will be given in this note. However, in fact, a finite "perceptronic net" (as defined in § 2) is a perceptron with constant thresholds and weights.

On the whole, we use the current terminology and notation. The deviations are, as a rule, in accordance with E. Čech's book [2]; e.g., an ordered pair of elements x, y is denoted by $\langle x, y \rangle$. The terminology and notation connected with the descriptive theory will be introduced in § 3.

§ 1.

1.1. <u>Definition</u>. A graph $\langle A, \varphi \rangle$ will be called a <u>perceptronic graph without loops</u> (or simply a <u>perceptronic graph</u>) if, for any non-void $X \subset A$, we have $X - \varphi [X] \neq \emptyset$.

<u>Remark</u>. The condition above is equivalent to "there exists no sequence $\{x_n \mid n \in N\}$ with $x_{n+1} \varphi x_n$ for all n ". The class of graphs fulfilling this condition is well known. However, in view of concepts introduced later on, a term ("perceptronic") different from the current ones, is used.

1.2. <u>Proposition</u>. Let $\mathcal{A} = \langle A, \varphi \rangle$ be a graph. If there exists a family of ordinals, $\{\xi(u) \mid u \in A\}$ such that $\xi(u) < \xi(v)$ whenever $u \varphi v$ then $\mathcal{A}$ is a perceptronic graph. If $\mathcal{A}$ is a perceptronic graph, then there exists exactly one family of ordinals $\{\lambda(u) \mid u \in A\}$ such that (1) $\lambda(u) < \lambda(v)$ whenever $u \varphi v$, (2) if $\xi(u)$,

$u \in A$, are ordinals and $\xi(u) < \xi(v)$ whenever $u \varphi v$, then $\lambda(u) \leq \xi(u)$ for every $u \in A$.

This proposition is well known.

Definition. The ordinal $\lambda(u)$ will be called the order of u in $\mathcal{A}$ and will be denoted by $ord(u, \mathcal{A})$ or simply by $ord\, u$. The set of all points $u \in A$ of order ξ will be called the ξ-layer of $\mathcal{A}$ and will be denoted by $L_\xi \mathcal{A}$.

1.3. Definition. A perceptronic graph $\mathcal{A} = \langle A, \varphi \rangle$ will be called star-infinite at a point $x \in A$ if, for any finite set K, $\sup \{ord\, y + 1 \mid y \varphi x,\ y\ \mathrm{non} \in K\} = ord\, x$, star-infinite if it is star-infinite at every $x \in A$.

Observe that if $\mathcal{A}$ is star-infinite at $x \in A$ with $ord\, x > 0$, then $\varphi^{-1}[x]$ is infinite.

As usual, we call $\mathcal{A} = \langle A, \varphi \rangle$ finite, countable, etc., if A is finite, countable, etc., respectively.

1.4. Definition. Let $\mathcal{A} = \langle A, \varphi \rangle$ be a perceptronic graph. Let $\mathcal{A}'$ be a partial graph of $\mathcal{A}$ (i.e. $\mathcal{A}' = \langle A, \sigma \rangle$ with $\sigma \subset \varphi$). We shall call $\mathcal{A}'$ normal with respect to $\mathcal{A}$ if $ord(x, \mathcal{A}') = ord(x, \mathcal{A})$ for every $x \in A$.

1.5. Example. Define graphs $\mathcal{G}_\alpha$ for countable ordinals α as follows. Put $\mathcal{G}_0 = \langle (0), \emptyset \rangle$. If $\mathcal{G}_\beta = \langle G_\beta, \varphi_\beta \rangle$ have been defined for all $\beta < \alpha$ and (i) $ord\, x \leq \beta$ if $x \in G_\beta$, (ii) $\beta \in G_\beta$, $L_\beta \mathcal{G}_\beta = (\beta)$, then construct $\mathcal{G}_\alpha = \langle G_\alpha, \varphi_\alpha \rangle$ in the following way:

(1) if $\alpha = \gamma + 1$, then $G_\alpha = N \times G_\gamma \cup (\alpha)$, $\langle n, \gamma \rangle \varphi_\alpha \alpha$ for all $n \in N$, $\langle m, x \rangle \varphi_\alpha \langle n, y \rangle$ iff $m = n$, $x \varphi_\gamma y$;

(2) if α is a limit number, then G_α consists of α and all $\langle \xi , x \rangle$ where $\xi < \alpha$, $x \in G_\xi$; $\langle \xi , \xi \rangle \varphi_\alpha \alpha$ for all $\xi < \alpha$; $\langle \xi , x \rangle \varphi_\alpha \langle \eta , y \rangle$ iff $\xi = \eta < \alpha$, $x \varphi_\xi y$.

Then G_α are star-infinite countable perceptronic graphs, and the relations φ_α are single-valued.

1.6. <u>Proposition</u>. Every star-infinite countable perceptronic graph $\langle A , \varphi \rangle$ has a star-infinite normal partial graph $\langle A , \varphi' \rangle$ such that φ' is single-valued.

<u>Proof</u>. Let $\{ \langle a_i , b_i \rangle \mid i \in N \}$ be a sequence consisting of all $\langle y , x \rangle \in \varphi$. We are going to define relations φ_k and elements c_k, $k \in N$, in the following way.

Put $\varphi_0 = \varphi$. Suppose that, for some k, $\varphi_0 , \ldots , \varphi_k$, and $c_0 , \ldots , c_{k-1}$ have been defined in such a way that (1) $\varphi = \varphi_0 \supset \ldots \supset \varphi_k$, (2) $(\varphi - \varphi_k)^{-1}[A]$ is finite, (3) for $0 \leqq i < k$ we have $c_i \in A$, $\mathrm{ord}\, c_i \geqq \mathrm{ord}\, a_i$, $\langle c_i , b_i \rangle \in \varphi_{i+1}$, $\langle c_i , y \rangle \;\mathrm{non} \in \varphi_{i+1}$ for $y \neq b_i$ and $c_i = a_i$ whenever $\langle a_i , b_i \rangle \in \varphi_i$, (4) for $0 \leqq i < j < k$ we have $\langle c_i , b_i \rangle \in \varphi_j$.

We are to define φ_{k+1}, c_k in such a way that (1) - (4) are satisfied with k replaced by $k+1$. If $\langle a_k , b_k \rangle \in \varphi_k$, we put $c_k = a_k$. If $\langle a_k , b_k \rangle \;\mathrm{non} \in \varphi_k$, we choose an element c_k such that $\langle c_k , b_k \rangle \in \varphi_k$, $\mathrm{ord}\, c_k \geqq \mathrm{ord}\, a_k$; this is possible since $\langle A , \varphi \rangle$ is star-infinite, hence, by (2), we have $\sup \{ \mathrm{ord}\, y + 1 \mid y \varphi_k b_k \} = \mathrm{ord}\, b_k > \mathrm{ord}\, a_k$. Now let φ_{k+1} consist of $\langle c_k , b_k \rangle$ and of all $\langle y , x \rangle \in \varphi_k$ such that $y \neq c_k$.

It is easy to see that $\varphi_0, \dots, \varphi_{k+1}$, $c_0, \dots, c_k$ satisfy (1) - (4).

We construct φ_k, c_k for all $k \in N$ and put $\varphi' = \bigcap \varphi_k$. It is easy to prove that φ' possesses the properties required.

§ 2.

2.1. We denote by $\bar{R}$ the extended line $(-\infty) \cup R \cup (+\infty)$ endowed with the usual order and topology. As usual, we put $0.(+\infty) = 0.(-\infty) = 0$, $c.(+\infty) = +\infty$ if $c > 0$, etc. The sum of a finite family $\{\xi_k\}$, $\xi_k \in \bar{R}$, is defined (in the usual way) iff there are no k, ℓ with $\xi_k = +\infty$, $\xi_\ell = -\infty$. If ξ, η, ζ are in $\bar{R}$ and either $\xi \leqq \eta \leqq \zeta$ or $\xi \geqq \eta \geqq \zeta$, we shall say that η is between ξ and ζ.

Mappings into $\bar{R}$ will be called functions. A function f with values in R will be said a real-valued function. A function f assuming values $0, 1$ only will be called a 01-function.

"Space" will mean a topological space. The set of all functions, real-valued functions, 01-functions on a space (or a set) P will be denoted by $\bar{F}(P)$, $F(P)$, $F_{01}(P)$, respectively; $\bar{C}(P)$, $C(P)$, $C_{01}(P)$ will designate the corresponding sets of continuous functions.

2.2. If $\{\xi(k) \mid k \in K\}$ is a countable family of elements of $\bar{R}$, then $\Sigma\{\xi(k) \mid k \in K\}$ will designate its sum (if it exists); more explicitly, $\eta = \Sigma\{\xi(k) \mid k \in K\}$, where $\eta \in \bar{R}$, means that, for every neighborhood U of

η in $\overline{R}$, there exists a finite set $X \subset K$ such that $\Sigma \{ \xi(y) \mid y \in Y \}$ is defined and belongs to U whenever Y is finite, $X \subset Y \subset K$. If $f_k \in \overline{F}(P)$, P being a space or a set, then $\Sigma \{ f_k \mid k \in K \}$ designates the function f defined as follows: $f(p) = \Sigma \{ f_k(p) \mid k \in K \}$ for all $p \in P$.

2.3. We denote by M the set of all monotone functions $f \in \overline{F}(\overline{R})$ such that $f(\xi) \in R$ whenever $\xi \in R$. The set $F_{01}(\overline{R}) \cap M$ will be denoted by M_{01}. If $\tau \in M_{01}$ and τ is non-constant, then there exists exactly one $c \in \overline{R}$ such that every neighborhood of c intersects both $f^{-1}[0]$ and $f^{-1}[1]$; this c will be called the <u>threshold</u> of τ.

2.4. <u>Definition</u>. If $\langle A, \varphi \rangle$ is a countable perceptronic graph, and $\tau = \{ \tau_a \mid a \in A \}$, $\beta = \{ \beta_{yx} \mid y \varphi x \}$ are families, $\tau_a \in M$, $\beta_{yx} \in R$, then $\mathfrak{A} = \langle A, \varphi, \tau, \beta \rangle$ is called a <u>pseudo-perceptronic net</u>.

<u>Conventions</u>. If $\mathfrak{A} = \langle A, \varphi, \tau, \beta \rangle$ is a pseudo-perceptronic net, then $\mathfrak{A}\varphi$ will denote the relation consisting of all $\langle y, x \rangle \in \varphi$ with $\beta_{yx} \neq 0$; $\varphi^{-\omega}$ will denote the relation consisting of all $\langle u, v \rangle$ such that, for some $n \in N$ and some $x_0, \ldots, x_n$, we have $u = x_0$, $x_0 \varphi x_1 \varphi \ldots \varphi x_n$, $x_n = v$; $(\mathfrak{A}\varphi)^{-\omega}$ is defined analogously.

2.5. <u>Convention</u>. In what follows, $\mathfrak{A}$ will always designate a pseudo-perceptronic net $\langle A, \varphi, \tau, \beta \rangle$.

2.6. <u>Definition</u>. Suppose that, for every $x \in A$ of order > 0, the following holds: if, for every

$y \in (\mathcal{A}\varphi)^{-1}[x]$, μ_y lies between $\tau_y(-\infty), \tau_y(+\infty)$, them $\{\beta_{yx}\mu_y \mid y \varphi x\}$ has a sum. Then $\mathcal{A}$ is called a quasi-perceptronic net.- If, in addition, $\tau_u \in M_{01}$ for all $u \in A$, then $\mathcal{A}$ is called a perceptronic net. - If, for every $x \in A$, either $\tau_x(\xi) = 0$ for all $\xi \in \bar{R}$ or $(\mathcal{A}\varphi)^{-1}[x]$ is finite, then $\mathcal{A}$ will be called finitary.

Remarks. 1) $\mathcal{A}$ is quasi-perceptronic iff, for every $x \in A$ of order > 0, both $\Sigma\{\beta_{yx}\tau_y(-\infty) \mid y \varphi x\}$ and $\Sigma\{\beta_{yx}\tau_y(+\infty) \mid y \varphi x\}$ exist and if one of them equals $+\infty$, then the other is distinct from $-\infty$. - 2) If, for every $x \in A$, $\tau_x \in M_{01}$ and $\{\beta_{yx} \mid y \varphi x\}$ has a sum, then $\mathcal{A}$ is a perceptronic net. In particular, $\mathcal{A}$ is a perceptronic net if all τ_x, $x \in A$, are in M_{01} and, for every $x \in A$, either all β_{yx} are non-negative or all β_{yx} are non-positive.

2.7. Convention. Terms and symbols introduced for perceptronic graphs $\mathcal{A} = \langle A, \varphi\rangle$ will be applied to corresponding nets $\mathcal{A} = \langle A, \varphi, \tau, \beta\rangle$, unless the contrary is explicitly stated. E.g., $L_\xi \mathcal{A}$ will mean $L_\xi \mathcal{A}$, etc.

Convention. We shall write L_ξ instead of $L_\xi \mathcal{A}$, etc.

2.8. Definition. A family $\{\lambda_z \mid z \in L_0 \mathcal{A}\}$, where $\lambda_z \in R$, will be called an initial numerical input (or simply input) of $\mathcal{A}$. If P is a space (or a set), then a family $\{h_z \mid z \in L_0 \mathcal{A}\}$, where $h_z \in F(P)$, will be called an initial input of real-valued functions (sometimes simply input) of $\mathcal{A}$.

2.9. Definition. Let $\{\lambda_z\}$ be an initial numerical input of $\mathcal{A}$. Let $u \in A$. Put $B = (\mathcal{A}\varphi)^{-\omega}[u]$. Assume that there exist families $\{\mu_x \mid x \in B\}$, $\{\nu_x \mid x \in B\}$ such that

(1) $\mu_x \in \bar{R}$, $\nu_x \in \bar{R}$, (2) if $x \in B \cap L_0$, then $\mu_x = \lambda_x$, (3) $\mu_x = \Sigma \{\beta_{yx} \nu_y \mid y \varphi x\}$ whenever $x \in B$, x *non* $\in L_0$, (4) $\nu_x = \tau_x(\mu_x)$ whenever $x \in B$. Then we shall say that the net $\mathcal{O}\!\mathcal{L}$ and the initial input $\{\lambda_z\}$ <u>generate the summary input</u> μ_u and the <u>output</u> ν_u <u>at</u> u .

The summary input and the output (at a point $u \in A$) generated by an initial input $\{h_z\}$ of functions (on a space or set P) are defined analogously.

<u>Convention</u>. Any function $f \in \bar{F}(P)$ generated, as an output (summary input) at a point $u \in A$, by $\mathcal{O}\!\mathcal{L}$ and an initial input of continuous functions on P will be called a <u>continuously generated output (input) function</u> of the net $\mathcal{O}\!\mathcal{L}$ at u , or, for short, an <u>output (input) function</u> of $\mathcal{O}\!\mathcal{L}$ at u .

2.10. <u>Proposition</u>. If a net $\mathcal{O}\!\mathcal{L}$ and an initial input $\{\lambda_z\}$ or $\{h_z\}$ (where $\lambda_z \in R$, $h_z \in F(P)$) generate a summary input $\mu_u \in \bar{R}$ (or $g_u \in \bar{F}(P)$) at a point $u \in A$, then both the input and the output at u are determined unequivocally (in more detail: if e.g. $\{\mu_x\}, \{\nu_x\}$ as well as $\{\mu'_x\}, \{\nu'_x\}$ satisfy conditions (1) to (4) in 2.9, then $\mu_x = \mu'_x$, $\nu_x = \nu'_x$ for all $x \in B$).

In fact, this proposition is almost evident. A formal proof proceeds by a standard transfinite induction.

2.11. <u>Proposition</u>. Let $\{\lambda_z\}$, where $\lambda_z \in R$, or $\{h_z\}$ $h_z \in F(P)$, be an initial input of $\mathcal{O}\!\mathcal{L}$. Let C denote the set of points $u \in A$ such that the initial input generates a summary input and an output at u . Then $(\mathcal{O}\!\mathcal{L}_\varphi)^{-1}[C] \subset C$.

The proof may be omitted.

2.12. Theorem. If $\mathfrak{A} = \langle A, \varphi, \tau, \beta \rangle$ is a quasi-perceptronic net or a finitary pseudo-perceptronic net, then every initial numerical input or an input of real-valued functions on a space P generates an input and an output at every point of A.

The proof proceeds by a standard transfinite induction.

2.13. If a countable perceptronic graph $\langle A, \varphi \rangle$ is given, we may consider various $\tau = \{\tau_a\}$, $\beta = \{\beta_{y,x}\}$ such that $\langle A, \varphi, \tau, \beta \rangle$ is a pseudo-perceptronic net, and also various inputs $\{\lambda_x\}$, $\{h_x\}$. In what follows, we shall investigate mainly the case when τ, β are fixed and $\{h_x\}$ are continuous functions on a given P.

However, first we consider two cases where τ or β may vary.

2.14. Definition. A quasi-perceptronic net $\mathfrak{A}$ will be called (1) monothetic if $\mathfrak{A}\varphi$ is single-valued, (2) normally monothetic if $\mathfrak{A}\varphi$ is single-valued and $\langle A, \mathfrak{A}\varphi \rangle$ is a normal partial graph of $\langle A, \varphi \rangle$.

2.15. Theorem. Let $\mathfrak{A} = \langle A, \varphi, \tau, \beta \rangle$ be a quasi-perceptronic net such that $\langle A, \mathfrak{A}\varphi \rangle$ is star-infinite. Then there exists a β' such that $\langle A, \varphi, \tau, \beta' \rangle$ is a normally monothetic quasi-perceptronic net and $\beta'_{y,x} = 0$ whenever $\beta_{y,x} = 0$.

The proof is similar to that of 1.6.

2.16. Given a perceptronic graph $\langle A, \varphi \rangle$, we may consider all output functions generated by $\langle A, \varphi, \tau, \beta \rangle$ at a given point $u \in A$, where $\beta = \{\beta_{y,x}\}$ is fixed whereas τ_a (and possibly also the initial input $\{h_x\}$, $h_x \in F(P)$) vary. If A is finite, then there is a certain connection with problems of so-called linear superpositions of functions. For a survey and bibliography concerning these we refer to [6]. Only one theorem from this field will be paraphrased in terms of perceptrons here.

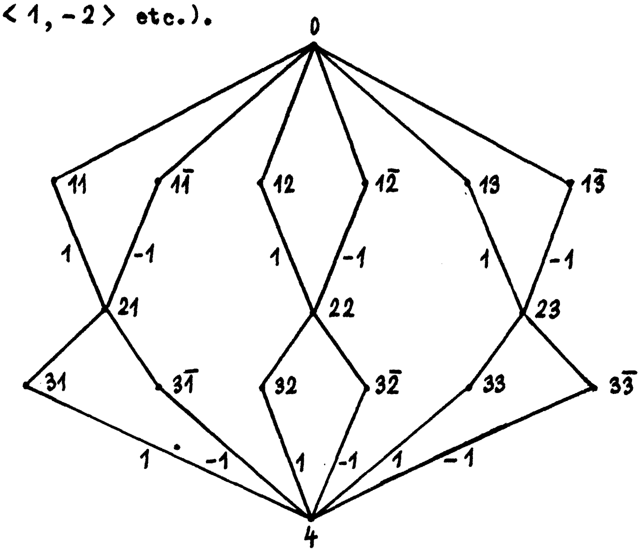
<u>Theorem</u>. Let A consist of points 0, $\langle 1, \pm i \rangle, \langle 2, i \rangle$, $\langle 3, \pm i \rangle$, 4, where $i = 1, 2, 3$. Let a relation φ on A be defined as follows:

$0 \varphi \langle 1, \pm i \rangle, \langle 1, \pm i \rangle \varphi \langle 2, i \rangle, \langle 2, i \rangle \varphi \langle 3, \pm i \rangle$, $\langle 3, \pm i \rangle \varphi 4$. If $\langle y, x \rangle \in \varphi$, and $y = \langle 1, -i \rangle$ or $y = \langle 3, -i \rangle$, then we put $\beta_{yx} = -1$; for all other pairs $\langle y, x \rangle \in \varphi$, we put $\beta_{yx} = 1$.

For $a = 0$, $a = \langle 2, i \rangle$, $a = 4$ let τ_a be fixed, $\tau_a(\xi) = \xi$ for all $\xi \in \bar{R}$. For other points $a \in A$, let τ_a belong to M and be continuous non-decreasing; otherwise let τ_a be arbitrary.

Let P be a set, and let h be a real-valued function on P. Then the set of all output functions generated by h at the point 4 of the perceptronic net described above (with $\tau = \{\tau_a\}$ satisfying the conditions mentioned) is equal to the set of all functions of the form $g \circ h$, where $g : R \longrightarrow R$ is continuous.

We omit the proof since the theorem is, in fact, a paraphrase of a theorem proved by N. Bari [1]. The perceptronic graph described in the theorem (and some weights β_{yx}) are represented on the following diagram (we write $1\bar{2}$ instead of $\langle 1, -2 \rangle$ etc.).



<u>Remark</u>. The above theorem is more complicated than the original one proved by N. Bari. The theorem given here may be of some interest, since it shows interrelations of seemingly quite different topics, and perhaps also since similar reformulations of known theorems could help to compare the degree of their complexity (expressed e.g. in terms of the complexity of the pertinent perceptronic graph).

§ 3.

3.1. We introduce some symbols and terms concerning descriptive set theory. Then we shall state some well known theorems and also some less known (or perhaps not occurring in the literature as yet) propositions which will be needed later on. Current propositions of the descriptive theory will be applied without reference. A detailed exposition can be found e.g. in K. Kuratowski's book [3].

Convention. If f is a function on P, and $c \in R$, then $[f > c]$ will designate the set of all $x \in P$ such that $f(x) > c$; and similarly, $[f < c]$, $[f = c]$, etc.

Convention. If $\mathcal{X}$ is a collection of sets, then $\sigma\mathcal{X}$ denotes the collection of all $\bigcup\{X_m \mid m \in N\}$, and $\delta\mathcal{X}$ denotes that of all $\bigcap\{X_m \mid m \in N\}$, where $X_m \in \mathcal{X}$.

Convention. A collection of sets $\mathcal{U}$ is called finitely multiplicative if $U_1 \in \mathcal{U}$, $U_2 \in \mathcal{U}$ implies $U_1 \cap U_2 \in \mathcal{U}$. The multiplicity of a family of sets $\{X_k \mid k \in K\}$ is the least cardinal m such that $\bigcap\{X_k \mid k \in B\} = \emptyset$ whenever $B \subset K$, $\operatorname{card} B > m$.

Definition. Let P be a space. Sets $[f > 0]$ where $f \in C(P)$ will be called sets of additive class (or σ-class) 0, their complements will be called sets of multiplicative class (or δ-class) 0. If α is a countable ordinal and sets of class β have been defined for all $\beta < \alpha$, the sets of class α are defined as follows: sets $\bigcup B_m$ and $\bigcap B_m$ where B_m is a set of class β_m, $\beta_m < \alpha$, are of additive class α and multiplicative class α, respectively. We shall say that X is a set of σ-class (δ-class) $< \alpha$ if it is a set of σ-class (δ-class) β for some $\beta < \alpha$.

Observe that according to the definition every set of σ-class (δ-class) β is a set of σ-class (δ-class) α for every $\alpha > \beta$.

Definition. A function f on P will be called a function of class $(\alpha, \uparrow)$ (of class $(\alpha, \downarrow)$) if all $[f > c]$ (all $[f < c]$) are sets of additive class α.

If f is both of class $(\alpha, \uparrow)$ and $(\alpha, \downarrow)$ (hence, if all $f^{-1}[G]$, $G \subset \bar{R}$ open, are of σ-class α), then f will be called a function of class α. If every $[f > c]$ and every $[f < c]$ is a set of σ-class α or δ-class α, then f will be called a function of class $(\alpha, *)$.

Remarks. 1) Every $f \in M$ (see 2.3) is a function of class $(0, *)$. - 2) Every function of class $(\alpha, *)$ is of class $\alpha + 1$.

3.2. We recall two important well known theorems of descriptive theory. Observe that these theorems are usually proved (e.g. in K. Kuratowski's book [3]) for metrizable spaces; however, the statements and proofs remain valid for arbitrary spaces if descriptive classes of sets and functions are defined as above (3.1).

A) Theorem (see e.g. K. Kuratowski [3], § 30. V). Every set of additive class α, $\alpha > 1$, is a union of countably many disjoint sets of multiplicative class α.

B) Theorem (see loc. cit., § 31). Let $\alpha > 0$ be a countable ordinal. Every pointwise limit of a sequence of functions of class α is a function of class $\alpha + 1$. If α is a limit ordinal, then every function of class α is a pointwise limit of a sequence of functions of classes $< \alpha$; if α is isolated, then every function of class α is such a limit of functions of class $\alpha - 1$.

3.3. Lemma. Let K be a countably infinite set. Let $\{\beta_k \mid k \in K\}$ be a family of non-negative real numbers and let $\{\gamma_k \mid k \in K\}$ be a family of countable ordinals. Put $\alpha = \sup\{\gamma_k + 1\}$ and assume $\Sigma\{\beta_i \mid i \in K$,

$\gamma_i \geq \gamma$, $\beta_i < \varepsilon \} = \infty$ for every $\gamma <$ $< \alpha$ and every positive real number ε . Let P be a set, $\mathcal{U}$ a collection of subsets of P, q a natural number. Assume that (1) $\mathcal{U}$ is finitely multiplicative, (2) if $U \in \mathcal{U}$, then $P - U \in \sigma\mathcal{U}$, (3) if $V \in \sigma\mathcal{U}$, then there exists a countable collection $\mathcal{V} \subset \mathcal{U}$ with a multiplicity $\leq q$ such that $\bigcup \mathcal{V} = V$. Assume that there are given sets $\Phi(\gamma)$, $\gamma < \alpha$, of 01 -functions on P such that $0 \in \Phi(0)$, $\Phi(\gamma) \subset \Phi(\gamma')$ whenever $\gamma < \gamma' < \alpha$ and that $\bigcup \{\Phi(\gamma) \mid \gamma < \alpha\}$ consists of all $f \in F_{01}(P)$ with $[f = 1] \in \mathcal{U}$.

Then, for any non-negative function $f \in \overline{F}(P)$ such that $[f > c] \in \sigma\mathcal{U}$ for each $c \in R$, there exists a family $\{f_k \mid k \in K\}$ such that $f_k \in \Phi(\gamma_k)$ and $f =$ $= \sum \beta_k f_k$.

<u>Proof</u>. Let Ψ denote the set of all non-negative $f \in$ $\in \overline{F}(P)$ such that $[f > c] \in \sigma\mathcal{U}$ for all $c \in R$; let Φ denote the set of all $g \in F_{01}(P)$ such that $[g = 1] \in$ $\in \mathcal{U}$,

I. We are going to prove: if $f \in \Psi$, $f \leq q + 1$, then there exists a sequence $\{g_n \mid n \in N\}$ such that $g_n \in \Phi$ and $g \leq f$, $f - g \leq q$, $f - g \in \Psi$, where $g = \sum g_n$.

In fact, by (3) there exists a sequence $\{Y_n\}$ such that $Y_n \in \mathcal{U}$, the multiplicity of $\{Y_n\}$ does not exceed q , $\bigcup Y_n = [f > q]$. Let $g_n \in F_{01}(P)$ be such that $[g_n = 1] = Y_n$. It is easy to see that $\{g_n\}$ possesses properties mentioned above; e.g., $f - g \in \Psi$ follows from the property (2) of the collection $\mathcal{U}$.

II. From the assertion above, it follows that if $f \in \Psi$ is bounded, then there exist families $\{g_{mn} \mid m \in N, n \in N\}$, $\{\lambda_m \mid m \in N\}$ such that $g_{mn} \in \Phi$, λ_m are non-negative reals, $f = \sum \{\lambda_m g_{mn} \mid m \in N, n \in N\}$.

III. If $f \in \Psi$ is not bounded, then, for $m \in N$, $p \in P$, we put: $f_m(p) = 0$ if $f(p) \leqq m$; $f_m(p) = f(p) - m$ if $m < f(p) < m + 1$; $f_m(p) = 1$ if $f(p) \geqq m + 1$. Then $f = \sum f_m$, $f_m \in \Psi$, $f_m \leqq 1$.

IV. By I - III, the following holds: if $f \in \Psi$, then there exist countable families $\{g_s \mid s \in S\}$, $\{\mu_s \mid s \in S\}$ such that $g_s \in \Phi$, $\mu_s \in R$, $\mu_s \geqq 0$, $f = \sum \mu_s g_s$.

V. Choose, for each $s \in S$, an index $k(s) \in K$ such that $g_s \in \Phi(\gamma_{k(s)})$. The assumptions on $\{\beta_k \mid k \in K\}$ imply that there exist disjoint sets $Z_s \subset K$ such that $\sum \{\beta_k \mid k \in Z_s\} = \mu_s$ and that, for every $s \in S$ and every $k \in Z_s$, we have $\gamma_k \geqq \gamma_{k(s)}$, hence $g_s \in \Phi(\gamma_k)$. Now put $f_k = g_s$ if $k \in Z_s$, and $f_k = 0$ if $k \in K$, $k \text{ non} \in \bigcup Z_s$. The family $\{f_k \mid k \in K\}$ possesses the properties required.

3.4. <u>Lemma</u>. Let K be a countably infinite set. Let $\{\beta_k \mid k \in K\}$ be a family of non-negative reals, and let $\{\gamma_k \mid k \in K\}$ be a family of countable ordinals. Put $\alpha = \sup \{\gamma_k + 1\}$ and suppose that $\sum \{\beta_k \mid k \in K, \gamma_k \geqq \gamma\} = \infty$ for each $\gamma < \alpha$. Let P be a set and let $\mathcal{U}$ be a collection of subsets of P. Suppose that there are given sets $\Phi(\gamma)$, $\gamma < \alpha$, of 01-functions on P such that

$0 \in \Phi(0)$, $\Phi(\gamma) \subset \Phi(\gamma')$ whenever $\gamma < \gamma' < \alpha$ and that $\bigcup \{\Phi(\gamma) \mid \gamma < \alpha\}$ consists of all 01-functions f on P such that $[f = 1] \in \mathcal{U}$.

Then (I) if $X \in \sigma \mathcal{U}$ and $c \in R$, then there exists a family $\{f_k \mid k \in K\}$ such that $f_k \in \Phi(\gamma_k)$ and $[\sum \beta_k f_k > c] = X$, (II) if $\mathcal{U}$ is such that every $V \in \sigma \mathcal{U}$ is a union of a point-finite countable collection $\mathcal{V} \subset \mathcal{U}$, then, for every $Y \in \delta\sigma \mathcal{U}$, there exists a family $\{f_k \mid k \in K\}$ such that $f_k \in \Phi(\gamma_k)$ and $[\sum \beta_k f_k = \infty] = Y$.

<u>Proof</u>. We are going to prove the assertion (I). Let $X = \bigcup \{X_n \mid n \in N\}$, $X_n \in \mathcal{U}$. Let g_n be 01-functions on P , $[g_n = 1] = X_n$. Then g_n is in some $\Phi(\delta_n)$, $\delta_n < \alpha$. The properties of the family $\{\beta_k\}$ imply that there exist disjoint finite sets $Z_n \subset K$ such that $\sum \{\beta_k \mid k \in Z_n\} > c$ and $\gamma_k \geqq \delta_n$ whenever $k \in Z_n$. Put $f_k = g_n$ if $k \in Z_n$, and $f_k = 0$ if $k \in K$, k non $\in \bigcup Z_n$. Clearly,
$[\sum \beta_k f_k > c] = \bigcup X_n$, $f_k = g_n \in \Phi(\delta_n) \subset \Phi(\gamma_k)$.

We now prove assertion (II). We have $Y = \bigcap Y_n$, where $Y_n \in \sigma \mathcal{U}$, $Y_n \supset Y_{n+1}$ for every $n \in N$. By the assumptions on $\mathcal{U}$, we have $Y_n = \bigcup \{Y_{nm} \mid m \in N\}$ where $Y_{nm} \in \mathcal{U}$ and the families $\{Y_{nm} \mid m \in N\}$ are point-finite. Let $g_{nm} \in F_{01}(P)$, $[g_{nm} = 1] = Y_{nm}$. Choose a δ_{nm} such that $g_{nm} \in \Phi(\delta_{nm})$. The assumptions on $\{\beta_k\}$ imply that there exist disjoint finite sets $Z_{nm} \subset K$ such that $\sum \{\beta_k \mid k \in Z_{nm}\} > 1$ and $\gamma_k \geqq \delta_{nm}$ whenever $k \in Z_{nm}$. We put $f_k = g_{nm}$ if $k \in Z_{nm}$,

$f_k = 0$ if $k \in K$, k non $\in \bigcup Z_{nm}$. It is easy to show that $\{f_k\}$ possesses the properties mentioned in the assertion (II).

3.5. <u>Lemma</u>. Let P be a space. If $X \subset P$ is a set of additive class 1, then there exist sets X_{nm} of multiplicative class 0 such that the multiplicity of the family $\{X_{nm} \mid n \in N, m \in N\}$ does not exceed 1 and $X = \bigcup \{X_{nm} \mid n \in N, m \in N\}$.

<u>Proof</u>. There exists a sequence $\{Y_n\}$ such that $X = \bigcup Y_n$ and Y_n are sets of σ-class 0. We may suppose that $Y_n \subset Y_{n+1}$ for each $n \in N$. There exist non-negative functions $g_n \in C(P)$ such that $[g_n = 0] = Y_n$. Put $Z_{n0} = [g_n \geq 1]$, $Z_{nm} = [\frac{1}{m} \geq g_n \geq \frac{1}{m+1}]$ for $m = 1, 2, \ldots$. Put $X_{00} = Y_0$, $X_{0m} = \emptyset$ for $m = 1, 2, \ldots$, $X_{nm} = Y_n \cap Z_{n-1,m}$ for $n \in N$, $n > 0$, $m \in N$. It is easy to see that $\{X_{nm}\}$ has the properties mentioned above.

§ 4.

4.1. <u>Convention</u>. If α is an ordinal, then $\alpha - 1$ designates the least ξ such that $\xi + 1 \geq \alpha$, and 2α designates the ordinal defined as follows: $2 . 0 = 0$, $2\alpha = \sup \{2\beta + 2 \mid \beta < \alpha\}$; thus $2\alpha = \alpha$ if α is a limit ordinal.

4.2. <u>Theorem</u>. Let $\mathcal{A} = \langle A, \varphi, \tau, \beta \rangle$ be a pseudo-perceptronic net. Let P be a space. If $x \in A$, $\operatorname{ord} x = \alpha$, then (1) every continuously generated input function at x

is of class 2α if α is finite, and of class $2\alpha + 1$ if α is infinite, (2) every continuously generated output function at x is of class $(2\alpha, *)$ if α is finite, and of class $(2\alpha+1, *)$ if α is infinite.

Proof. The assertion is evident if $\alpha = 0$. Let α be a countable ordinal and assume that the assertion has been proved for all points $y \in A$ of order $< \alpha$. Let $x \in A$, $\operatorname{ord} x = \alpha$; let g be an input function at x generated by an initial input $\{h_x\}$, $h_x \in C(P)$. Then $g = \sum \{\beta_{yx} f_y \mid y \varphi x\}$, where f_y is an output function at y generated by $\{h_x\}$. Put $\gamma_y = \operatorname{ord} y$.

By the assumption, f_y is a function of class $2\gamma_y + 1$ if γ_y is finite, and of class $2\gamma_y + 2$ if γ_y is infinite. Then, by 3.2 B, g is a function of class $(2(\alpha - 1) + 1) + 1 = 2\alpha$ if α is finite; if α is infinite, then g is a function of class $\sup\{2\gamma_y + 2 \mid y \varphi x\} + 1$, hence of class $2\alpha + 1$.

4.3. For some kinds of nets, the above estimate of the class of input and output functions can be improved; see the following theorems 4.4, 4.5. However, for a certain rather general class of nets, the estimate in 4.2 is, in essence, precise; see Theorem 4.7 below.

4.4. Theorem. Let $\mathfrak{A} = \langle A, \varphi, \tau, \beta \rangle$ be a perceptronic net. Suppose that $\beta_{yx} \geqq 0$ for all $\langle y, x \rangle \in \varphi$ and that all τ_x, $x \in A$, are functions of class $(0, \downarrow)$. Let P be a space. If $x \in A$, $\operatorname{ord} x = \alpha$, then (1) if $\alpha > 0$, then every continuously generated input function at x is a function of class $(2\alpha - 1, \uparrow)$ if α is finite, and a function of class $(2\alpha, \uparrow)$ if α is infinite; (2) every continuously generated output function at x is a 01-function of class $(2\alpha, \downarrow)$ if α is finite, and of class $(2\alpha+1, \downarrow)$ if α is infinite.

Proof. The assertion is trivial if $\alpha = 0$. Let α be a countable ordinal and assume that (1),(2) have been proved for all ordinals $< \alpha$. Let $x \in A$, $\mathrm{ord}\, x = \alpha$. Let g be an input function at x generated by an initial input $\{h_z\}$, $h_z \in C(P)$. Then $g = \Sigma\{\beta_{yx} f_y \mid y \,\varphi\, x\}$, where f_y is an output function at y. By the assumption, f_y are 01-functions of class $(2\,\mathrm{ord}\, y, \downarrow)$. Since β_{yx} are non-negative, this implies that every $[g < c]$ is of σ-class γ, where $\gamma = \sup\{\mathrm{ord}\, y \mid y \,\varphi\, x\} + 1$ if α is isolated, $\gamma = \sup\{\mathrm{ord}\, y \mid y \,\varphi\, x\}$ if α is a limit number.

In the first case we get $\gamma = 2\alpha - 1$ if α is finite, $\gamma = 2\alpha$ if α is infinite; in the second case, $\gamma = 2\alpha$. This proves that g is of class $(2\alpha - 1, \uparrow)$ or $(2\alpha, \uparrow)$ according to whether α is finite or infinite. Thus, (1), and hence also (2), holds at x.

Remark. Assuming, in addition, that every function τ_u is non-increasing, we obtain a stronger result: input functions at x are of class $(\mathrm{ord}\, x, \uparrow)$, output functions are of class $(\mathrm{ord}\, x, \downarrow)$.

4.5. Theorem. Let $\mathfrak{A} = \langle A, \varphi, \tau, \beta \rangle$ be a pseudoperceptronic net. Suppose that every τ_u, $u \in A$, is continuous. If $x \in A$, $\mathrm{ord}\, x = \alpha$, then every input or output function at x is of class α if α is finite, and of class $\alpha + 1$ if α is infinite.

We omit the proof, which is similar to the proofs of 4.2 and 4.4.

4.6. Definition. Let $\mathcal{O}\!\mathcal{L} = \langle A, \varphi, \tau, \beta \rangle$ be a perceptronic net. Let $x \in A$. We shall say that $\mathcal{O}\!\mathcal{L}$ is (1) sum-infinite at x if, for every $\gamma < ord\, x$, $\Sigma \{\beta_{yx} \mid y \varphi x,\ ord\, y \geqq \gamma\}$ is infinite, (2) finely sum-infinite at x if, for every $\gamma < ord\, x$ and every real $\varepsilon > 0$, $\Sigma \{\beta_{yx} \mid y \varphi x, ord\, y \geqq \gamma; |\beta_{yx}| < \varepsilon\}$ is infinite, (3) coarsely sum-infinite at x if it is sum-infinite at x without being finely sum-infinite. If $\mathcal{O}\!\mathcal{L}$ is sum-infinite (finely sum-infinite, coarsely sum-infinite) at every $x \in A$ of order > 0, we shall call $\mathcal{O}\!\mathcal{L}$ sum-infinite (finely sum-infinite, coarsely sum-infinite).

4.7. Theorem. Let $\mathcal{O}\!\mathcal{L}$ be a normally monothetic sum-infinite perceptronic net. Assume that every β_{yx} is non-negative, and that every τ_x, $x \in A$, is right-continuous non-decreasing and has a positive threshold. Assume that all τ_x, $x \in L_0$, have a finite threshold and that τ_x has an infinite threshold whenever $\mathcal{O}\!\mathcal{L}$ is coarsely sum-infinite at x. Let a space P be given.

Then for any $x \in A$ of order α we have

(1) if $\alpha > 0$ and $\mathcal{O}\!\mathcal{L}$ is finely sum-infinite at x, then the set of all continuously generated input functions at x coincides with that of all non-negative functions on P of class $(2\alpha - 1, \uparrow)$ if α is finite, and of class $(2\alpha, \uparrow)$ if α is infinite;

(2) the set of all continuously generated output functions at x coincides with that of all 01-functions on P of class $(2\alpha, \downarrow)$ if α is finite, and of class $(2\alpha + 1, \downarrow)$ if α is infinite.

Proof. The assertion holds for $\alpha = 0$, since τ_x, $x \in L_0$ are of class $(0, \downarrow)$ and have finite thresholds.

Suppose that $\alpha > 0$ is a countable ordinal and that the assertion has been proved for all points y of order $< \alpha$. Let $x \in A$ be of order α; we are going to prove (1),(2) for the point x.

Put $\psi\alpha = 2\alpha - 1$ if $\alpha > 0$ is finite, $\psi\alpha = 2\alpha$ if α is infinite.

Denote by $\mathcal{U}$ the collection of all sets $U \subset P$ of δ-class $< \psi\alpha$. Clearly, $\mathcal{U}$ is finitely multiplicative. If $U \in \mathcal{U}$, then $P - U$ is a set of some σ-class $\delta < \psi\alpha$, hence it is the union of a sequence of sets of δ-class $< \delta$ and therefore belongs to $\sigma\mathcal{U}$. If a set V is in $\sigma\mathcal{U}$, then it is a set of σ-class $\psi\alpha$. If $\alpha > 1$, then, by 3.2 A, V is the union of a disjoint sequence of sets of δ-class $< \psi\alpha$; if $\alpha = 1$, then $\psi\alpha = 1$, and, by 3.5, we have $V = \bigcup\{V_n \mid n \in N\}$, where V_n are sets of δ-class 0 and the multiplicity of $\{V_n\}$ does not exceed 1. Thus, $\mathcal{U}$ possesses properties indicated in 3.3.

Clearly, $\sigma\mathcal{U}$ consists of all sets $W \subset P$ of σ-class $\psi\alpha$, hence the set of all $f \in \overline{F}(P)$ such that, for every $c \in R$, we have $[f > c] \in \sigma\mathcal{U}$ coincides with that of all $f \in \overline{F}(P)$ of class $(\psi\alpha, \uparrow)$.

Put $D = (\mathcal{O}p)^{-1}[x]$; for every $d \in D$, put $\gamma_d = ord\, d$. Since $\mathcal{O}$ is normally monothetic, we have $sup(\gamma_d + 1) = \alpha$. For every $\gamma < \alpha$, let $\Phi(\gamma)$ denote the set of all 01-functions on P of class $(\psi\gamma + 1, \downarrow)$. It is easy to see that $\bigcup\{\Phi(\gamma) \mid \gamma < \alpha\}$ consists of all 01-functions f on P such that

$[f = 1] \in \mathcal{U}$.

We are now going to prove that assertion (1) holds for the point x . Since, by the assumption, $\mathcal{O}\!\!\mathcal{L}$ is finely sum-infinite at x , it is easy to see that the assumptions of Lemma 3.3 are fulfilled (with K replaced by D).

Now let $g \in \overline{F}(P)$ be a non-negative function of class $(\psi\alpha, \uparrow)$. Then $[g > c] \in \sigma\mathcal{U}$ for every $c \in R$, hence, by Lemma 3.3, there exists a family $\{f_d \mid d \in D\}$ such that $g = \Sigma\{\beta_{dx} f_d \mid d \in D\}$, and, for each $d \in D$, $f_d \in \Phi(\gamma_d)$, hence f_d is of class $(\psi\gamma_d + 1, \downarrow)$.

For every $d \in D$, put $B_d = L_0 \cap (\mathcal{O}\!\!\mathcal{L}\varphi)^{-\omega}[d]$. Since $\mathcal{O}\!\!\mathcal{L}$ is monothetic, $\{B_d\}$ is a disjoint family. Since it is assumed that (1),(2) hold for all points of order $< \alpha$, there exists, for every $d \in D$, an initial input $\{h_{dz} \mid z \in L_0\}$, $h_{dz} \in C(P)$, which generates the output f_d at d ; we may assume that $h_{dz} = 0$ if $z \in L_0$, z *non* $\in B_d$. Now put $h^*_z = h_{dz}$ if $z \in B_d$, $h^*_z = 0$ if $z \in L_0 - \cup B_d$. The initial input $\{h^*_z\}$ generates the output f_d at d , hence the summary input g at x . We have proved that every non-negative function of class $(\psi\alpha, \uparrow)$ is an input function at x . The fact that every input function at x is of class $(\psi\alpha, \uparrow)$ follows easily from Theorem 4.4. Thus we have shown that assertion (1) holds for x .

Now we prove that (2) holds for x . By the assumptions in the theorem, two cases are possible: (i) $\mathcal{O}\!\!\mathcal{L}$ is finely sum-infinite at x , and the threshold of τ_x is finite, (ii) the threshold of τ_x is infinite, hence $\tau_x(\xi) = 0$ for $\xi < \infty$, $\tau_x(\infty) = 1$.

Consider the first case. We may suppose that the threshold of τ_x is equal to 1. Let f be a 01-function on P of class $(\psi\alpha + 1, \downarrow)$. Put $X = [f = 1]$. It is easy to show that there exists a function $g \in \overline{F}(P)$ of class $(\psi\alpha, \uparrow)$ such that $0 \leq g \leq 1$, $[g = 1] = X$. According to already proved, g is an output function at x. Thus $f = \tau_x \circ g$ is an output function at x.

Consider the second case. Let f be a 01-function on P of class $(\psi\alpha + 1, \downarrow)$. Put $X = [f = 1]$. It is easy to see that $X \in \sigma\delta\,\mathcal{U}$. Hence, by 3.4, there exists a family $\{f_d \mid d \in D\}$ such that $f_d \in \Phi(\gamma_d)$ and $[\Sigma\beta_d f_d = \infty] = X$. For every $d \in D$, f_d is a 01-function of class $(\psi\gamma_d + 1, \downarrow)$, hence an output function at d. Since $\mathcal{A}$ is monothetic, this implies that $g = \Sigma\beta_d f_d$ is an input function at x, hence $f = \tau_x \circ g$ is an output function at x.

In both cases, it is easy to see that, conversely, every output function at x is of class $(\psi\alpha + 1, \downarrow)$. This concludes the proof.

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Matematický ústav
Karlova universita
Sokolovská 83
Praha 8
Československo

(Oblatum 16.5.1972)