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A NOTE ON THE RIEMANN CURVATURE TENSOR

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In Paper [2] the problem was discussed whether, and how, a Riemann metric can be derived from a "generalized" curvature tensor, under a natural assumption of regularity. The purpose of this Note is to extend our results to a wider class of curvature tensors.

We shall start with some preparatory lemmas.

Lemma 1. Let V be a real vector space with a positive scalar product g. Let $G \subset O(V)$ be a connected Lie group of orthogonal transformations of V and $\mathfrak{g} \subset \mathfrak{o}(V)$ its Lie algebra. Then for any symmetric bilinear form h on V the following is true:

h is invariant with respect to G if and only if for any $A \in \mathfrak{g}$ and $X, Y \in V$

$$h(AX, Y) + h(X, AY) = 0 . \tag{1}$$

Proof. See [1], Chapter I.

Lemma 2. (See [1], Appendix 5.) Let G be a subgroup of $O(m)$ which acts irreducibly on the m-dimensional coordinate space R^m. Then any symmetric bilinear form

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on $\mathbb{R}^n$ which is invariant by G is a multiple of the standard scalar product

$$(x, y) = \sum_{i=1}^{n} x^i y^i$$

Let $\mathcal{L}$ be a set of linear endomorphisms of a vector space V. Put

$$(2)\ \Theta(\mathcal{L}) = \{h \in S^2(V) \mid h(AX, Y) + h(X, AY) = 0;\ X, Y \in V,\ A \in \mathcal{L}\}$$

where $S^2(V)$ denotes the space of all symmetric bilinear forms on V.

We say that $\mathcal{L}$ <u>generates a Lie algebra</u> $\mathfrak{g} \subset \mathfrak{gl}(V)$ if $\mathfrak{g}$ is the least Lie subalgebra of $\mathfrak{gl}(V)$ containing $\mathcal{L}$. Finally, $G(\mathcal{L})$ will denote the connected subgroup of $GL(V)$ whose Lie algebra is generated by $\mathcal{L}$.

<u>Proposition 1</u>. Let V be a vector space with a (positive) scalar product g and $G \subset O(V)$ an irreducible Lie group of orthogonal transformations of V. Let $\mathcal{L} \subset \mathfrak{o}(V)$ be a set of linear endomorphisms generating the Lie algebra $\mathfrak{g}$ of G. Then

(i) $\dim \Theta(\mathcal{L}) = 1$, i.e., $\Theta(\mathcal{L}) = (g)$.

(ii) If $X \in V$ and $AX = 0$ for any $A \in \mathcal{L}$, then $X = 0$.

<u>Proof</u>. ad (i). If $\mathcal{L} = \mathfrak{g}$, the assertion is nothing else than an infinitesimal version of Lemma 2 (cf. Lemma 1). In a general case we have $\Theta(\mathfrak{g}) \subset \Theta(\mathcal{L})$. Put $\mathcal{L}' = \{A \in \mathfrak{g} \mid \Theta(\mathcal{L}) \subset \Theta(\{A\})\}$. Because $\Theta(L') = \bigcap \Theta(\{A\})$ $(A \in \mathcal{L})$, we get $\Theta(\mathcal{L}') \supset \Theta(\mathcal{L})$.

It suffices to show that $\mathcal{L}' = \mathfrak{g}$. Clearly, if

$A, B \in \mathcal{L}$, then $\alpha A + \beta B \in \mathcal{L}'$. Now, for any $X \in V$, $h \in \Theta(\mathcal{L})$, $A, B \in \mathcal{L}$, $h([A,B]X, X) = h(ABX, X) - h(BAX, X) = -h(BX, AX) + h(AX, BX) = 0$, and hence $[A,B] \in \mathcal{L}'$.

ad (ii). Let first $\mathcal{L} = \mathfrak{g}$. Then if a non-zero $X \in V$ exists with $AX = 0$ for any $A \in \mathfrak{g}$, the corresponding group G pointwise preserves the vector subspace $(X) \subset V$ and hence G is not irreducible - a contradiction.

Now, let $\mathcal{L} \subset \mathfrak{g}$ be general, and let $X \in V$ be such that $AX = 0$ for any $A \in \mathcal{L}$. Then the same is true for any $B \in \mathfrak{g}$. This completes the proof.

Let B be a tensor of type $(1,3)$ on a vector space V, i.e., a bilinear map of $V \times V$ into $\mathfrak{gl}(V)$. Then $\mathcal{B} = \{B(X,Y) \mid X, Y \in V\}$ is a subset of $\mathfrak{gl}(V)$ and we shall put

$$G(B) \overset{def}{=\!=} G(\mathcal{B}), \quad \Theta(B) \overset{def}{=\!=} \Theta(\mathcal{B}) .$$

Following [2], a linear map $B : V \wedge V \longrightarrow \mathfrak{gl}(V)$ is called <u>regular</u> if the endomorphism $B(X \wedge Y)$ is non-trivial for any $X \wedge Y \neq 0$. (We can write also $B(X,Y)$ instead of $B(X \wedge Y)$ as B corresponds to a unique anti-symmetric bilinear map of $V \times V$ into $\mathfrak{gl}(V)$.)

Further, suppose that a scalar product g on V exists satisfying $g(B(U,T)Y, X) = -g(B(U,T)X, Y)$, $g(B(U,T)X, Y) = g(B(X,Y)U, T)$, for any $U, T, X, Y \in V$. Then B is called <u>a curvature structure with respect to</u> g. Now, we have

Proposition 2. Let V be a vector space provided with a scalar product g and let $B : V \wedge V \longrightarrow gl(V)$ be a regular curvature structure with respect to g. Then the group $G(B)$ is an irreducible subgroup of $O(V)$.

Proof. The inclusion $G(B) \subset O(V)$ is obvious because $\mathcal{B} \subset o(V)$. We show that $G(B)$ is irreducible. According to [2], Lemma 1, for any two vectors $X \perp Y$ of V there are transformations $B(U_i \wedge T_i)$ $(U_i, T_i \in V, i = 1, \ldots, k)$ such that $\sum_{i=1}^{k} B(U_i \wedge T_i)X = Y$. If the group $G(B)$ were reducible, the corresponding Lie algebra generated by $\{B(U \wedge T) \mid U, T \in V\}$ would possess a proper invariant subspace $V' \subset V$, a contradiction.

Let (M, g) be a Riemann manifold of class C^∞ having the curvature tensor R. Following C. Teleman [4], the space (M, g) is called non-divisible if, at each point $x \in M$, the group $G(R_x)$ is irreducible. It is obvious that each non-divisible Riemann manifold is irreducible (see [1], Ch.III.,IV.).

More generally, we shall call a tensor field B of type $(1, 3)$ on (M, g) non-divisible if the group $G(B_x)$ is irreducible for each $x \in M$.

Further, the tensor field B is called a curvature structure with respect to g (or on (M, g)) if so is each algebraic tensor B_x $(x \in M)$. For example, the Riemann curvature tensor R of (M, g) and the corresponding Weyl tensor of conformal curvature C are curvature structures on (M, g).

According to Proposition 2, <u>any regular curvature structure on (M, g) is non-divisible</u>. (Here "regular" means "regular at each point $x \in M$".)

One can re-write Proposition 1 as follows:

<u>Proposition 3</u>. Let (M, g) be a Riemann space (of class C^∞) and B a non-divisible curvature structure on (M, g). Then

(i) $\dim(B_x) = 1$ for each $x \in M$, i.e., $\Theta(B) = \cup\Theta(B_x)(x \in M)$ is a line bundle; and g is a section of $\Theta(B)$

(ii) If $B(X,Y)Z = 0$ for any vector fields X, Y on M then Z is a null field.

Now, we can see easily that <u>Theorem 2 and all the paragraphs 3 - 7 of [2] remain true if we replace the word "regular" by the word "non-divisible" everywhere</u>. Particularly, we get the following theorems (the reader is referred to the original paper [2] for details).

<u>Theorem 1</u>. (C. Teleman, [4].) Let (M, g) be a connected non-divisible Riemann space of dimension $n \geq 3$, and let Φ be a curvature tensor-preserving diffeomorphism of (M, g) onto a Riemann space (M', g'). Then Φ is a homothety.

<u>Corollary</u>. (See K. Nomizu and K. Yano, [3].) Let (M, g) be a connected, analytic, irreducible, locally symmetric Riemann space of dimension $n \geq 3$ and let Φ be a curvature tensor-preserving diffeomorphism of (M, g)

onto a Riemann space (M', g'). Then Φ is a homothety.

Proof of the corollary: one can see easily that, for any point $x \in M$, $G(R_x)$ is the restricted homogeneous holonomy group of (M, g) at x. Thus (M, g) is non-divisible.

Theorem 2. (Cf.[2], paragraph 5 for details.) Let B be a non-divisible tensor field of type $(1, 3)$ on a C^∞-manifold M, $\dim M \geq 3$. Then one can decide whether or not B is locally a Riemann curvature tensor only by algebraic operations and differentiations.

Theorem 3. Let M be a C^∞-manifold, $\dim M \geq 3$. A local reconstruction of a non-divisible Riemannian metric g on M from its curvature tensor R requires only algebraic operations, differentiations and the integration of an exact differential.

References

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