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A CONSEQUENCE OF A THEOREM OF L. FUCHS

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In his paper [2] L. Fuchs has proved that any direct summand of a separable torsion free abelian group is likewise separable. As a simple consequence of this result we get the following proposition generalizing a well known Baer theorem [1, Theorem 49.1].

Theorem. A separable torsion free abelian group G is completely decomposable if and only if G belongs to some Baer class Γ_α .

Proof. The necessity being obvious, it remains to prove the sufficiency of the below pronounced condition only. We start with the formulation of the following two propositions from [3]:

Lemma 1. If A_i $(i = 1, \dots, n)$ are torsion free groups such that $A_i \in \Gamma_{\alpha_i}$ $(i = 1, \dots, n)$ then there exists an ordinal $\beta \leq max(\alpha_1, \dots, \alpha_n)$ with $A_1 \oplus \dots \oplus A_n \in \Gamma_\beta$. (See [3, Lemma 4].)

Lemma 2. Let B be a pure subgroup of finite rank in a torsion free group A . If $A \in \Gamma_\beta$ then $A/B \in \Gamma_\gamma$ for some $\gamma \leq \beta$. (See [3, Lemma 5].)

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Now, let G be a separable group belonging to some Γ_α ; by induction on α we shall prove that G is completely decomposable. For $\alpha = 1$, G is countable and the complete decomposability of G follows by a Baer theorem ([1, Theorem 49.1]). Thus let $1 < \alpha$ and assume our assertion has been proved for all separable groups belonging to Γ_β with $\beta < \alpha$. By the definition of Baer classes there exists a pure subgroup S of finite rank in G such that $G/S = \overline{G} = \sum_{i \in I} \oplus \overline{G}_i$, where $\overline{G}_i \in \Gamma_{\alpha_i}$, $\alpha_i < \alpha$ $(i \in I)$.

The group G being separable, S is contained in a completely decomposable direct summand H of finite rank in G ; thus we have

(1) $$G = H \oplus K \quad .$$

Since H is completely decomposable it suffices to show that K is so as well. First of all we may write

(2) $$K \cong G/H \cong (G/S)/(H/S) \quad .$$

If we put $H/S = \overline{H}$ then $\overline{H}$ is a pure subgroup of finite rank in $\overline{G} = \sum_{i \in I} \oplus \overline{G}_i$ and hence we get $\overline{H} \subseteq \overline{G}_{i_1} \oplus \dots \oplus \overline{G}_{i_m}$ for suitable indices $i_1, \dots, i_m \in I$. Defining $\overline{F} = \overline{G}_{i_1} \oplus \dots \oplus \overline{G}_{i_m}$ and $J = I \setminus (i_1, \dots, i_m)$, we have $\overline{G} = \overline{F} \oplus \sum_{i \in J} \overline{G}_i$ and $\overline{H} \subseteq \overline{F}$. By Lemma 1 it is $\overline{F} \in \Gamma_\beta$ where $\beta \leq max(\alpha_{i_1}, \dots, \alpha_{i_m}) < \alpha$. In view of (2) we infer that

(3) $$K \cong \bar{G}/\bar{H} \cong \bar{F}/\bar{H} \oplus \sum_{i \in J} \oplus \bar{G}_i .$$

Since $\bar{F} \in \Gamma_\beta$, Lemma 2 implies $\bar{F}/\bar{H} \in \Gamma_\gamma$ with $\gamma \leq \beta < \alpha$. By the Fuchs theorem from [2] and (1) K is separable; hence in view of (3) by the same Fuchs theorem we conclude that the groups $\bar{F}/\bar{H}$ and $\bar{G}_i$ $(i \in J)$ are likewise so. Thus by inductive hypothesis the groups $\bar{F}/\bar{H}$, $\bar{G}_i$ $(i \in J)$ and simultaneously K are completely decomposable. This finishes the proof.

References

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