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# ON CERTAIN SUM IN NUMBER THEORY

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Let $r$ be a natural number and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be given real numbers, $M_1, M_2, \dots, M_r$ positive integers. For a real $t$ denote by $\langle t \rangle$ the distance of $t$ from the nearest integer, i.e. $\langle t \rangle = \min_{p \text{ int.}} |t - p|$ and, for a positive integer $k$, let

$$P_k = \max_{j=1,2,\dots,r} \langle \alpha_j M_j k \rangle .$$

In the papers [2] - [7] there it is shown that $O$-estimates in the theory of lattice points in ellipsoids with weight can, in an important special case, be reduced to $O$-estimates of the function

$$(1) \qquad F(x) = F_{\varphi,\beta}(x;\alpha_j) = \sum_{k \le \sqrt{x}} k^{\varphi} \min{}^{\beta}\left(\frac{\sqrt{x}}{k}, \frac{1}{P_k}\right)$$

( $\varphi$ and $\beta$ are non-negative real numbers, $x$ is a real number $\ge 0$ ; for $B = 0$ we put $\min(A, \frac{1}{B}) = A$ ). In [3] , for example, it is proved that for $\alpha_1 = \alpha_2 = \dots = \alpha_r = \alpha$, $\varphi = 0$, $\beta = \frac{r}{2} - 1$, $r \ge 6$ it holds for every $\varepsilon > 0$

$$x^{\frac{r}{4}-\frac{1}{2}} F(x) = O(x^{(\frac{r}{4}-\frac{1}{2})\frac{2\gamma+1}{\gamma+1}+\varepsilon}),$$

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AMS, Primary 10H25
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Ref. Ž. 1.945, 1.947

where $\gamma$ is the least upper bound of all the numbers $\gamma_1$ satisfying the inequality

$$\langle \alpha k \rangle \leqq k^{-\gamma_1}$$

for infinitely many natural $k$ .

The aim of the present paper is the study of the function (1). namely the investigation of their $O$ - and $\Omega$ -estimates, especially in dependence on the character of the system $\alpha_1, \alpha_2, \dots, \alpha_k$ . (Although the direct applications in the theory of lattice points have the $O$ - estimates of the function (1), it is evidently worth while also to study its $\Omega$ -estimates and other asymptotic properties.) Special cases of particular theorems or proofs (for $\varphi = 0$ ) can be found in the papers [2] - [5], where they are not, of course, stated explicitely.

In the sequel, let the letter $c$ denote (in general, different) positive constants depending only on $\alpha_j$ and $M_j$, $j = 1, 2, \dots, k$, $\varphi$ and $\beta$ . We write shortly $A \ll B$ instead of $|A| \leqq cB$ ; if, in addition, it is $B \ll A$ , we write $A \asymp B$ . The symbols $O$, $o$ and $\Omega$ have their usual meaning and refer to the limiting process $x \to +\infty$ ; the constants involved in their definitions are of the "type $c$ " with one exception: $O$ -relations (as well as $o$ and $\Omega$ relations) involving a positive parameter $\varepsilon$ can have constants still depending on this $\varepsilon$ . $k$ and $m$ denote always positive integers, $u$ and $v$ non-negative integers. Let the number $x$ be big enough, i.e. $x > c$ . Instead of the function (1) we write more exactly $F_{\varphi,\beta}(x; \alpha_j, M_j)$

or $F(x; \alpha_1, \alpha_2, \dots, \alpha_k)$ etc.

For completeness sake, let us bring the following two simple statements.

Theorem 1. It is always

$$x^{\frac{\varphi+1}{2}} << F(x) << x^{\beta/2} \quad \text{for } \varphi < \beta - 1 ,$$

$$x^{\frac{\varphi+1}{2}} << F(x) << x^{\beta/2} \lg x \quad \text{for } \varphi = \beta - 1 ,$$

$$x^{\frac{\varphi+1}{2}} << F(x) << x^{\frac{\varphi+1}{2}} \quad \text{for } \varphi > \beta - 1 .$$

Proof. It is $P_k < 1$, $k \leq \sqrt{x}$ , therefore

$$F(x) \geq \sum_{k \leq \sqrt{x}} k^{\varphi} >> x^{\frac{\varphi+1}{2}}$$

But clearly, it is also

$$F(x) \leq x^{\beta/2} \sum_{k \leq \sqrt{x}} k^{\varphi-\beta} ,$$

whence the assertion follows easily.

The upper estimates generally cannot be improved, as it is shown by the next theorem.

Theorem 2. Let the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ be rational and let $H$ be the least common denominator of the numbers $\alpha_1 M_1, \alpha_2 M_2, \dots, \alpha_k M_k$ . Then it is

$$F(x) = \frac{x^{\beta/2}}{H^{\beta-\varphi}} \sum_{n=1}^{\infty} \frac{1}{n^{\beta-\varphi}} + O(x^{\frac{\varphi+1}{2}}) \quad \text{for } \varphi < \beta - 1 ,$$

$$F(x) = \frac{x^{\beta/2}}{2H} \lg x + O(x^{\frac{\varphi+1}{2}}) \quad \text{for } \varphi = \beta - 1 ,$$

$$F(x) = K x^{\frac{\varphi+1}{2}} + O(x^{\varphi/2}) + O(x^{\beta/2}) \quad \text{for } \varphi > \beta - 1$$

with a suitable positive constant $K = c$ .

Proof. If $k \equiv j \pmod{H}$ , then

$P_k = P_j \quad (j = 1, 2, \ldots, H), P_H = 0, P_j > 0 \quad (j = 1, 2, \ldots, H-1).$

Thus it is sufficient (for $j = 1, 2, \ldots, H$ ) to consider the sums

$$\sum_{\substack{k \leq \sqrt{x} \\ k \equiv j \pmod H}} k^{\varphi} \min{}^{\beta}\left(\frac{\sqrt{x}}{k}, \frac{1}{P_j}\right).$$

For $j = H$ we get

$$x^{\beta/2} \sum_{\substack{k \leq \sqrt{x} \\ H \mid k}} k^{\varphi-\beta} = x^{\beta/2} H^{\varphi-\beta} \sum_{m \leq \frac{\sqrt{x}}{H}} m^{\varphi-\beta} ,$$

which can, for $\varphi < \beta - 1$ or $\varphi = \beta - 1$ or $\varphi > \beta - 1$ , be brought to

$$x^{\beta/2} H^{\varphi-\beta} \left(\sum_{m=1}^{\infty} m^{\varphi-\beta} + O\left(x^{\frac{\varphi-\beta+1}{2}}\right)\right)$$

or

$$x^{\beta/2} H^{\varphi-\beta} \left(\lg \frac{\sqrt{x}}{H} + c + O\left(\frac{1}{\sqrt{x}}\right)\right)$$

or

$$x^{\beta/2} H^{\varphi-\beta} \left(H^{\beta-\varphi-1} \frac{x^{\frac{\varphi-\beta+1}{2}}}{\varphi - \beta + 1} + c + O\left(x^{\frac{\varphi-\beta}{2}}\right)\right) ,$$

respectively. But for $j = 1, 2, \ldots, H-1$ we get

$$\frac{1}{P_j^{\beta}} \sum_{\substack{k \leq P_j\sqrt{x} \\ k \equiv j \pmod H}} k^{\varphi} + x^{\beta/2} \sum_{\substack{P_j\sqrt{x} < k \leq \sqrt{x} \\ k \equiv j \pmod H}} k^{\varphi-\beta} = \sum_{\substack{k \leq \sqrt{x} \\ k \equiv j \pmod H}} k^{\varphi} \min{}^{\beta}\left(\frac{\sqrt{x}}{k}, \frac{1}{P_j}\right)$$

which, for $\beta \geq \varphi + 1$ , shall be trivially estimated by

$$\frac{1}{P_j^{\beta}} \sum_{k \leq \sqrt{x}} k^{\varphi} << x^{\frac{\varphi+1}{2}} ,$$

whereas for $\beta < \varphi + 1$ we shall use the obvious expressions

$$\sum_{\substack{k \leq P_j\sqrt{x} \\ k \equiv j \pmod H}} k^{\varphi} = \frac{P_j^{\varphi+1} x^{\frac{\varphi+1}{2}}}{H(\varphi + 1)} + O\left(x^{\varphi/2}\right) ,$$

$$\sum_{\substack{P_j\sqrt{x}<k\le\sqrt{x}\\ k\equiv j \pmod{H}}} k^{\varphi-\beta} = \frac{1-P_j^{\varphi-\beta+1}}{H(\varphi-\beta+1)}\, x^{\frac{\varphi-\beta+1}{2}} + O(x^{\frac{\varphi-\beta}{2}}) .$$

From the relations brought above the assertion of the theorem immediately follows.

But if we discard the cases covered by this theorem, then for $\varphi < \beta - 1$ the upper estimate of Theorem 1 can be improved, but, in general, only to $o(x^{\beta/2})$ .

Theorem 3. Let $\varphi < \beta - 1$ and let $\Phi(x) = \Phi(x ; \alpha_1 ; \alpha_2, \dots, \alpha_k)$ be a positive function of $k + 1$ variables in the domain $x > 0$, $[\alpha_1, \alpha_2, \dots, \alpha_k] \in \mathfrak{M} = \langle 0, \frac{1}{M_1}) \times \langle 0, \frac{1}{M_2}) \times \dots \times \langle 0, \frac{1}{M_k})$ such that it always hold

$$0 \le \Phi(x ; \alpha_j) \ll F(x ; \alpha_j)$$

Then the next is true:

1) If at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ is irrational, then it is

$$\Phi(x ; \alpha_1, \alpha_2, \dots, \alpha_k) = o(x^{\beta/2}) .$$

2) Let the function $\Phi(x ; \alpha_1, \alpha_2, \dots, \alpha_k)$ for every $x > 0$ be continuous in $\mathfrak{M}$ . Let there exist a set $\mathfrak{N} \subset \mathfrak{M}$ dense in $\mathfrak{M}$ and such that for each $k$-tuple $[\alpha_1, \alpha_2, \dots, \alpha_k] \in \mathfrak{N}$ it is

$$\limsup_{x \to +\infty} \Phi(x ; \alpha_1, \alpha_2, \dots, \alpha_k)\, x^{-\beta/2} > 0 .$$

Then for every positive non-increasing function $\varphi(x)$, $\lim_{x\to+\infty} \varphi(x) = 0$ there exists a set $\mathfrak{N}_\varphi \subset \mathfrak{M}$ of the first category in $\mathfrak{M}$ such that for each $k$-tuple $[\alpha_1, \alpha_2, \dots, \alpha_k] \in \mathfrak{M} - \mathfrak{N}_\varphi$ it is

$$\Phi(x ; \alpha_1, \alpha_2, \dots, \alpha_k) = o(x^{\beta/2}), \quad \Phi(x ; \alpha_1, \alpha_2, \dots, \alpha_k) = \Omega(x^{\beta/2}\varphi(x))$$

In particular, it can be put $\Phi(x) = F(x)$ .

<u>Proof</u>. For the proof of the first part of this theorem note that $P_k \neq 0$ for all $k$, and, for each $x > c$, determine a natural number $\psi(x)$ so that

$$\sum_{k \leq \psi(x)} \frac{k^\varphi}{P_k^\beta} \leq \frac{x^{\beta/2}}{lg\, x} < \sum_{k \leq \psi(x)+1} \frac{k^\varphi}{P_k^\beta} .$$

The function $\psi(x)$ is non-decreasing, $\lim_{x\to+\infty} \psi(x) = +\infty$, therefore

$$F(x) \leq \sum_{k \leq \psi(x)} \frac{k^\varphi}{P_k^\beta} + x^{\beta/2} \sum_{k > \psi(x)} k^{\varphi-\beta} = o(x^{\beta/2}) .$$

To prove the second part of the assertion, we shall use the method of categories, the usefulness of which in the number theory has been brought to attention by V. Jarník (cf. its analogous application in [2], pp.447-449). From the first part of the theorem it follows that if $[\alpha_1, \alpha_2, \dots, \alpha_k] \in \mathfrak{N}$ then the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ are rational. Let $\mathfrak{N}_1$ be the set of all $[\alpha_1, \alpha_2, \dots$ $\dots, \alpha_k] \in \overline{\mathfrak{M}}$ with $\alpha_1, \alpha_2, \dots, \alpha_k$ rational.

For each $n$ let $\mathfrak{M}_n$ be the set of all those $[\beta_1, \beta_2, \dots, \beta_k] \in \text{int}\, \mathfrak{M}$ , for which there exists $x = x(n, \beta_1, \beta_2, \dots, \beta_k) > n$ such that

$$\Phi(x; \beta_1, \beta_2, \dots, \beta_k) > x^{\beta/2} \varphi(x)\, n .$$

For a fixed $x$ , the function $\Phi(x; \alpha_1, \alpha_2, \dots, \alpha_k)$ (as a function of variables $\alpha_1, \alpha_2, \dots, \alpha_k$ ) is continuous in $\mathfrak{M}$ , hence the sets $\mathfrak{M}_n$ are open. If we choose $[\alpha_1, \alpha_2, \dots, \alpha_k] \in \mathfrak{N}$ , then, by assumption, it is

$$\Phi(x; \alpha_1, \alpha_2, \dots, \alpha_k) \geq c\, x^{\beta/2}$$

for arbitrary large $x$. So, for a given $n$, there surely exists $x > n$ such that

$$\frac{\Phi(x;\alpha_1,\alpha_2,\dots,\alpha_k)}{x^{\beta/2}\varphi(x)} \geqq \frac{c}{\varphi(x)} > n .$$

Thus it is $\mathfrak{N} \subset \mathfrak{M}_n$ for all $n$. The sets $\mathfrak{M}_n$ are open and dense in $\mathfrak{M}$, and therefore the sets $\mathfrak{M} - \mathfrak{M}_n$ (and clearly also the sets $\overline{\mathfrak{M}} - \mathfrak{M}$) are nowhere dense in $\overline{\mathfrak{M}}$, $\mathfrak{N}_1$ is of the first category in $\overline{\mathfrak{M}}$. Since $\overline{\mathfrak{M}}$ is a complete space, for

$$[\beta_1,\beta_2,\dots,\beta_k] \in \overline{\mathfrak{M}} - \mathfrak{N}_1 \cup \bigcup_n (\overline{\mathfrak{M}} - \mathfrak{M}_n) = (\mathfrak{M} - \mathfrak{N}_1) \cap \bigcap_n \mathfrak{M}_n = \mathfrak{M} - \mathfrak{N}_\varphi \neq \emptyset$$

(the set $\mathfrak{N}_\varphi$ is therefore of the first category in $\mathfrak{M}$), we have:

a) for each $n$ there exists $x > n$ such that

$$\Phi(x;\beta_1,\beta_2,\dots,\beta_k) > n\,x^{\beta/2}\varphi(x)$$

i.e.

$$\Phi(x;\beta_1,\beta_2,\dots,\beta_k) = \Omega(x^{\beta/2}\varphi(x)) .$$

b) $$\Phi(x;\beta_1,\beta_2,\dots,\beta_k) = o(x^{\beta/2})$$

(since $[\beta_1,\beta_2,\dots,\beta_k] \in \mathfrak{M} - \mathfrak{N}_1$).

Since the function $F(x;\alpha_1,\alpha_2,\dots,\alpha_k)$ is obviously continuous in $\mathfrak{M}$ for each fixed $x$, it follows from the theorem 2 that in the second part of the theorem there can be put $\Phi(x) = F(x)$.

<u>Remark 1</u>. Since

$$\frac{k}{\sqrt{x}} \min\left(\frac{\sqrt{x}}{k}, \frac{1}{P_k}\right) \leqq 1$$

we can write for $0 \leqq t \leqq \min(\varphi,\beta)$

$$F_{\varphi,\beta}(x;\alpha_j) \leqq x^{\frac{t}{2}} F_{\varphi-t,\beta-t}(x;\alpha_j) .$$

Hence for $\varphi \leqq \beta$ it can be put $t = \varphi$ and thus it suffices to find the estimates for the case $\varphi = 0$ only.

The generally valid estimate $F(x) \gg x^{\frac{\varphi+1}{2}}$ (see Theorem 1), cannot, in general, be much improved even in the case $\varphi < \beta - 1$. Namely, it holds

<u>Theorem 4</u>. Let $1 < \beta - \varphi \leqq \kappa - 1$. Then for almost all the systems $[\alpha_1, \alpha_2, \ldots, \alpha_\kappa]$ (in the sense of the Lebesgue measure in $E_\kappa$) it is

$$F(x) \ll x^{\frac{\varphi+1}{2}} lg^{\tau} x ,$$

where $\tau = 3\kappa - 1$ for $\beta - \varphi < \kappa - 1$, $\tau = 3\kappa + 2$ for $\beta - \varphi = \kappa - 1$.

<u>Proof</u>. From the assumptions, the inequality $\kappa \geqq 2$ follows and since for almost all $[\alpha_1, \alpha_2, \ldots, \alpha_\kappa]$ it is $P_k \neq 0$ for every $k$, it is sufficient, by Remark 1, to prove for almost all $[\alpha_1, \alpha_2, \ldots, \alpha_\kappa]$ the estimate

$$\sum_{k \leqq \sqrt{x}} \frac{1}{P_k^{\beta}} \ll \sqrt{x}\, lg^{\tau} x .$$

This estimate is stated, for $\beta = \kappa - 1 + \varphi$, in [4], p.619; using the inequality $\beta - \varphi < \kappa - 1$ in the final part of the proof, we shall also obtain the above sharpening for these (pairs) $\beta$ and $\kappa$.

In the following two theorems, we shall bring an improvement of both $O$ - and $\Omega$ -estimates for certain special systems $[\alpha_1, \alpha_2, \ldots, \alpha_\kappa]$.

<u>Theorem 5</u>. Let $\varphi < \beta - 1$, $\gamma > 0$ and let the inequality

$$P_k << k^{-\gamma}$$

be fulfilled for infinitely many $k$. Then

$$F(x) = \Omega(x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}}) .$$

Proof. Let $k_1, k_2, \dots$ be an increasing sequence of positive integers such that $P_{k_m} << k_m^{-\gamma}$, $m = 1, 2, \dots$. Let $x_m = k_m^{2(\gamma+1)}$, i.e. $k_m < \sqrt{x_m}$, and therefore

$$F(x_m) \geq k_m^{\varphi} \min^{\beta}\left(\frac{\sqrt{x_m}}{k_m}, \frac{1}{P_{k_m}}\right) >> k_m^{\beta\gamma+\varphi} = x_m^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}}) ,$$

q.e.d.

Theorem 6. Let $\varphi < \beta - 1$, $\gamma > 0$ and let the inequality

$$P_k >> k^{-\gamma}$$

hold for all $k$. Then

$$F(x) = O(x^{\frac{\beta\gamma+\varphi+1}{2(\gamma+1)}}) .$$

Proof. Let $\alpha = \dfrac{1}{2(\gamma+1)} < \dfrac{1}{2}$. Clearly, it is

$$F(x) << \sum_{k \leq x^{\alpha}} \frac{k^{\varphi}}{P_k^{\beta}} + \sum_{k > x^{\alpha}} \frac{x^{\beta/2}}{k^{\beta-\varphi}} << \sum_{k \leq x^{\alpha}} k^{\beta\gamma+\varphi} + x^{\beta/2} \sum_{k > x^{\alpha}} k^{\varphi-\beta} << x^{\frac{\beta\gamma+\varphi+1}{2(\gamma+1)}},$$

q.e.d.

Remark 2. It can be easily seen that the simplification of the remark 1 gives no better results in Theorem 6.

Remark 3. Let $\varphi < \beta - 1$, and let $\gamma = \gamma(\alpha_1, \alpha_2, \dots, \alpha_k)$ be the least upper bound of all the numbers $\tau$, $\tau > 0$, for which the inequality

$$P_k << k^{-\tau}$$

has infinitely many solutions. From the preceding two theorems it follows that for every $\varepsilon > 0$ the estimates

$$F(x) = O(x^{\frac{\beta\gamma+\rho+1}{2(\gamma+1)}+\varepsilon}), \quad F(x) = \Omega(x^{\frac{\beta\gamma+\rho}{2(\gamma+1)}-\varepsilon})$$

hold (for $\gamma = +\infty$ we define the value of both fractions to be $\beta/2$ ), i.e. it holds

$$\frac{\beta\gamma+\rho}{2(\gamma+1)} \leqq \limsup_{x\to+\infty} \frac{\lg F(x)}{\lg x} \leqq \frac{\beta\gamma+\rho+1}{2(\gamma+1)} .$$

So for $\gamma = +\infty$ we get the final result

$$\limsup_{x\to+\infty} \frac{\lg F(x)}{\lg x} = \frac{\beta}{2} .$$

Remark 4. Let us list some properties of continuous fractions to be used in the sequel without mentioning. Let $\alpha$ be an irrational number and

$$\alpha = (a_0; a_1, a_2, \dots)$$

its (regular) continued fraction expansion. If $p_n/q_n$ denotes the $n$-convergent of $\alpha$, then $q_{-1} = 0$, $q_0 = 0$,

$$q_n = a_n q_{n-1} + q_{n-2} ,$$

$n = 1, 2, \dots$ . In this notation, it holds (cf. e.g.[1]), pp. 240-242, [3], pp.380-382):

a) If $u \geqq v \geqq 0$, then it is

$$q_u \geqq q_v \, 2^{[\frac{u-v}{2}]}$$

b) Let $1 \leqq k = u q_n + v < q_{n+1}$, $v < q_n$ . Then

$$\text{(2)} \qquad \langle \alpha k \rangle \asymp \frac{u}{q_{n+1}} \qquad (k = u q_n < q_{n+1}) ;$$

$$\text{(3)} \qquad \langle \alpha k \rangle \asymp \frac{a_{n+1} - u}{q_{n+1}} \qquad (k = u q_n + q_{n-1} < q_{n+1}).$$

If none of the above cases has place, then there exists a natural number $j$ ($j \leq q_n/2$) such that

$$(4) \qquad \langle \alpha k \rangle \asymp \frac{j}{q_n} \qquad (k = u q_n + v < q_{n+1}, v \neq 0, q_{n-1}).$$

The number $j$ can be chosen so that it depends only on $v$, $n$ and $\alpha$ (but is independent on $u$), and, conversely, to each positive integer $j$ ($j \leq q_n/2$) we can find for every $u \leq a_{n+1}$ at most two values $v$ so that (4) holds.

Remark 5. Let $T \geq 1$, $T = c$. Using Theorem 1, we can easily see that

$$\sum_{\sqrt{x} < k \leq \sqrt{Tx}} k^{\rho} \min^{\beta}\left(\frac{\sqrt{Tx}}{k}, \frac{1}{P_k}\right) << x^{\beta/2} \sum_{\sqrt{x} < k \leq \sqrt{Tx}} k^{\rho-\beta} << x^{\frac{\rho+1}{2}} << F(x).$$

From this we easily obtain

$$F(x) \leq F(Tx) << F(x).$$

Let $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$. If $m$ is a positive integer, then $\langle m\alpha \rangle \leq m \langle \alpha \rangle$, therefore $F_0(x) << << F(x)$, where $F_0(x)$ denotes the function $F(x)$ for $M_1 = M_2 = \dots = M_k = 1$. Let $T = \max_{j=1,2,\dots,k} M_j$. Then (using the preceding remark)

$$F(x) \leq \sum_{j=1}^{k} \sum_{k \leq \sqrt{x}} k^{\rho} \min^{\beta}\left(\frac{\sqrt{x}}{k}, \frac{1}{\langle \alpha M_j k \rangle}\right) \leq k F_0(T^2 x) << F_0(x)$$

and, on the whole, we get in this case

$$F(x) \asymp F_0(x).$$

The following and the main theorem of the whole present paper improve the $O$-estimate of Theorem 6.

Theorem 7. Let $\rho < \beta - 1$, $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$, $\gamma > 0$, and let for all natural $k$ be

(5) $$\langle \alpha k \rangle \gg k^{-\gamma} .$$

Then

$$F(x) \ll x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}} \qquad \text{for } \varphi < \beta - 2 ,$$

$$F(x) \ll x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}} + x^{\frac{\varphi+1}{2}} \lg x \qquad \text{for } \varphi = \beta - 2 ,$$

$$F(x) \ll x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}} + x^{\frac{\varphi+1}{2}} \qquad \text{for } \varphi > \beta - 2 .$$

Proof. From the assumption it follows that $\gamma \geq 1$ and the number $\alpha$ is irrational. By Remark 5, we can assume $M_1 = M_2 = \dots = M_n = 1$ . By Remark 1, we could accomplish the proof of the theorem only for $\varphi = 0$ , but we would practically reach neither a simplification nor any better results this way. From (5) and (2) (for $k = q_m$ ) it follows

(6) $$q_{m+1} \ll q_m^{\gamma} .$$

Determine positive integers $R$ and $N$ so that

(7) $$q_N \leq \sqrt{x} < q_{N+1} , \qquad q_{R-1} \leq x^{\frac{1}{2(\gamma+1)}} < q_R .$$

Then by (6) it is

(8) $$q_R \ll x^{\frac{\gamma}{2(\gamma+1)}} < \sqrt{x} .$$

Let

$$I_k = k^{\varphi} \min^{\beta} \left( \frac{\sqrt{x}}{k} , \frac{1}{\langle \alpha k \rangle} \right) .$$

Hence it is

$$F(x) = \sum_{k \leq \sqrt{x}} I_k$$

and

$$(9) \qquad I_k \leqq x^{\beta/2} k^{\rho-\beta} \quad ,$$

$$(10) \qquad I_k \leqq \frac{k^{\rho}}{\langle \alpha k \rangle^{\beta}} \quad .$$

Clearly, the estimate (9) suits the case $\langle \alpha k \rangle \leqq \leqq \frac{k}{\sqrt{x}}$ , and, the estimate (10) will be used in the remaining cases.

For $R \leqq m \leqq N$ let

$$S_m = \sum I k \ ,$$

where we are summing up over all $k \leqq \sqrt{x}$, $q_m \leqq k < q_{m+1}$. Let the members with $k$ of the form (2) or (3) be comprised into $S_m^1$ , all others into $S_m^2$ . In $S_m^1$ it is $k \geqq u q_m$ , hence

$$(11) \qquad S_m^1 << \sum_{u=1}^{\infty} \frac{x^{\beta/2}}{(u q_m)^{\beta-\rho}} << x^{\beta/2} q_m^{\rho-\beta} \quad .$$

To each $k$ in $S_m^2$ a positive integer $j \leqq q_m/2$ can be assigned (see Remark 4) so that (4) holds. If $j \leqq \frac{(u+1)q_m^2}{\sqrt{x}}$, we shall use (9), otherwise (10). Since $k < (u+1) q_m$ , we get, by Remark 4

$$S_m^2 << \sum_u \Big( x^{\beta/2} \sum_{j \leqq \frac{(u+1)q_m^2}{\sqrt{x}}} k^{\rho-\beta} + \sum_{j > \frac{(u+1)q_m^2}{\sqrt{x}}} \Big(\frac{q_m}{j}\Big)^{\beta} k^{\rho} \Big) <<$$

$$<< \sum_u \Big( x^{\beta/2} \sum_{j \leqq \frac{(u+1)q_m^2}{\sqrt{x}}} (u q_m)^{\rho-\beta} + u^{\rho} q_m^{\rho+\beta} \sum_{j \geqq \frac{(u+1)q_m^2}{\sqrt{x}}} j^{-\beta} \Big) <<$$

$$<< \sum_u \Big( x^{\beta/2} q_m^{\rho-\beta} \frac{u^{\rho-\beta+1} q_m^2}{\sqrt{x}} + u^{\rho} q_m^{\rho+\beta} \Big(\frac{u q_m^2}{\sqrt{x}}\Big)^{1-\beta} \Big) <<$$

$$<< x^{\frac{\beta-1}{2}} q_m^{\rho-\beta+2} \sum_u u^{\rho-\beta+1} ,$$

where the summation runs over all positive integers $u$,
$u \leq \frac{\min(\sqrt{x},\, q_{n+1})}{q_n}$. Hence it is

$$(12)\quad S_n^2 \ll \begin{cases} x^{\frac{\beta-1}{2}} q_n^{\varphi-\beta+2} & \text{for } \varphi < \beta-2\,, \\ x^{\frac{\beta-1}{2}} q_n^{\varphi-\beta+2} \lg 2\, \frac{\min(q_{n+1},\sqrt{x})}{q_n} & \text{for } \varphi = \beta-2\,, \\ x^{\frac{\beta-1}{2}} \min^{\varphi-\beta+2}(q_{n+1}, \sqrt{x}\,) & \text{for } \varphi > \beta-2\,. \end{cases}$$

By (11) and (7) we have

$$(13)\quad \sum_{n=R}^{N} S_n^1 \ll \sum_{n=R}^{N} \frac{x^{\beta/2}}{q_n^{\beta-\varphi}} \ll x^{\beta/2} q_R^{\varphi-\beta} \ll x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}} \,.$$

By (12) and (8) it is

$$(14)\quad \sum_{n=R}^{N} S_n^2 \ll x^{\frac{\beta-1}{2}} \sum_{n=R} q_n^{\varphi-\beta+2} \ll x^{\frac{\beta-1}{2}} q_R^{\varphi-\beta+2} \ll x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}}$$

for $\varphi < \beta - 2$ ,

$$(15)\quad \sum_{n=R}^{N} S_n^2 \ll x^{\frac{\beta-1}{2}} \sum_{n=R}^{N} \lg 2\, \frac{\min(q_{n+1},\sqrt{x})}{q_n} \ll x^{\frac{\beta-1}{2}} \lg x = x^{\frac{\varphi+1}{2}} \lg x$$

for $\varphi = \beta - 2$ , and, finally,

$$(16)\quad \sum_{n=R}^{N} S_n^2 \ll x^{\frac{\beta-1}{2}} \sum_{n=R}^{N} \min^{\varphi-\beta+2}(q_{n+1},\sqrt{x}) \ll x^{\frac{\beta-1}{2}+\frac{\varphi-\beta+2}{2}} = x^{\frac{\varphi+1}{2}}$$

for $\varphi > \beta - 2$ .

Since by (3),(10),(7) and (8) it is

$$(17)\qquad I_{q_{R-1}} \ll q_{R-1}^{\varphi}\, q_R^{\beta} \ll x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}} \,,$$

it remains to estimate

$$\sum_{\substack{k<q_R \\ k\neq q_{R-1}}} I_k$$

In this sum, we can use the relation (4) (for $n = R$). For $\frac{\sqrt{x}}{k} \leq \frac{q_R}{j}$ we shall use (9), otherwise (10). So we have (with obvious summation domains)

$$\sum_{\substack{k<q_R \\ k\neq q_{R-1}}} I_k << x^{\beta/2} \sum k^{\varphi-\beta} + \sum k^{\varphi} \left(\frac{q_R}{j}\right)^{\beta} .$$

In the first sum there is $k > \frac{j\sqrt{x}}{q_R}$, in the second $k < \frac{j\sqrt{x}}{q_R}$, therefore, with the aid of (8) we have

$$\sum_{\substack{k<q_R \\ k\neq q_{R-1}}} I_k << x^{\beta/2} \left(\frac{\sqrt{x}}{q_R}\right)^{\varphi-\beta} \sum_{j=1}^{\infty} j^{\varphi-\beta} + q_R^{\beta} \sum_{j=1}^{\infty} \left(\frac{j\sqrt{x}}{q_R}\right)^{\varphi} \frac{1}{j^{\beta}} <<$$

$$<< x^{\varphi/2} q_R^{\beta-\varphi} << x^{\frac{\beta\gamma+\varphi}{2(\gamma+1)}} .$$

The relations (13) - (17) together with the estimate just proved give the assertion.

Theorem 5 and 7 enable us, together with Theorem 1, to summarize the result on one theorem only.

<u>Theorem 8</u>. Let $\varphi < \beta - 1$, $\alpha_1 = \alpha_2 = \ldots = \alpha_k = \alpha$, and let $\gamma$ be the last upper bound of all the numbers $\tau$, $\tau > 0$, for which the inequality

$$\langle \alpha k \rangle << k^{-\tau}$$

has infinitely many solutions. Then

$$\limsup_{x \to +\infty} \frac{\lg F(x)}{\lg x} = \begin{cases} \dfrac{\beta\gamma + \rho}{2(\gamma+1)} & \text{for } \gamma > \dfrac{1}{\beta-\rho-1}, \\ \dfrac{\rho+1}{2} & \text{for } \gamma \leq \dfrac{1}{\beta-\rho-1} \end{cases}$$

(for $\gamma = +\infty$ the fraction equals to $\beta/2$ ).

Proof is easily obtained from Theorems 1,5 and 7 and from Remark 5.

Remark 6. In the papers [2] and [3] , actually the function

$$F_1(x) = \sum_{k \leq \sqrt{x}} \lg 2k \min^{\beta}\left(\frac{\sqrt{x}}{k}, \frac{1}{P_k}\right)$$

is used. The results stated in the theorems 1 - 8 for the function $F(x)$ can be transferred accordingly also to the function $F_1(x)$ . Essentially, only the member $\lg x$ will occur everywhere (with the exception of the $O$ -estimate for $\beta > 1$ , and, in the same case, of the main member in the theorem 2). The theorem 7 can be transferred most conveniently with the aid of the estimate $\lg 2k \ll \lg x$ .

Remark 7. The application of the results of this paper in the theory of lattice points in ellipsoids with weight will be presented in the papers [6] and [7] .

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