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ON PROBLEMS CONCERNING EXTENSION OF LINEAR OPERATIONS ON LINEAR SPACES x)

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The aim of this paper is the formulation of the so-called Φ -extensibility of linear operators (i.e. linear transformations of a linear space into another one) which is a generalization of the traditional extension of linear operators, resp. functionals preserving the norm. A necessary and sufficient condition for extensibility of bounded linear operators is proved (it is the condition analogous to that in [3]).

A theorem is proved on extension of complex linear operators that is a generalization of the well known Suchomlinoff's result concerning the extension of complex linear functionals preserving the norm (see [2]). We shall call P, Q the linear space over a field K. The symbol R denotes a subspace of the space P. The elements of P, resp. Q, resp. K will be denoted by small Latin letters from the end of the alphabet x, y, z etc., resp.

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x) This paper is a more exact extension of the results in [4].

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from its beginning, i.e. a, b, c etc., resp. by small Greek letters etc. Linear operators from P into Q operators only in the following will be marked by capital letters A, B, C etc.

For the domains of operators the symbol def is used, i.e. for example a space which is a domain of the operator A will be denoted $def\ A$.
Linear envelopes of subsets of a linear space will be denoted by brackets.

Definition 1. Let Φ be a mapping from P into $exp\ Q$ (i.e. the set of all subsets of the linear space Q). We shall say the operator A to be Φ-admissible, if the following condition is satisfied:

$$x \in def\ A \Longrightarrow A(x) \in \Phi(x) \ .$$

Definition 2. Let Φ be a mapping from P into Q. The operator A be called Φ-extensionable, if there is an operator B such that

$$def\ B = P \ ,$$

$$x \in def\ A \Longrightarrow A(x) = B(x) \ ,$$

$$x \in P \Longrightarrow B(x) \in \Phi(x) \ .$$

Definition 3. Let Φ be a mapping from P into Q. This mapping is called linearly covering P in respect to Q, if the following statement is satisfied:

Let A be a Φ-admissible operator, then there is an element $a \in Q$ for every $y \in P$ so that

$$A(x) + \alpha \cdot a \in \Phi(x + \alpha y)$$

for all $x \in def\ A$ and $\alpha \in K$.

Theorem 1. Let Φ be a mapping from P into Q. Then the following statements are equivalent:

(i) Every Φ -admissible operator is a Φ -extensionable operator;

(ii) The mapping Φ is linearly covering P in respect to Q .

Proof. Let (i) be true. Let A be a Φ -admissible operator and $y \in P$. From (i) it follows that A has a Φ -admissible extension B such that $def\, B = P$. Suppose that $a = B(y)$. Then

$A(x) + \alpha a = B(x) + \alpha B(y) = B(x + \alpha y) \in \Phi(x + \alpha y)$

and so (ii) is satisfied.

Let (ii) be true. Let A be a Φ -admissible operator. Let $\mathcal{L}$ be a set of all Φ -admissible operators. According to the assumption the set is not empty because $A \in \mathcal{L}$. Let us introduce the relation of a partial order on $\mathcal{L}$ as follows:

$D \prec E \quad (D, E \in \mathcal{L})$ if:

$def\, D \subset def\, E\,, \ x \in def\, D \Longrightarrow D(x) = E(x)$

is fulfilled.

Such system $\mathcal{L}$ satisfies the assumption of Zorn's Lemma because if $\{F_i\}_{i \in I}$ is a monotone subsystem of the system $\mathcal{L}$, then we define the operator F in the following way:

$$def\, F = \bigcup_{i \in I} def\, F_i \quad ,$$

$x \in def\, F \Longrightarrow F(x) = F_i(x)$ for such i that $x \in F_i$.
It is obvious that the definition is correct and that $F_i \prec F$ for $i \in I$ (obviously $F \in \mathcal{L}$).
And so there is $B \in \mathcal{L}$ such that $A \prec B$ and if $B \prec C$, then $B = C$. We shall prove by contradiction that

$def\, B = P$. Let $def\, B \neq P$. It means that there is such $y \in P$ that $def\, B \subsetneqq [def\, B \cup y] \subset P$. Because $B \in \mathcal{L}$, there is $b \in Q$ such that $B(x) + \alpha b \in \Phi(x + \alpha y)$ for all $x \in def\, B$ and $\alpha \in K$. It is possible to write every element $z \in [def\, B \cup y]$ uniquely in the form

$$x + \alpha y, \; x \in def\, B, \; \alpha \in K .$$

We define the operator C on $[def\, B \cup y]$ by this way: $C(z) = B(x) + \alpha b$, where $z = x + \alpha y$, $x \in def\, B$, $\alpha \in K$. It is easy to see:

$$x \in def\, B \Longrightarrow C(x) = B(x) ,$$

$$z \in def\, C \Longrightarrow C(z) \in \Phi(z) ,$$

$$B \neq C .$$

Hence $C \in \mathcal{L}$, $B \prec C$, $B \neq C$, however, it is a contradiction. Thus (ii) is satisfied and the proof is complete.

Convention. In the following K will denote the field of real or complex numbers. Let P, Q be normed linear spaces. We denote the norm on P by ${}^1\| \cdot \|$, the norm of Q by ${}^2\| \cdot \|$. The symbol $S(a; \varepsilon)$ is used for the set

$$\{ b \in Q ;\; {}^2\| a - b \| \leqq \varepsilon \}, \quad \varepsilon > 0$$

(e.g. a closed sphere in Q with the centre a and radius ε).

Definition 4. Let $k \geqq 0$. Let P, Q be normed linear spaces. The linear space Q is called k-productively centred in respect to P, if the following is satisfied:

Let A be such that

$$S(A(x_1), k\, {}^1\| x_1 + y \|) \cap S(A(x_2), k\, {}^1\| x_2 + y \|) \neq \emptyset$$

for all $x_1, x_2 \in def\, A$ and $y \in P$, then

$\bigcap_{x \in def\, A} S(A(x), k\, {}^1\|x+y\|) \neq \emptyset$ for all $y \in P$.

Theorem 2. Let $k \geq 0$. Let P, Q be normed linear spaces. Then the following statements are equivalent:

(i) The mapping Φ from a linear space P to $exp\, Q$ defined by

$$x \in P \Longrightarrow \Phi(x) = \{a \in Q;\ {}^2\|a\| \leqq k\, {}^1\|x\|\}$$

is linearly covering P in respect to Q;

(ii) The linear space Q is k-productively centred in respect to P.

Proof. Let (i) be valid. Let A be such that

$$S(A(x_1), k\, {}^1\|x_1+y\|) \cap S(A(x_2), k\, {}^1\|x_2+y\|) \neq \emptyset$$

for all $x_1, x_2 \in def\, A$ and $y \in P$.

From the relation

$$S(A(x), k\, {}^1\|x\|) \cap S(0,0) \neq \emptyset,\ x \in def\, A$$

(in the previous relation we denote $x_1 = x$, $x_2 = y = 0$ - zero in P) it follows that

$${}^2\|A(x)\| \leqq k\, {}^1\|x\|,\quad x \in def\, A.$$

Thus the operator A is Φ-admissible. According to (i) the condition is satisfied that there is $a \in Q$ for every $y \in P$ such that

$${}^2\|A(x) + \alpha a\| \leqq k\, {}^1\|x + \alpha y\| \text{ for } x \in def\, A \text{ and } \alpha \in K.$$

It follows from the last relation (denoting $\alpha = 1$) that

$$-a \in \bigcap_{x \in def\, A} S(A(x), k\, {}^1\|x+y\|) \text{ for all } y \in P$$

(generally for different y there are, of course, different $-a$). Thus, it is true that

$$\bigcap_{x \in def\, A} S(A(x), k\, {}^1\|x+y\|) \neq \emptyset$$

for all $y \in P$ and (ii) is satisfied.

Let (ii) be true. Let A be Φ -admissible. We will show that

$$S(A(x_1), k\, {}^1\|x_1 + y\|) \cap S(A(x_2), k\, {}^1\|x_2 + y\|) \neq \emptyset$$

for all $x_1, x_2 \in def\, A$ and $y \in P$.

It is sufficient to show that the sum of radiuses of such two spheres is greater or equals the distance of their centres which is correct under the assumption, because

$$k\,({}^1\|x_1 + y\| + {}^1\|x_2 + y\|) \geqq k\, {}^1\|x_1 - x_2\| \geqq$$
$$\geqq {}^2\|A(x_1 - x_2)\| = {}^2\|A(x_1) - A(x_2)\| \quad .$$

So there is $-a \in Q$ for every $y \in P$ such that

$$-a \in \bigcap_{x \in def A} S(A(x), k\, {}^1\|x + y\|),$$

in other words:

$${}^2\|A(x) + a\| \leqq k\, {}^1\|x + y\| \quad \text{for} \quad x \in def\, A \; .$$

From there it follows that for all $\alpha \in K$, $\alpha \neq 0$:

$$|\alpha| \cdot {}^2\|A(\tfrac{x}{\alpha}) + a\| \leqq |\alpha|\; k\, {}^1\|(\tfrac{x}{\alpha}) + y\|,\; x \in def\, A$$

so that

$${}^2\|A(x) + \alpha a\| \leqq k\, {}^1\|x + \alpha y\|,\; x \in def\, A,\; \alpha \in K,\; \alpha \neq 0.$$

Since the last relation is trivial for $\alpha = 0$, (i) is satisfied and the proof is complete.

<u>Definition 5</u>. The linear space Q is called productively centred in respect to P, if it is k -productively centred for every $k \geqq 0$.

<u>Remark 1</u>. As a result of Theorem 1.2 and Definition 5 there follows the statement: Let P, Q be normed linear spaces. Let Q be productively centred to P. Then every bounded operator from P into Q may be extended on the whole P preserving the norm.

Theorem 3. The linear space of real numbers is productively centred in respect to every normed linear space over the field of real numbers.

Proof. Theorem 3 is a result of a more general statement for the linear space of real numbers: Let $\mathcal{S}$ be an arbitrary system of closed spheres in the linear space of real numbers such that any two elements of this system have a non empty intersection. Then the intersection of all these spheres is a non empty set. The proof of this statement is easy. We denote $\mathcal{S} = \{I_\mu\}_{\mu \in N}$, $I_\mu = \langle p_\mu, q_\mu \rangle$. If we denote $p = \sup_{\mu \in N} p_\mu$, $q = \inf_{\mu \in N} q_\mu$, then it follows $p \leq q$. Suppose, on the contrary, that $p > q$. Then there is μ_1, μ_2 such that $p_{\mu_1} > q_{\mu_2}$ by another way $I_{\mu_1} \cap I_{\mu_2} = \emptyset$, on the contrary to the hypothesis. Hence it follows $I = \langle p, q \rangle$ and $I \subset I_\mu$ for every $\mu \in N$, so $\bigcap_{\mu \in N} I_\mu \neq \emptyset$ and the proof is complete.

Remark 2. As the result of Remark 1 and Theorem 3, there follows the Hahn-Banach theorem on extension of real bounded linear functionals preserving the norm.

Convention. Let P be a normed linear space over the field of complex numbers. By the symbol ${}_{R}P$ we denote the linear space P as a normed linear space over the field of real numbers, analogously for subspaces and linear envelopes.

Definition 6. Let Q be a linear space over the field of complex numbers. We call this linear space a pure complex linear space, if:

1. There is introduced a so-called involution(see [1]) on a

linear space Q , i.e. a mapping J from Q into Q such that

$$J(\alpha a + \beta b) = \bar{\alpha} J(a) + \bar{\beta} J(b) ;$$

$$J(J(a)) = a ;$$

2. on the linear space Q there is introduced a norm such that

$${}^{2}\| J(a) \| = {}^{2}\| a \| ,$$

$${}^{2}\| a \| = \max_{t \in \Delta} {}^{2}\| \mathrm{Re}\, a \cdot \cos t + \mathrm{Im}\, a \cdot \sin t \|$$

(Δ is a set of real numbers).

By the symbol $\mathrm{Re}\, a$, resp. $\mathrm{Im}\, a$ we denote the so-called real part, resp. imaginary part of the element a . Every element $a \in Q$ may be written uniquely in the form

$$\mathrm{Re}\, a + i\, \mathrm{Im}\, a , \quad \mathrm{Re}\, a , \mathrm{Im}\, a \in \mathrm{Re}\, Q$$

- is a subspace of the space ${}_{k}Q$ for every its element it follows $J(a) = (a)$.

Theorem 4. Let P be a normed linear space over the field of complex numbers. Let Q be a pure complex linear space. Let $k \geq 0$. Let a mapping ${}_{k}\Phi$ from ${}_{k}P$ into $\exp \mathrm{Re}\, Q$ defined by the following

$$x \in {}_{k}P \Longrightarrow {}_{k}\Phi(x) = \{ a \in \mathrm{Re}\, Q ; {}^{2}\| a \| \leq k \, {}^{1}\| x \| \}$$

be the linearly covering ${}_{k}P$ in respect to $\mathrm{Re}\, Q$. Then the mapping Φ from P into $\exp Q$ defined by

$$x \in P \Longrightarrow \Phi(x) = \{ a \in Q ; {}^{2}\| a \| \leq k \, {}^{1}\| x \| \}$$

is linearly covering P in respect to Q .

Proof. At first we shall prove the following lemmas.

Lemma 1. Let P be a linear space over the field of complex numbers. Let Q be a pure complex linear space.

Then

(i) for an arbitrary operator A it follows that

$x \in def\, A \Longrightarrow Im\, A(x) = -Re\, A(ix)$, $Re\, A$ is the operator from ${}_{r}P$ into Q ;

(ii) if B is the operator from ${}_{r}P$ into $Re\, Q$ then $A(x) = B(x) - iB(ix)$, $x \in def\, B$ the operator from P into Q is defined and $B = Re\, A$.

The proof is

Lemma 2. Let P be a normed linear space over the field of complex numbers. Let Q be a pure complex linear space. Let $k \geqq 0$. Then:

if $x \in def\, C \Longrightarrow {}^{2}\|C(x)\| \leqq k\, {}^{1}\|x\|$,

then $x \in {}_{r}def\, C \Longrightarrow {}^{2}\|Re\, C(x)\| \leqq k\, {}^{1}\|x\|$,

and inversely.

Proof. This statement is trividal in regard to the first direction, see Definition 6.

Let $x \in {}_{r}def\, C$. Then we have ${}^{2}\|Re\, C(x)\| \leqq k\, {}^{1}\|x\|$. Because $x \cdot e^{-it} \in {}_{r}def\, C$ for all real t, it follows

${}^{2}\|Re\, C(x) \cos t - Re\, C(ix) \sin t\| \leqq k\, {}^{1}\|x \cdot e^{-it}\| = k\, {}^{1}\|x\|$

for all real t and so

$$ {}^{2}\|C(x)\| = \max_{t \in \Delta} {}^{2}\|Re\, C(x) \cos t + Im\, C(x) \sin t\| = $$
$$ = \max_{t \in \Delta} {}^{2}\|Re\, C(x) \cos t - Re\, C(ix) \sin t\| \leqq k\, {}^{1}\|x\| $$

and the proof is complete.

Lemma 3. Let P be a linear space over the field of complex numbers. Then it follows

$$ r\,[\,r\,[R \cup y] \cup iy\,] = [R \cup y]\,. $$

The proof is easy.

Now we prove Theorem 4.

Let A be Φ -admissible. From the lemma it follows that $\mathrm{Re}\, A$ is ${}_{\kappa}\Phi$ -admissible, i.e. there is $a_1 \in \mathrm{Re}\, Q$ for every $y \in P$ such that

$$x \in \mathrm{def}\, A \Longrightarrow {}^{2}\| \mathrm{Re}\, A(x) + \beta a_1 \| \leqq k\, {}^{1}\| x + \beta y \|$$

for all real β.

$\mathrm{Re}\, A(x) + \beta a_1$ is an ${}_{\kappa}\Phi$ -admissible operator on ${}_{\kappa}[\mathrm{def}\, A \cup y]$ into $\mathrm{Re}\, Q$, i.e. for every $i y$ there is $a_2 \in \mathrm{Re}\, Q$ such that

$$x \in {}_{\kappa}\mathrm{def}\, A \Longrightarrow {}^{2}\| \mathrm{Re}\, A(x) + \beta a_1 + \gamma a_2 \| \leqq k\, {}^{1}\| x + \beta y + \gamma i y \|$$

for all real β, γ.

$\mathrm{Re}\, A(x) + \beta a_1 + \gamma a_2$ is the ${}_{\kappa}\Phi$ -admissible operator on $\kappa[{}_{\kappa}\mathrm{def}\, A \cup y] \cup i y]$ into $\mathrm{Re}\, Q$.

We define the operator B as follows:

$$\mathrm{def}\, B = [\mathrm{def}\, A \cup y],$$

if $z = x + (\beta + i\gamma) y$, $x \in \mathrm{def}\, A$, $(\beta + i\gamma) \in K$, then

$$B(z) = A(x) + (\beta + i\gamma)(a_i - i a_2).$$

It follows that $\mathrm{Re}\, B(z) = \mathrm{Re}\, A(x) + \beta a_1 + \gamma a_2$.

According to the preceding we have that

$$z \in [\mathrm{def}\, A \cup y] \Longrightarrow {}^{2}\| B(z) \| \leqq k\, {}^{1}\| z \|,$$

in other words,

${}^{2}\| A(x) + \alpha a \| \leqq k\, {}^{1}\| x + \alpha y \|$ for all $x \in \mathrm{def}\, A$ and $\alpha \in K$ $(a = a_1 - i a_2)$.

So Φ is linearly covering P in respect to Q, q.e.d.

<u>Theorem 5</u>. Let P be a normed linear space over the field of complex numbers. Let Q be a pure complex linear space. Let $\mathrm{Re}\, Q$ be productively centred in respect to ${}_{\kappa}P$. Then every operator from P into Q is extension-

able on the whole P preserving the norm.

Proof. This theorem is an easy result of Theorem 1, 2, 4.

Remark 3. Theorem 5 is a generalization of the well known Suchomlinoff's result concerned with the extension of complex linear functionals preserving the norm.

R e f e r e n c e s

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