Jaroslav Nešetřil

Every group is a maximal subgroup of the semigroup of relations

Commentationes Mathematicae Universitatis Carolinae, Vol. 12 (1971), No. 1, 19--21

Persistent URL: http://dml.cz/dmlcz/105323

Commentationes Mathematicae Universitatis Carolinae

12,1 (1971)

EVERY GROUP IS A MAXIMAL SUBGROUP OF THE SEMIGROUP OF RELATIONS

Jaroslav NEŠETŘIL, Praha

The aim of this note is to extend a result of [2], namely to prove the following theorem:

Theorem: The class of maximal subgroups of semigroups of binary relations includes all groups.

This generalizes [2], Theorem 4.7 to infinite groups.x) We preserve the notation of [2] and refer to the results proved there, too.

Concerning graphs we use the notation of [1].

Proof of the theorem: Let G be an infinite group (the proof for finite case would be similar; since the finite case is solved in [2], we make this assumption for sake of brevity). By [1], there is a graph (X, R) such that $C(X, R) \simeq G$, where $C(X, R)$ is the monoid of all compatible mappings (i.e. homomorphisms) into itself. By constructions given in [1], we can assume the following about the graph (X, R):

----

x) Using a different method this generalization was obtained independently by A.H. Clifford, R.J. Plemmons and B.M. Schein.

AMS, Primary 20M20, 08A05 Ref.Ž. 2.721.4

a) $|X| = |R|$ (this follows from the fact that (X, R) can be chosen without isolated points).

b) Let $V(x) = \{y \mid (x, y) \in R\}$, then $x \neq y$ implies $V(x) \nsubseteq V(y)$ and $V(y) \nsubseteq V(x)$. Similarly for $\bar{V}(x) = \{y \mid (y, x) \in R\}$.

c) $V(x) \neq \emptyset$, $V(x) \neq X$ for every $x \in X$. Similarly for $\bar{V}(x)$.

Let $\varphi : X \longrightarrow R$ be a bijection. Define the relation α on $X_{01} = X \times \{0, 1\}$ ($0, 1 \notin X$) by:

$$((x, 0), (y, 0)) \in \alpha \iff ((x, 1), (y, 1)) \in \alpha \iff x = y,$$

$((x, 0), (y, 1)) \in \alpha \iff x$ is incident with $\varphi(y)$,

$$((x, 1), (y, 0)) \notin \alpha .$$

By b),c), α is reduced. Further, α is idempotent as can be easily seen. Thus by Lemma 3.4 [2] (and by its remark), the maximal subgroup H_α of $\mathcal{B}_X$ containing α is given by $H_\alpha \simeq G_\alpha = \{\varphi \in S_{X_{01}} \mid \exists \sigma \in S_{X_{01}} \alpha\varphi = \sigma\alpha\}$. But in this special case we have $G_\alpha = \{\varphi \mid \alpha\varphi = \varphi\alpha\}$. Similarly as in the proof of [2], Lemma 4.2, $G_\alpha \simeq \{\varphi \in S_X \mid \exists \sigma \in S_X, R\varphi = \sigma R\} = G_R$.
But obviously $G_R = A(X, R) = C(X, R) \simeq G$, by the assumption ($A(X, R)$ is the group of all automorphisms of the graph (X, R)).

I thank to Z. Hedrlín, who turned my attention to the paper [2].

R e f e r e n c e s

[1] Z. HEDRLÍN and A. PULTR: Symmetric relations (undirec-

ted graphs) with given semigroups, Monatsh.für Math.69(1965),318-322.

[2] J.S. MONTAGUE and R.J. PLEMMONS: Maximal subgroups of the semigroups of relations. J.of Algebra 13(1969),575-587.

Matematicko-fyzikální fakulta
Karlova universita
Praha 8, Sokolovská 83
Československo

(Oblatum 23.9.1970)