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COMPARABILITY OF BOREL PROBABILITY MEASURES ON EUCLIDEAN LINE

Pavel ČIHÁK, Praha

In this paper, the necessary and sufficient conditions for comparability of Borel probability measures on the euclidean line by ordering introduced by G. Choquet will be found. The proof does not use neither the theorem on desintegration of measures [2],Chap.6) nor the theory of lifting [7]. The problem of finding the α, μ -conditionally maximal measure is solved by a special α, μ -stochastic operator $E_{\alpha,\mu}$. It is also proved that this operator is a $\mathcal{U}$ -exposed element of the set of all α, μ -stochastic operators.

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1. Notations. The same notation as in [3] will be used. Moreover, let $\mathbb{R}^* = (\infty) \cup \mathbb{R} \cup (-\infty)$ be the two point compactification of the euclidean line $\mathbb{R}$.
Let $C(\mathcal{X})$ be the space of all continuous functions on a

compact space X with the customary norm $\| \ \|$. Put $C^+(X) = \{f \in C(X) ; f \geq 0\}$. Let $\mathcal{P}(X)$ be the set of all Borel regular probability measures on all Borel subsets of the set X.

If $\mu \in \mathcal{P}(X)$, then we put $\mu(f) = \int_X f(x) \cdot d\mu(x)$ for all $f \in C(X)$. Let L be the set of all affine functions on the space R. The symbol C_L ($\check{C}_L$ resp.) denotes the set of all continuous (convex resp.) functions defined on R of the linear envelope of $C(R^*)$ and $L^+ = \{\ell^+ ; \ell \in L\}$. The symbol $\mathcal{P}_L$ denotes the set of all measures $\mu \in \mathcal{P}(R^*)$ such that all functions $f \in C_L$ are μ-integrable.

If $\lambda, \mu \in \mathcal{P}_L$, then $\mu \succ \lambda$ denotes that $\mu(f) \geq \lambda(f)$ for all $f \in \check{C}_L$. $L^p(\mu)$ denotes the Lebesgue space of classes of μ-equivalent functions on R for $\mu \in \mathcal{P}_L$ and $1 \leq p \leq \infty$ (see [2]).

If f is a convex function on R, then f' denotes the right derivative of f.

If $\mu \in \mathcal{P}_L$ and E is a Borel subset of the space R, then the symbol $\mu | E$ denotes such measure that $(\mu | E)(F) = \mu(E \cap F)$ for each Borel set F.

The letter ℓr denotes such function that $\ell r(x) = x$ for all $x \in R$. The symbol $\otimes$ denotes the tensor product of two functions.

2. Conditions for comparability of measures

(2.1) Lemma. Suppose $\lambda, \mu \in \mathcal{P}_L$, $\mu \succ \lambda$, λ is supported by a compact subset of the euclidean space R.

Then there exists a linear map $B : C_L \longrightarrow L^{\infty}(\lambda)$ such that

$B \geq 0, B(\ell) = \ell$ for all $\ell \in L$ and

$\mu(f) = \lambda(B(f))$ for all $f \in C_L$.

Proof. Suppose that the measure λ is supported by a compact set X . There are two elements a_1 and a_2 of the set R such that the interval $\langle a_2, a_1 \rangle$ contains the set X . Put

$\beta_0 = \mu(\langle a_2, a_1 \rangle)$, $\beta_1 = \mu((a_1, \infty))$ and $\beta_2 = \mu((-\infty, a_2))$.

Define $\mu_0 = \frac{1}{\beta_0} \mu | \langle a_1, a_2 \rangle$, $\mu_1 = \frac{1}{\beta_1} \mu | (a_1, \infty)$ and $\mu_2 = \mu | (-\infty, a_2)$ for $\beta_i > 0$ and $\mu_i = \mu$ for $\beta_i = 0$ and $i \in \{0, 1, 2\}$. Then $\mu_i \in \mathcal{P}_L$ for each i . Put $b_i = \mu_i(b)$ for $i = 1, 2$. If $\beta_1 = 0$, then $b_1 > a_1$ and $\mu_1 \vdash \delta_{b_1}$ and if $\beta_2 > 0$, then $b_2 < a_2$ and $\mu_2 \vdash$ $\vdash \delta_{b_2}$ (see [6]).

We have $\mu = \beta_0 \mu_0 + \beta_1 \mu_1 + \beta_2 \mu_2$. Define $\eta = \beta_0 \mu_0 + \beta_1 \delta_{b_1} + \beta_2 \delta_{b_2}$. Then $\eta \in \mathcal{P}_L$ and the measure η is supported by a compact set Y , $Y \subset (b_1) \cup \langle a_2, a_1 \rangle \cup (b_2)$. Clearly, $\mu \vdash \lambda$. If $f \in \check{C}_L$, then there is a function $f_1 \in \check{C}_L$ such that $f_1(x) = f(x)$ for each $x \in \langle a_1, a_2 \rangle$, $f_1(x) = f(a_1) + f'(a_1) \cdot (x - a_1)$ for each $x > a_1$ and $f_1(x) = f(a_2) + f'(a_2) \cdot (x - a_2)$ for each $x < a_2$. Hence $f \geq f_1$, $\eta(f) \geq \eta(f_1) = \mu(f_1) \geq \lambda(f_1) =$ $= \lambda(f)$ and $\eta \vdash \lambda$.

Define the map $B_1 : C_L \longrightarrow C(Y)$ such that

$B_1(f)(x) = f(x)$ for all $x \in \langle a_2, a_1 \rangle$ and

$B_1(f)(b_i) = \mu_i(f)$ for $\beta_i > 0$, $i = 1, 2$ for all

$f \in C_L$.

Clearly, B_1 is a linear map,

$B_1 \geq 0$, $B_1(\ell) = \ell$ for all $\ell \in L$ and

$\eta(B_1(f)) = \beta_0 \mu_0(f) + \beta_1 \mu_1(f) + \beta_2 \mu_2(f) = \mu(f)$ for all $f \in C_L$.

Now, put $Q = \{(\delta_x, \nu); \nu \in \mathcal{P}(Y), \nu(b) \in X\}$. Then the set Q is a compact subset of the space $C'(X) \times C'(Y)$ by $C(X)$-weak topology on the dual space $C'(X)$ and $C(Y)$-weak topology on the dual space $C'(Y)$, since Q is the graph of the continuous map $\nu \longrightarrow \delta_{\nu(b)}$ and X is a compact subset of $C'(X)$.

If $(\lambda, \eta) \notin \overline{co}(Q)$, then there exist functions $g \in C(Y)$ and $h \in C(X)$ such that $\lambda(h) - \eta(g) > 0 \geq \geq \delta_x(h) - \nu(g)$ for all $(\delta_x, \nu) \in Q$ by Hahn-Banach theorem. Put $\check{g}(y) = \inf\{\nu(g);\ \nu \in \mathcal{P}(Y),\ \nu(b) = y\}$ for each $y \in co(Y)$. Then $g \geq \check{g} \geq \min g(Y)$ on the set Y and we shall show that the function $\check{g}$ is convex continuous on $co(Y)$. Clearly, if $y_1, y_2 \in co(Y)$ and $t \in (0,1)$, then $t \cdot \nu_1(g) + (1-t) \cdot \nu_2(g) \geq \check{g}(y_0)$ for each $\nu_1, \nu_2 \in \mathcal{P}(Y)$ such that $\nu_1(b) = y_1$, $\nu_2(b) = y_2$, where $t \cdot y_1 + (1-t) \cdot y_2 = y_0$. Hence $t \cdot \check{g}(y_1) + (1-t) \cdot \check{g}(y_2) \geq \check{g}(y_0)$, the function $\check{g}$ is convex and according to [5] $\check{g}$ is continuous. Clearly, $X \subset co(Y)$ and $\check{g}(x) \geq h(x)$ for each $x \in X$. We obtain the following inequalities:

$\eta(g) \geq \eta(\check{g}) \geq \lambda(\check{g}) \geq \lambda(h)$ and $\lambda(h) - \eta(g) \leq 0$, which is a contradiction. Hence $(\lambda, \eta) \in \overline{co}(Q)$.

According to the Choquet's theorem [1] there exists a measure $M \in \mathcal{P}(Q)$ such that

$$(\lambda, \eta) = \int_Q (\delta_x, \nu) \cdot dM(\delta_x, \nu),$$

i.e.

$\lambda(h) - \eta(g) = \int_Q (\sigma_x(h) - \nu(g)) \cdot dM(\sigma_x, \nu)$ for all $h \in C(X)$ and $g \in C(Y)$.

Put

$m(h \otimes g) = \int_Q \sigma_x(h) \cdot \nu(g) \cdot dM(\sigma_x, \nu)$ for all $h \in C(X)$.

Then

$m(h \otimes 1) = \int_Q \sigma_x(h) \cdot dM(\sigma_x, \nu) = \lambda(h)$ for all $h \in C(X)$

and

$m(1 \otimes g) = \int_Q \nu(g) \cdot dM(\sigma_x, \nu) = \eta(g)$ for all $g \in C(Y)$.

Now, define a linear functional m_g on $C(X)$ such that $m_g(h) = m(h \otimes g)$ for all $h \in C^+(X)$ and $g \in C^+(Y)$.

Then

$$0 \leq m_g(h) \leq \|g\| \cdot m(h \otimes 1) = \|g\| \cdot \lambda(h) .$$

According to the Lebesgue-Nikodym theorem ([2], Chap. 5) there exists one and only one element $B_0(g) \in L^\infty(\lambda)$ such that $m_g = B_0(g) \cdot \lambda$. The map $B_0 : C(Y) \longrightarrow L^\infty(\lambda)$ is linear, nonnegative $B_0 \geq 0$ and $\eta(g) = \lambda(B_0(g))$ for all $g \in C(Y)$.

Moreover, if $\ell \in L$, then

$m(h \otimes \ell) = \int_Q \sigma_x(h) \cdot \nu(\ell) \cdot dM(\sigma_x, \nu) = \int_Q \sigma_x(h) \cdot \ell(x) \cdot dM(\sigma_x, \nu) = \lambda(\ell \cdot h)$;

hence $B_0(\ell) = \ell$ for all $\ell \in L$.

Now, put $B = B_0 \circ B_1$, i.e. $B(f) = B_0(B_1(f))$ for all $f \in C_L$. Then $B : C_L \longrightarrow L^\infty(\lambda)$, B is a linear nonnegative map,

$B(\ell) = \ell \in L^\infty(\lambda)$ for all $\ell \in L$ and

$\mu(f) = \lambda(B(f))$ for all $f \in C_L$.

(2.2) <u>Note</u>. If $f \in \check{C}_L$, then $B(f) \geq f$ on R λ-almost everywhere.

Proof. If $\xi \in R$ and $f \in \check{C}_L$, then there is an affine function $\ell_\xi \in L$ such that $f \geq \ell_\xi$ on R and $f(\xi) = \ell_\xi(\xi)$. Hence

$B(f) \geq B(\ell_\xi) = \ell_\xi$ λ -a.e. on the set R.

Put $\bar{f}(x) = \sup\{\ell_\xi(x) ; \xi \in R, \xi \text{ rational number}\}$. Then the function $\bar{f}$ is convex, continuous and $\bar{f} = f$ on the set of all rational numbers of R. Hence $\bar{f} = f$ on R. Moreover, $B(f) \geq \bar{f} = f$ λ -a.e. on R.

(2.3) Lemma. Suppose $\lambda, \mu \in \mathcal{P}_L$, $\mu \vdash \lambda$, λ is supported by a compact subset of the euclidean space R. Let E be a Borel subset of R. Let $s(E)$ be a real number such that

(2.4) $$\int_{x > s(E)} d\mu(x) \leq \lambda(E) . \qquad \text{Then}$$

(2.5) $$\int_E x . d\lambda(x) \leq \int_{x > s(E)} x . d\mu(x) + (\lambda(E) - \int_{x > s(E)} d\mu(x)) . s(E) .$$

Moreover,

(2.6) $$\int_E x . d\lambda(x) = \int_E x . d\mu(x) \quad \text{for } E = R .$$

Proof. Suppose that $\lambda(E) > 0$, $s = s(E)$, where $s(E)$ is given in (2.4) and $\int_{x > s} d\mu(x) > 0$. Put $u = (\lambda(E))^{-1}$ and $v = (\int_{x > s} d\mu(x))$. Now we shall define a measure $\mu_E : \mu_E(f) = \frac{1}{\lambda(E)} \cdot \int_E B(f)(x) d\lambda(x)$ for all $f \in C_L$, where B is the map constructed in Lemma (2.1). Then $\mu_E(f) \leq \frac{1}{\lambda(E)} \cdot \lambda(B(f)) = \frac{1}{\lambda(E)} \cdot \mu(f)$ for all $f \in C^+(R^*)$. According to the Lebesgue-Nikodym theorem there exists an element $p \in L^1(\mu)$ such that $\mu_E = p \cdot \mu$, $0 \leq p \leq \frac{1}{\lambda(E)} = u$ and $\mu(p) = 1$. We obtain the following inequalities:

$$\frac{1}{\lambda(E)}\int_E x\,.\,d\lambda(x) = \mu_E(b) = \mu(p\,.\,b) = \int_{x>s} p(x)\,.\,x\,.\,d\mu(x) +$$
$$+ \int_{x\le s} p(x)\,.\,x\,.\,d\mu(x) \le \int_{x>s} p(x)\,.\,x\,.\,d\mu(x) +$$
$$+ s\,.\int_{x\le s} p(x)\,.\,d\mu(x) = \int_{x>s} v\,.\left[\frac{p(x)}{v}\,.\,x + \left(1-\frac{p(x)}{v}\right).\,s\right]d\mu(x) \le$$
$$\le \int_{x>s} v\,.\left[\frac{u}{v}\,.\,x + \left(1-\frac{u}{v}\right).\,s\right] d\mu(x) = \frac{1}{\lambda(E)}\int_{x>s} x\,.\,d\mu(x) +$$
$$+ \left(1-\frac{1}{\lambda(E)}\,.\int_{x>s} d\mu(x)\right).\,s\ ,$$

since $0 \le p \le u \le v$ and $\left(\frac{u}{v} - \frac{p}{v}\right).(x-s) \ge 0$ for $x \ge s$.

Clearly, if $\lambda(E) = 0$, then the inequality (2.5) is trivial; if $\lambda(E) > 0$ and $\int_{x>s} d\mu(x) = 0$, then $\frac{1}{\lambda(E)}\int_E x\,.\,d\lambda(x) \le s\,.\int_R p(x)\,.\,d\mu(x) = s = \frac{1}{\lambda(E)}\int_{x>s} x\,.\,d\mu(x) + \left(1-\frac{1}{\lambda(E)}\,.\int_{x>s} d\mu(x)\right).\,s$; if $E = R$, then $\lambda(b) = \lambda(B(b)) = \mu(b)$, i.e. (2.6).

(2.7) <u>Note</u>. Define the set $S(E) = \{s \in R^*;\ \int_{x>s} d\mu(x) \le \lambda(E) \le \int_{x\ge s} d\mu(x)\}$. Put $\sigma(E) = \inf S(E)$ and $\tau(E) = \sup S(E)$. Then $S(E)$ is a closed nonvoid interval in R^*, $S(E) = \langle \sigma(E), \tau(E) \rangle$ and $\mu((\sigma(E), \tau(E))) = 0$. This follows from regularity of the measure μ. For $E = \langle r, \infty \rangle$, $r \in R^*$, we put $S(r) = S(E)$, $\sigma(r) = \sigma(E)$, and $\tau(r) = \tau(E)$.

(2.8) <u>Theorem</u>. Suppose that $\lambda, \mu \in \mathcal{P}_1$, λ is supported by a compact subset of R. Let $s = s(r)$ be any function defined for λ-almost every $r \in R$ such that $s(r) \in S(r)$. Then $\mu \vdash \lambda$ if and only if

$$(2.9)\quad \int_{x\ge r} x\,.\,d\lambda(x) \le \int_{x>s(r)} x\,.\,d\mu(x) + \left(\int_{x\ge r} d\lambda(x) - \int_{x>s(r)} d\mu(x)\right).\,s(r)$$

for λ -a.e. $\kappa \in R$ and

(2.10) $$\int_{-\infty}^{\infty} x . d\lambda(x) = \int_{-\infty}^{\infty} x . d\mu(x) .$$

Proof. 1° The conditions (2.9) and (2.10) are necessary by (2.3).

2° Suppose that the conditions (2.9) and (2.10) hold for $\kappa \in H$, where $\lambda(H) = 1$. Define a convex wedge $K = \{f \in \check{C}_L ; \mu(f) \geq \lambda(f)\}$. Clearly, $K \supset L$. We shall prove that each convex function ℓ_ξ, where $\ell_\xi(x) = x - \xi$ for $x > \xi$, $\ell_\xi(x) = 0$ for $x \leq \xi$ and $\xi \in R$, is an element of K.

Suppose $\xi \in R$, $\varepsilon > 0$. There is an element $\kappa \in H$ such that

$$\lambda(\ell_\xi) - \varepsilon = \int_{x \geq \xi} (x - \xi) . d\lambda(x) - \varepsilon \leq \int_{x \geq \kappa} (x - \xi) . d\lambda(x) .$$

Let $s = s(\kappa)$ be the element of R^* given in (2.8). Put $A(\kappa) = \int_{x \geq \kappa} d\lambda(x)$. If $s(\kappa) \neq \pm \infty$, then

$$\int_{x \geq \kappa} (x - \xi) . d\lambda(x) = \int_{x \geq \kappa} x . d\lambda(x) - \xi \int_{x \geq \kappa} d\lambda(x) \leq \int_{x > s} x . d\mu(x) +$$

$$+ (A(\kappa) - \int_{x > s} d\mu(x)) . s - \xi . A(\kappa) = \int_{x > s} (x - \xi) . d\mu(x) +$$

$$+ (A(\kappa) - \int_{x > s} d\mu(x)) . (s - \xi) \leq \int_{x > s} (x - \xi) . d\mu(x) \leq \mu(\ell_\xi)$$

for the case $s \leq \xi$ and

$$\int_{x \geq \kappa} (x - \xi) . d\lambda(x) \leq \int_{x \geq s} (x - \xi) . d\mu(x) + (A(\kappa) -$$

$$- \int_{x \geq s} d\mu(x)) . (s - \xi) \leq \int_{x \geq s} (x - \xi) . d\mu(x) \leq \mu(\ell_\xi)$$

for the case $\xi < s$.

If $s(\kappa) = \infty$, then $A(\kappa) = 0$ and $\int_{\iota} (x - \xi) . d\lambda(x) = 0 \leq \mu(\ell_\xi)$.

If $b(\kappa) = -\infty$, then $A(\kappa) = 1$ and

$\int_{x \geq \kappa} (x-\xi).d\lambda(x) = \lambda(b) - \xi = \mu(b) - \xi = \mu(b-\xi) \leq \mu(b_\xi)$.
Hence $\lambda(b_\xi) \leq \mu(b_\xi)$ for all $\xi \in R$.

Finally, let $\langle a_2, a_1 \rangle$ be such interval that $\lambda(\langle a_2, a_1 \rangle) = 1$, let f be an arbitrary element of $\check{C}_1$ and let p be the right derivative of the function f. Then the function p is nondecreasing and $f(x) = f(a_2) + \int_{a_2}^{x} p(\xi).d\xi$ for all $x \in R$ by [5]. Define a function f_1: $f_1(x) = f(a_2) + (x-a_2).p(a_2) + \int_{a_2}^{a_1} b_\xi(x).dp(\xi)$ for all $x \in R$. Put $l(x) = f(a_2) + (x-a_2).p(a_2)$. Then
$f_1(x) = f(a_1) + (x-a_1).p(a_1) \leq f(x)$ for all $x > a_1$,
$f_1(x) = f(x)$ for all $x \in \langle a_2, a_1 \rangle$ and
$f_1(x) = f(a_2) + (x-a_2).p(a_2) \leq f(x)$ for all $x < a_2$.
Hence the function f_1 is convex, moreover $f_1 \in \check{C}_1$ and $f_1 \leq f$ on R.

Using the Fubini theorem, we obtain the following inequalities:

$$\mu(f) \geq \mu(f_1) = \mu(l) + \int_R \left(\int_{a_2}^{a_1} b_\xi(x).dp(\xi) \right).d\mu(x) =$$

$$= \mu(l) + \int_{a_2}^{a_1} \mu(b_\xi).dp(\xi) \geq \lambda(l) + \int_{a_2}^{a_1} \lambda(b_\xi).dp(\xi) =$$

$$= \lambda(f_1) = \lambda(f) ,$$

since $b_\xi \in K$ for all $\xi \in R$. Hence $f \in K$ and $K = \check{C}_1$ i.e. $\mu \succ \lambda$. The proof is complete.

(2.11) <u>Note</u>. If $\mu \succ \lambda$, where $\lambda, \mu \in \mathcal{P}_1$, and λ is supported by a compact subset of R, and f is a convex function on the set R such that $f \in L^1(\mu)$, then $\mu(f) \geq \lambda(f)$.

<u>Proof</u>. There is a nondecreasing sequence $f_n \in \check{C}_1$,

$n = 1, 2, \dots$, converging pointwise to f . We obtain that $\mu(f_n) \longrightarrow \mu(f)$, $\lambda(f_n) \longrightarrow \lambda(f)$ and $\mu(f_n) \geq \lambda(f_n)$. Hence $\mu(f) \geq \lambda(f)$.

3. Conditional maximality of measures

In the following section we shall suppose that $\mu \in \mathcal{P}_1$, $\alpha = (\alpha_j)_{j=1}^m$, $\alpha_j > 0$ for all j , $\sum_{j=1}^m \alpha_j = 1$.

(3.1) Definition. A measure σ will be called α, μ -conditionally maximal iff

1° $\sigma = \sum_{j=1}^m \alpha_j \delta_{c_j}$, where $c_1 \geq c_2 \geq \dots \geq c_m$,

2° $\mu \vdash \sigma$ and

3° if $\lambda = \sum_{j=1}^m \alpha_j \delta_{a_j}$, $a_1 \geq a_2 \geq \dots \geq a_m$, $\mu \vdash \lambda$, then $\sigma \vdash \lambda$.

(3.2) Theorem. There exists one and only one α, μ -conditionally maximal measure σ .

Proof. Put $A(k) = \sum_{j=1}^k \alpha_j$ for $k = 1, 2, \dots, m$. Let $s(k)$ be an element of $\mathbb{R}^*$ such that $\int_{x > s(k)} d\mu(x) \leq A(k) \leq \int_{x \geq s(k)} d\mu(x)$ for $k = 1, 2, \dots, m$ (see Note (2.7)).

According to Theorem (2.8) $\mu \vdash \lambda$, where $\lambda = \sum_{j=1}^m \alpha_j \delta_{a_j}$, $a_1 \geq a_2 \geq \dots \geq a_m$, if and only if $\sum_{j=1}^k \alpha_j a_j \leq \int_{x > s(k)} x . d\mu(x) + (A(k) - \int_{x > s(k)} d\mu(x)) . s(k)$ for $k = 1, 2, \dots, m$ and $\sum_{j=1}^m \alpha_j a_j = \int_{\mathbb{R}} x . d\mu(x)$.

These conditions of comparability can be written in a simple form:

$a . v^k \leq \langle b, w^k \rangle$ for $k = 1, 2, \dots, m-1$ and

$a \cdot v^m = \langle b, w^m \rangle$, where

$$v^k = (v_j^k)_{j=1}^m = \frac{1}{A(k)}(\alpha_1, \alpha_2, \dots, \alpha_k, 0, 0, \dots, 0) \; ,$$

$$w^k = \frac{1}{A(k)} \cdot \chi_{(s(k),\infty)} + \frac{1}{\mu(s(k))} \cdot \left(1 - \frac{1}{A(k)} \cdot \mu((s(k),\infty)) \cdot \chi_{(s(k))}\right)$$

(we put $\frac{0}{0} = 0$) and $\langle \; , \; \rangle$ is the duality between the spaces $L^1(\mu)$ and $L^\infty(\mu)$. Clearly, $0 \le w^k \le \frac{1}{A(k)}$, $w^k \in L^\infty(\mu)$ and $\mu(w^k) = 1$ for $k = 1, 2, \dots, m$. There is one and only one real vector $c = (c_j)_{j=1}^m$ such that

(3.3) $\qquad c \cdot v^k = \langle b, w^k \rangle$ for $k = 1, 2, \dots, m$.

Define a matrix $V = (v_j^k)_{k,j}$ of the type (m, m) and an operator $W = (w^k)_k$ of the type (m, R) , i.e. each row w^k is a real function defined on the set R for $k = 1, 2, \dots, m$.

Then the conditions (3.3) can be written in a simple form

(3.4) $\qquad Vc = Wb$.

Since there exists the inverse matrix V^{-1} , we obtain $c = V^{-1}(Wb) = (V^{-1} \cdot W)b = E_{\alpha,\mu} b$, i.e. $c_j = \langle b, e^j \rangle = \mu(e^j \cdot b)$ for $j = 1, 2, \dots, m$, where $E_{\alpha,\mu} = (e^j)_j = V^{-1} \cdot W$ is an operator of the type (m, R) . Clearly, $V \cdot E_{\alpha,\mu} = W$, $E^*_{\alpha,\mu} \cdot V^* = W^*$ and $E^*_{\alpha,\mu} v^k = w^k$ for $k = 1, 2, \dots, m$, where $E^*_{\alpha,\mu}$ and W^* are the adjoint operators and V^* is the adjoint matrix.

Now, we can establish the functions e^k for $k = 1, 2, \dots$ $\dots, m$, i.e. the row of the operator $E_{\alpha,\mu}$:

$$e^1 = \frac{1}{\alpha_1} \cdot \Big[\frac{1}{\mu(s(1))} \cdot (A(1) - \mu((s(1), \infty))) \cdot \chi_{(s(1))} +$$
$$+ \chi_{(s(1),\infty)} \Big] \; ,$$

$$e^k = \frac{1}{\mu(s(k))} \cdot \chi_{(s(k))} \qquad \text{for } s(k) = s(k-1) \ ,$$

$$e^k = \frac{1}{\alpha_k} \cdot \Big[\frac{1}{\mu(s(k))} \cdot (A(k) - \mu((s(k), \infty))) \cdot \chi_{(s(k))} + \chi_{(s(k)), s(k-1))} +$$
$$+ \frac{1}{\mu(s(k-1))} \cdot (\mu(\langle s(k-1), \infty)) - A(k-1)) \cdot \chi_{(s(k-1))} \Big]$$

for $s(k) < s(k-1)$

for $k = 2, 3, \dots, m-1$ and

$$e^m = \frac{1}{\alpha_m} \Big[\chi_{(-\infty, s(m-1))} + \frac{1}{\mu(s(m-1))} \cdot (\mu(\langle s(m-1), \infty) -$$
$$- A(m-1)) \cdot \chi_{(s(m-1))} \Big]$$

(where we put $\frac{0}{0} = 0$).

Clearly: $0 \leq e^k \leq \frac{1}{\alpha_k}$, $e^k \in L^\infty(\mu)$, $\mu(e^k) = 1$, $\sum_{j=1}^{m} \alpha_j e^j = 1$, μ -a.e. and the functions e^k are supported by the invervals $\langle s(k), s(k-1) \rangle$ for $k = 1, 2, \dots, m$, where $s(0) = \infty$.

If $k \in \{1, 2, \dots, m-1\}$, then $c_k = \mu(b \cdot e^k) = \int_{x \geq s(k)} x \cdot e^k(x) \cdot d\mu(x) \geq s(k) \cdot \int_{x \geq s(k)} e^k(x) \cdot d\mu(x) = s(k) = s(k) \cdot \int_{x \geq s(k)} e^{k+1}(x) \cdot d\mu(x) \geq \int_{x \geq s(k)} x \cdot e^{k+1}(x) \cdot d\mu(x) = \mu(b \cdot e^{k+1}) = c_{k+1}$.

Hence $\infty > c_k \geq s(k) \geq c_{k+1} > -\infty$ and $c_1 \geq c_2 \geq \dots \geq c_m$.

Put $\sigma = \sum_{j=1}^{m} \alpha_j \delta_{c_j}$. According to Theorem (2.8) $\mu \succ \sigma$.

If $\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}$, $a_1 \geq a_2 \geq \dots \geq a_m$, $\mu \succ \lambda$, then $a \cdot v^k \leq \langle b, w^k \rangle = c \cdot v^k$ for $k = 1, 2, \dots, m-1$ and $a \cdot v^m = \langle b, w^m \rangle = c \cdot v^m$ by (2.8), hence according to Theorem (2.8) $\sigma \succ \lambda$. If $\sigma' = \sum_{j=1}^{m} \alpha_j \delta_{c'_j}$, where $c'_1 \geq c'_2 \geq \dots \geq c'_m$, is also α, μ -conditionally maximal measure, then $\sigma' \succ \sigma$ and $\sigma \succ \sigma'$. According to Theorem

(2.7) in [3] we have $\sigma' = \sigma$. Hence σ is the unique α, μ-conditionally maximal measure.

(3.5) Remark. The classes $e^k \in L^\infty(\mu)$ do not depend on the choice of $s(k) \in S(k) = \{s \in R^*; \int_{x>s} d\mu(x) \le \sum_{j=1}^{k} \alpha_j \le \int_{x\ge s} d\mu(x)\}$ for $k = 1, 2, \dots, m$.

4. α, μ -stochastic operators

(4.1) Definition. An operator $Q = (q^j)_j$ of the type (m, R) will be called α, μ-stochastic iff

$q^j \in L^\infty(\mu)$, $\mu(q^j) = 1$, $0 \le q^j$ for all j and

$\sum_{j=1}^{m} \alpha_j q^j = 1$ μ -a.e.

(4.2) Define the set $V_\alpha = \{d = (d_j)_{j=1}^m ; d_j = \alpha_j \cdot x_j$ for all j, $x_1 \ge x_2 \ge \dots \ge x_m\}$.

Then V_α is a convex wedge generated by the vectors $v^1, v^2, \dots, v^m, -v^m$ introduced in (3.2).

(4.3) Define the set $\mathring{V}_\alpha = \{\gamma \in R_m ; \gamma \cdot d \ge 0$ for all $d \in V_\alpha\}$.

Then $\mathring{V}_\alpha$ is a convex cone generated by the following vectors:

$\gamma^1 = (\alpha_2, -\alpha_1, 0, \dots, 0)$, $\gamma^2 = (0, \alpha_3, -\alpha_2, 0, \dots, 0)$, $\dots$,

$\gamma^{m-1} = (0, \dots, 0, \alpha_m, -\alpha_{m-1})$.

(4.4) Theorem. Let b be a nondecreasing function defined on the set R such that $b \in L^1(\mu)$. Put $\eta = \int_R \delta_{b(x)} \cdot d\mu(x)$ (see [2],Chap.6). Let λ be a measure such that $\lambda = \sum_{j=1}^{m} \alpha_j \delta_{a_j}$, where $a_1 \ge a_2 \ge \dots \ge a_m$. Put $a = (a_j)_{j=1}^m$. Then the following conditions are equivalent:

1^o $\eta \succ \lambda$.

2^o There exists an α, μ -stochastic operator $Q = (q^j)_j$ such that $Qb = a$ (i.e. $\mu(q^j \cdot b) = a_j$ for $j = 1, 2, \dots, m$) .

3^o $E_{\alpha,\mu} b - a \in \mathring{V}_\alpha$.

Proof. $1^o \Longrightarrow 3^o$: Let $s(k)$ be an element of R^* such that $\int_{x > s(k)} d\mu(x) \leq A(k) \leq \int_{x \geq s(k)} d\mu(x)$, where $A(k) = \sum_{j=1}^{k} \alpha_j$ for $k = 1, 2, \dots, m$.
Suppose $\eta \succ \lambda$ and put $u(k) = b(s(k))$. Then

$$\int_{x > s(k)} d\eta(t) \leq \int_{x > s(k)} d\mu(x) \leq A(k) \leq \int_{x \geq s(k)} d\mu(x) \leq \int_{t \geq u(k)} d\eta(t)$$

for $k = 1, 2, \dots, m$.

According to Theorem (2.8), we obtain the following:

$$a \cdot v^k \leq \frac{1}{A(k)} \cdot \int_{t > u(k)} t \cdot d\eta(t) + (1 - \frac{1}{A(k)} \cdot \int_{t > u(k)} d\eta(t)) \cdot$$

$$\cdot u(k) \leq \langle b, w^k \rangle = \langle b, E^*_{\alpha,\mu} v^k \rangle = E_{\alpha,\mu} b \cdot v^k$$

for $k = 1, 2, \dots, m-1$ and

$$a \cdot v^m = \int_R t \cdot d\eta(t) = \langle b, w^m \rangle = \langle b, E^*_{\alpha,\mu} v^m \rangle = E_{\alpha,\mu} b \cdot v^m .$$

Hence $E_{\alpha,\mu} b - a \in \mathring{V}_\alpha$ by (4.2) and (4.3).

$2^o \Longrightarrow 1^o$: Let $Q = (q_j)_{j=1}^m$ be an α, μ -stochastic operator such that $Qb = a$ and let f be an element of $\check{C}_L$. Then

$$\lambda(f) = \sum_{j=1}^{m} \alpha_j \cdot f(a_j) = \sum_{j=1}^{m} \alpha_j \cdot f(\mu(q^j \cdot b)) \leq$$

$$\leq \sum_{j=1}^{m} \alpha_j \cdot \mu(q^j \cdot f \circ b) = \mu(f \circ b) = \eta(f)$$

since $f(\mu(q^j \cdot b)) = l_j(\mu(q^j \cdot b)) = \mu(q^j \cdot l_j \circ b) \leq \mu(q^j \cdot f \circ b)$ for all j , where l_j is an affine function in L such that $\ell_j(x) = f(a_j) + f'(a_j).(x - a_j)$ for all $x \in R$ and where the symbol $\circ$ denotes the composition of two maps.

$3^o \Longrightarrow 1^o$: Clearly, the operator $E_{\alpha,\mu}$ defined in (3.5) is α,μ -stochastic. Put $\sigma = \sum_{j=1}^{m} \alpha_j \sigma_{a_j}$, $c = (c_1, c_2, \dots, c_m) = E_{\alpha,\mu} b$. Then $\eta \vdash \sigma$ by the previous. The condition 3^o implies that $\sigma \vdash \lambda$ by [3]. Hence $\eta \vdash \lambda$.

$1^o \Longrightarrow 2^o$: Suppose $\eta \vdash \lambda$. According to Lemma (2.1) there is a linear map $B : C_L \longrightarrow L^\infty(\lambda)$ such that $B \geq 0$, $B(\ell) = \ell$ for each $\ell \in L$ and $\eta(f) = \lambda(B(f))$ for each $f \in C_L$. Put $A = \{a_1, a_2, \dots, a_m\}$. Then the map B can be considered as a map from C_L to $C(A)$. According to the Lebesgue-Nikodym theorem there are elements $p^j \in L^\infty(\eta)$ such that $B(f)(a_j) = \eta(p^j.f)$, $0 \leq p^j \leq \frac{1}{\alpha_j}$ for $j = 1, 2, \dots, m$ and for all $f \in C_L$. Put $q^j = p^j \circ b$. Then $B(f)(a_j) = \mu(q^j.f \circ b)$ for all $f \in C_L$, $\mu(q^j.b) = a_j$, $0 \leq q^j \leq \frac{1}{\alpha_j}$, $q^j \in L^\infty(\mu)$, $\sum_{j=1}^{m} \alpha_j q^j = 1$ and $\mu(q^j) = 1$. Hence the operator $Q = (q^j)_{j=1}^{m}$ is α,μ-stochastic and $Qb = a$.

(4.5) <u>Theorem</u>. Let $D_{\alpha,\mu}$ be the set of all α,μ-stochastic operators. Define $U_\alpha = \{d ; d = (d_j)_{j=1}^{m}$, $d_j = \alpha_j . x_j$ for all j, $x_1 > x_2 > \dots > x_m\}$ and $U^\mu = \{b ; b(x) = \int_{\xi \leq x} h(\xi) . d\mu(\xi)$ for all $x \in R$, $h \in L^1(\mu)$, $h > 0$ μ - a.e.$\}$. Then the operator $E_{\alpha,\mu}$ is the unique $U_\alpha \otimes U^\mu$ -exposed element of the set $D_{\alpha,\mu}$, where $U_\alpha \otimes U^\mu = \{d \otimes b ; d \in U_\alpha, b \in U^\mu\}$ (i.e. $E_{\alpha,\mu} b . d > Q b . d$ for all $d \in U_\alpha$, all $b \in U^\mu$ and all $Q \in D_{\alpha,\mu}$, $Q \neq E_{\alpha,\mu}$).

Proof. Suppose $Q \in D_{\alpha,\mu}$, $d \in \mathcal{U}_\alpha$ and $b \in \mathcal{U}^\mu$. Then according to Theorem (4.4) the following inequality holds:

$$E_{\alpha,\mu} b . d \geq Q b . d \quad . \tag{4.6}$$

There are positive numbers $t_1, t_2, \dots, t_{m-1}$ and a number t_m such that $d = \sum_{k=1}^{m} t_k . v^k$ and a function $h \in L^1(\mu)$ such that h is μ-a.e. positive and $b = \int_R \psi_\xi . h(\xi) . d\mu(\xi)$, where $\psi_\xi(x) = 0$ for all $x \in \xi$ and $\psi_\xi(x) = 1$ for all $x \geq \xi$. According to Theorem (4.4), we obtain the inequalities

$$E_{\alpha,\mu} \psi_\xi . v^k \geq Q \psi_\xi . v^k \text{ for } k = 1, 2, \dots, m-1 \tag{4.7}$$

and $E_{\alpha,\mu} \psi_\xi . v^m = Q \psi_\xi . v^m$ for all $\xi \in R$.

If the equality holds in (4.6), then

$$\sum_{k=1}^{m} \int_R (E_{\alpha,\mu} \psi_\xi . v^k - Q \psi_\xi . v^k) . h(\xi) . t_k . d\mu(\xi) = 0 .$$

Since the matrix V^{-1} exists, using (4.7) we obtain the equality

$$\int_R (E_{\alpha,\mu} - Q)(x) . \psi_\xi(x) . d\mu(x) = 0 \text{ for } \mu\text{-a.e. } \xi \in R ,$$

i.e.

$$\int_{x \geq \xi} (E_{\alpha,\mu} - Q)(x) . d\mu(x) = 0 \text{ for } \mu\text{-a.e. } \xi \in R .$$

It follows from the Lebesgue-Nikodym theorem that $E_{\alpha,\mu} = Q$ μ-almost everywhere. The proof is complete.

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