Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 11 (1970), No. 4, 595--618

Persistent URL: http://dml.cz/dmlcz/105303

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11,4 (1970)

TRANSFORMATIONS DETERMINING UNIQUELY A MONOID

Marie MÜNZOVÁ , Praha

If a transformation $f: X \longrightarrow X$ of a finite set X has a suitable structure, then there exists a monoid $M = (X, \cdot)$ having X for its underlying set and such that f is its left translation expressible in the form $f(x) = a \cdot x$ for some a and all x in X. It may happen that such a monoid M is unique. In this case we shall call f a <u>determining transformation</u> of the monoid M. Our aim is to describe all finite transformations determining, in this sense, some monoid. To this purpose, we are constantly using the basic results on translations of semigroups established in [1] and [2].

Let us assume that $f: X \longrightarrow X$ is a left translation of some monoid M (such transformations are called potential translations in [1]). If we are in the position that only f is given and the problem is to find a monoid M with f being its left translation, we can proceed in two steps:

1) first to find the whole system $L(M)$ of all left translations of M;

2) then to choose a suitable identity element e in X.

These two steps are based on a statement, the proof of which can be found in [2], characterizing the systems $L(M)$ and $R(M)$ of all the left and all the right translations of a monoid M in terms of transformations.

Statement 1: Let L be a system of transformations of a set X. There exists a monoid $M = (X, \cdot)$ with $L(M) = L$ if and only if one of the following conditions holds:

(A) L is a transformation monoid and there exists an element e in X such that for every x in X there is one and only one f in L such that $f(e) = x$;

(B) there exists an element e in X such that for every x in X there exist f in L and g in the centralizer $\mathcal{C}(L)$ of L (i.e. g commutes with all f from L) such that $f(e) = g(e) = x$.

Any point e satisfying (A) or (B), and only such a point, becomes an identity of a monoid on X with regard to the multiplication $x \cdot y = f_x(y)$ where f_x is the unique transformation in L with $f_x(e) = x$.

If we have a system S of transformations of a set X, we call any point e in X such that for every x there exists f in S such that $f(e) = x$ a <u>source</u> of S. The above statement deepens the well known fact that $L(M)$ and $R(M)$ centralize each other and the identity element of M is a common source of both $L(M)$ and $R(M)$.

To describe the structure of a given finite transformation $f: X \to X$ we shall use the following notions and characteristics:

The set $D_f(x) = \{f^k(x) \mid k = 0, 1, 2, \ldots\}$ is called the <u>path</u> of the element x, $x \in X$. If for x and y in X is $D_f(x) \cap D_f(y) \neq 0$, then x and y are E_f-<u>equivalent</u> and we write $x E_f y$. This means explicitly that for some $m, n \geq 0$ we have $f^m(x) = f^n(y)$. E_f-classes form the decomposition of X into <u>components</u> of $f: X \to X$. $E_f(x)$

denotes the component containing x (then we can write $y \in E_f(x)$ instead of $x E_f y$). Transformations with just one component are connected transformations. A transformation which has more than one component is called disconnected transformation.

Let $\mu(x)$ and $\kappa(x)$, $\mu(x) \geq 0$, $\kappa(x) \geq 1$, denote the least integer for which the identity $f^{\kappa(x)+\mu(x)}(x) = f^{\mu(x)}(x)$ holds.

The element x with $\mu(x) = 0$ is called cyclic; all cyclic elements in $D_f(x)$ form the cycle $Z(x)$ of x, and, clearly $\kappa(x) = |Z(x)|$ is the order of the cycle of x. (We use bars to designate the number of elements of a set.)

An element e is a top element of f if $\mu(e) \geq \mu(x)$ and $\kappa(x)$ divides $\kappa(e)$ for all x in X. An element b is a bottom element if $\mu(b) = 0$ and $\kappa(b)$ divides $\kappa(x)$ for all x in X. (Of course, f may have neither top nor bottom elements.)

Let f be a finite disconnected transformation with a top element e, the component which contains the top element is called the main component of the transformation f.

By $f^{-k}(x)$, $k \geq 0$, is designated the set of all t in X with $f^{k}(t) = x$.

Now we can formulate anew the results of [2].

Statement 2: Let f be a finite transformation. Then

1) there exists a monoid M such that $f \in L(M)$ if and only if f has a top element;

2) there exists a commutative monoid M such that $f \in L(M)$ if and only if f has both a top and a bottom element.

Let $f : X \to X$ be a finite transformation with a top element e. For every x in $E_f(e)$ we shall define its difference $d(x)$ (with regard to the top element)

(2) $$d(x) = u(e) + m(x) - u(x)$$

where $m(x)$ is the integer uniquely determined by the conditions

(3) $$f^{u(e)+m(x)}(e) = f^{u(x)}(x), \quad 0 \leq m(x) < n(e) .$$

Let $E_f(x)$ be a component of a finite transformation f; for y in $E_f(x)$ we can define the difference y with regard to x (designated by $d(x,y)$) in the same way as above.

Further, we shall need the following lemma:

Lemma 1: Let $f : X \to X$ be a finite transformation with a top element; let x be an element from $E_f(e)$ and k be an arbitrary integer $k \geq 0$. Then for $y = f^k(x)$ it holds

(4) $$f^{d(y)} = f^{d(x)+k} .$$

Proof: If $u(x) > k$, then $u(y) = u(x) - k$ and $m(y) = m(x)$, therefore

$$d(y) = u(e) + m(y) - u(y) = u(e) + m(x) - (u(x) - k) = d(x) + k .$$

If $u(x) \leq k$, then $u(y) = 0$; it follows that

$$f^{d(y)}(e) = f^{u(e)+m(y)-u(y)}(e) = f^{u(e)+m(x)-u(x)+k}(e) = f^{d(x)+k}(e) .$$

We get the assertion using the easily verified equivalence

$$f^{m}(x) = f^{n}(x) \Longleftrightarrow (m = n) \text{ or } (m \geq u(x) \text{ and } n(x) \mid m - n) .$$

We know now the structure of finite transformations being members of some system $L(M)$ of left translations of a monoid M. For us it is important to know the form of a monoid M such that $L(M)$ contains f. The answer is given by the following construction.

Let $f : X \to X$ be a finite transformation with a top element e. Denote $E_f(e)$ its main component (i.e. the E_f -

-class containing e) and $Y = X - E_f(e)$ its complement. (If f is a connected transformation then Y is void.)

Construction 1: Let $f: X \to X$ be a finite transformation with a top element e. The family of transformations $L(M_p) = \{f_x, x \in X\}$ and $R(M_p) = \{g_y, y \in X\}$ are systems of all the left and all the right translations of the monoid M_p, where

(5) $L(M_p)$: for $x \in E_f(e)$ it is $f_x(t) = f^{d(x)}(t)$ $(d(e) = 0)$ for $t \in E_f(e)$;

(6) for $x \in E_f(e)$ and $t \in Y$ it is $f_x(t) = f^{d(x)}(t)$;

(7) for $x \in Y$ it is $f_x(t) = p(x)$.

(8) $R(M_p)$: $g_e = 1_x$;

(9) for $t \in E_f(e)$, $y \in E_f(e)$ put $g_y(t) = f^{d(t)}(y)$;

(10) for $t \in E_f(e)$, $y \in Y$ put $g_y(t) = f^{d(t)}(y)$;

(11) for $t \in Y$ put $g_y(t) = p(t)$;

where $p: Y \to Y$ is a transformation such that

(12) $p(f(x)) = f(p(x))$ for every x in Y and

(13) $p \circ p = p$.

Demonstration: By (5) e is a source of $L(M_p)$, because $g_y(e) = y$ for all y in X. Hence e is a source of $L(M_p)$ and $R(M_p)$. Now we must demonstrate that $\mathcal{C}(L(M_p)) = R(M_p)$.

Commutativity of every f_x with g_e is obvious by (8).

1) For $x \in E_f(e)$, $t \in E_f(e)$, $y \neq e$, $y \in E_f(e)$ and $y \in Y$:

$$f_x \circ g_y(t) = f_x(f^{d(t)}(y)) = f^{d(x)+d(t)}(y),$$

$$g_y \circ f_x(t) = g_y(f^{d(x)}(t)) = f^{d(x)+d(t)}(y)$$ by lemma 1 for $t \neq e$;

for $t = e$:

$$f_x \circ g_y(e) = f_x(y) = f^{d(x)}(y) = g_y(x) = g_y \circ f_x(e).$$

2) For $x \in E_p(e)$, $t \in Y$, $y \neq e$:

$f_x \circ g_y(t) = f_x(p(t)) = f^{d(x)}(p(t)) = p(f^{d(x)}(t))$ by (12);

$g_y \circ f_x(t) = g_y(f^{d(x)}(t)) = p(f^{d(x)}(t))$, because $f^{d(x)}(t) \in Y$.

3) For $x \in Y$, $t \in E_f(e)$, $y \neq e$:

$f_x \circ g_y(t) = f_x(f^{d(t)}(y)) = p(x)$, because $f^{d(t)}(y) \in Y$;

for $t = e$ $\quad g_y \circ f_x(e) = g_y(x) = p(x)$,

for $t \neq e$ $\quad g_y \circ f_x(t) = g_y(p(x)) = p(p(x)) = p(x)$ by (13).

4) For $x \in Y$, $t \in Y$, $y \neq e$:

$f_x \circ g_y(t) = f_x(p(t)) = p(x)$, because $p(t) \in Y$;

$g_y \circ f_x(t) = g_y(p(x)) = p(p(x)) = p(x)$ by (13).

The demonstration is complete.

Now we can formulate a simple consequence of the construction 1.

Corollary: If $f : X \to X$ is a determining transformation, then:

1) f has one and only one top element;

2) the $p : Y \to Y$ satisfying (12) and (13) is the identical transformation;

3) if f has at most two components, then M_p is a commutative monoid.

Proof: 1) and 2) is evident.

3) If f has at most two components, then f has a bottom elemnt; this means that f belongs to $L(M)$ where M is a commutative monoid; $f \in L(M_p)$ therefore $M_p = M$.

Now we shall confine ourselves to connected transformations.

Theorem 1: A finite connected transformation $f : X \to X$ is a determining transformation if and only if

(i) there is a unique top element e ,

(ii) for every x in X it is $f(x) \in D_f(e)$,

(iii) $|f^{-1}(f^2(x))| = 1$ for all $x \in E_f(e) \setminus D_f(e)$,

(iv) if there exist $x_i, x_j \in E_f(e) \setminus D_f(e)$ such that for some b in X it is $b \in f^{-1}(f^{d(x_j)+1}(x_i))$ and $b \neq f^{d(x_j)}(x_i)$, then $d(b) = d(x_j)$, $f(x_i)$ is not in the cycle $Z(e)$ and $\kappa(e)$ does not divide $d(x_j)$.

Proof: Designate by T the set $T = E_f(e) \setminus D_f(e)$. First we shall show the necessity of these conditions. Conditions (i) and (ii) are settled by corollary of the construction 1.

Let us suppose that the conditions (iii) or (iv) are not fulfilled; then we are able to construct a monoid M which is different from M_n (as given in the construction 1) and such that $f \in L(M)$.

I. There exist elements x_i and x_j in T or $x_j = f(e)$ such that $b \in f^{-1}(f^{d(x_j)+1}(x_i))$, $b \neq f^{d(x_j)}(x_i)$ and $d(b) \neq d(x_i)$, $d(b) \neq d(x_j)$. We can suppose that $b \neq x_i$, $b \neq x_j$.

Construction 2: Let $f: X \to X$ be the transformation described above. Then there exists a monoid M such that $L(M) = \{\tilde{f}_x, x \in X\}$, $R(M) = \{\tilde{g}_y, y \in X\}$ defined as follows:

(14) $L(M)$: $\tilde{f}_x = f_x$ for $x \neq x_i$;

(15) $\tilde{f}_{x_i}(t) = f_{x_i}(t)$ for $t \neq x_j$, $\tilde{f}_{x_i}(x_j) = b$;

(16) $R(M)$: $\tilde{g}_y = g_y$ for $y \neq x_j$;

(17) $\tilde{g}_{x_j}(t) = g_{x_j}(t)$ for $t \neq x_i$, $\tilde{g}_{x_j}(x_i) = b$;

where f_x and g_y are transformations from the construction I.

Demonstration: By (14) and (16), e is the source of both $L(M)$ and $R(M)$. We must show that transformations $\tilde{g}_y$ commute with $\tilde{f}_x$ for $x \in X$ and $y \in X$. Because $\tilde{f}_x$ and $\tilde{g}_y$, for $x \neq x_i$, $y \neq x_j$ are the transformations from the construction 1, we know that $\tilde{f}_x \circ \tilde{g}_y = \tilde{g}_y \circ \tilde{f}_x$ for $x \neq x_i$, $y \neq x_j$.

1) $\tilde{f}_{x_i}$ commutes with $\tilde{g}_{x_j}$:

a) For $t = e$:

$$\tilde{f}_{x_i} \circ \tilde{g}_{x_j}(e) = \tilde{f}_{x_i}(x_j) = b, \quad \tilde{g}_{x_j} \circ \tilde{f}_{x_i}(e) = \tilde{g}_{x_j}(x_i) = b .$$

b) For $t = x_i$:

let $x_i \neq x_j$ $\tilde{g}_{x_j} \circ \tilde{f}_{x_i}(x_i) = \tilde{g}_{x_j}(f^{d(x_i)}(x_i)) = f^{2d(x_i)}(x_j) = f^{2d(x_i)+d(x_j)}(e)$,

$f^{d(x_i)}(x_i) \in D_f(e)$ because $d(x_i) \geq 1$ (f has only one top element). Let $x_i = x_j$, then $\tilde{g}_{x_j} \circ \tilde{f}_{x_i}(x_i) = \tilde{g}_{x_j}(b) =$
$= f^{d(b)}(x_j) = f^{2d(x_j)}(x_j) = f^{3d(x_j)}(e)$;

$$\tilde{f}_{x_i} \circ \tilde{g}_{x_j}(x_i) = \tilde{f}_{x_i}(b) = f^{d(x_i)}(b) = f^{2d(x_i)+d(x_j)}(e) .$$

c) For $t = x_j$ and

for $x_i = x_j$ it is $\tilde{f}_{x_i} \circ \tilde{g}_{x_j}(x_j) = \tilde{f}_{x_i}(b) = f^{d(x_i)}(b) = f^{3d(x_i)}(e) = f^{2d(x_i)}(x_i)$;

for $x_i \neq x_j$ it is $\tilde{f}_{x_i} \circ \tilde{g}_{x_j}(x_j) = \tilde{f}_{x_i}(f^{d(x_j)}(x_j)) = f^{d(x_i)+d(x_j)}(x_j)$ by lemma 1, $\tilde{g}_{x_j} \circ \tilde{f}_{x_i}(x_j) = \tilde{g}_{x_j}(b) = f^{d(b)}(x_j) = f^{d(x_j)+d(x_i)}(x_j)$.

d) For $t \in X$, $t \neq x_j$, $t \neq x_i$, $t \neq e$:

$$\tilde{f}_{x_i} \circ \tilde{g}_{x_j}(t) = \tilde{f}_{x_i}(g_{x_j}(t)) = f_{x_i}(g_{x_j}(t)) = \tilde{g}_{x_j}(f_{x_i}(t)) = \tilde{g}_{x_j} \circ \tilde{f}_{x_i}(t) ,$$

because $g_{x_j}(t) \neq x_i, x_j, e$ and $f_{x_i}(t) \neq x_i, x_j, e$.

2) $\tilde{g}_{x_j}$ commutes with $\tilde{f}_x$, $x \in X$. This has been proved for $x \neq x_i$; by (14) $\tilde{f}_x = f_x$ for $x \neq x_i$.

a) For $t = e$:

$$\tilde{g}_{x_j} \circ f_x(e) = \tilde{g}_{x_j}(x) = f^{d(x)}(x_j); \quad f_x \circ \tilde{g}_{x_j}(e) = f_x(x_j) = f^{d(x)}(x_j) .$$

b) For $t = x_i$:

$\tilde{g}_{x_j} \circ f_x(x_i) = \tilde{g}_{x_j}(f^{d(x)}(x_i)) = \tilde{g}_{x_j}(x) = f^{d(x)}(x_j) = f^{d(x)+d(x_i)}(x_j) = f^{d(x)+d(x_i)+d(x_j)}(e)$,

$f_x \circ \tilde{g}_{x_j}(x_i) = f_x(b) = f^{d(x)}(b) = f^{d(x)-1}(f(b)) = f^{d(x)+d(x_j)}(x_i) = f^{d(x)+d(x_i)+d(x_j)}(e)$, because we can suppose that $x \neq e$.

c) For $t \in X$, $t \neq e$, $t \neq x_i$:

$\tilde{g}_{x_j} \circ f_x(t) = \tilde{g}_{x_j}(f_x(t)) = g_{x_j}(f_x(t)) = f_x(g_{x_j}(t)) = f_x \circ \tilde{g}_{x_j}(t)$, because $f_x(t) \neq x_i, e$.

3) $\tilde{g}_y, y \in X$ commute with $\tilde{f}_{x_i}$. This has been proved for $y = x_j$. By (16) $\tilde{g}_y = g_y$ for $y \neq x_j$.

a) For $t = e$:

$$g_y \circ \tilde{f}_{x_i}(e) = g_y(x_i) = f^{d(x_i)}(y), \quad \tilde{f}_{x_i} \circ g_y(e) = \tilde{f}_{x_i}(y) = f^{d(x_i)}(y) .$$

b) For $t = x_j$:

$g_y \circ \tilde{f}_{x_i}(x_j) = g_y(b) = f^{d(b)}(y) = f^{d(x_i)+d(x_j)}(y)$ by lemma 1,

$\tilde{f}_{x_i} \circ g_y(x_j) = \tilde{f}_{x_i}(f^{d(x_j)}(y)) = f^{d(x_i)}(f^{d(x_j)}(y)) = f^{d(x_i)+d(x_j)}(y)$, because we can suppose that $y \neq e$ (g_e is an identity).

c) For $t \in X$, $t \neq x_j$, $t \neq e$:

$$g_y \circ \tilde{f}_{x_i}(t) = g_y(f_{x_i}(t)) = f_{x_i}(g_y(t)) = \tilde{f}_{x_i} \circ g_y(t) ,$$ because $g_y(t) \neq x_j, e$.

Thus the construction 2 has been confirmed.

II. If $b \in f^{-1}(f^{d(x_j)+1}(x_i))$, $b \neq f^{d(x_j)}(x_i)$ and $d(b) = d(x_j)$, then $f(b)$ must be in the cycle $Z(a)$, because

(18) $\qquad f(b) = f^{d(x_j)+1}(x_i) = f^{d(b)+1}(x_i)$, $\qquad$ hence

(19) $\qquad f^{d(b)+1}(e) = f^{d(b)+1+d(x_i)}(e)$.

Condition (19) means that $\kappa(e)$ divides $d(x_i)$ $(d(x_i) \neq 0)$.

$f^{d(b)+1}(e) = f^{d(x_j)+1}(e) = f(x_j)$, hence $x_j \in f^{-1}(f^{d(x_j)+1}(e))$.

We can see that $f^{-1}(f^{d(x_j)+1}(x_i)) = \{y \in X \mid d(y) = d(x_j)\} \cup \{f^{d(x_j)}(x_i)\}$.

We can suppose that $|\{y \in X \mid d(y) = d(x_j)\} \cup \{f^{d(x_j)}(x_i)\}| = 2$, because $y \neq x_j$ and $d(y) = d(x_j)$ have no influence on the number of monoids which contain f as a left translation.

A) Let us suppose that $x_j \in f^{-1}(f^{d(x_j)+1}(x_i))$ and $f(x_i)$ is in the cycle $Z(e)$.

<u>Construction 3</u>: Let $f: X \to X$ be a finite connected

transformation with a top element e described above. Then $L(M)=\{\tilde{f}_x, x\in X\}$ is the system of all the left translations of a commutative monoid M, defined as follows:

(20) $L(M):$ $\tilde{f}_x = f_x$ for $x\neq x_i$, $x\neq x_j$;

(21) $\tilde{f}_{x_i}(x_j)=x_j$; $\tilde{f}_{x_i}(x_i)=x_i$; $\tilde{f}_{x_i}(t)=f_{x_i}(t)$, $t\neq x_i, x_j$;

(22) $\tilde{f}_{x_j}(x_i)=x_j$; $\tilde{f}_{x_j}(t)=f_{x_j}(t)$; $t\neq x_i$;

where f_x, $x\in X$ are transformations from the construction 1.

Demonstration: We must show that $R(M)=L(M)$. By (20) the source of $L(M)$ is the element e.

1) $\tilde{f}_x$ commutes with $\tilde{f}_y$, $x,y\in X$, $x\neq x_i, x_j$, $y\neq x_i, x_j$.

a) For $t=e$:

$$\tilde{f}_x\circ\tilde{f}_y(e)=\tilde{f}_x(y)=f^{d(x)}(y)=f^{d(x)+d(y)}(e)=f^{d(y)}(f^{d(x)}(e))=\tilde{f}_y(f^{d(x)}(e))=\tilde{f}_y\circ\tilde{f}_x(e).$$

b) For $t\neq e$:

$$\tilde{f}_x\circ\tilde{f}_y(t)=\tilde{f}_x(f^{d(y)}(t))=f^{d(x)+d(y)}(t)=f^{d(y)}(f^{d(x)}(t))=\tilde{f}_y(f^{d(x)}(t))=\tilde{f}_y\circ\tilde{f}_x(t).$$

2) For $\tilde{f}_{x_i}$ and $\tilde{f}_{x_j}$; $\tilde{f}_{x_i}$ commutes with $\tilde{f}_{x_j}$.

a) For $t=e$:

$$\tilde{f}_{x_i}\circ\tilde{f}_{x_j}(e)=\tilde{f}_{x_i}(x_j)=x_j, \quad \tilde{f}_{x_j}\circ\tilde{f}_{x_i}(e)=\tilde{f}_{x_j}(x_i)=x_j.$$

b) For $t=x_i$:

$$\tilde{f}_{x_i}\circ\tilde{f}_{x_j}(x_i)=\tilde{f}_{x_i}(x_j)=x_j, \quad \tilde{f}_{x_j}\circ\tilde{f}_{x_i}(x_i)=\tilde{f}_{x_j}(x_i)=x_j.$$

c) For $t=x_j$:

$\tilde{f}_{x_i}\circ\tilde{f}_{x_j}(x_j)=\tilde{f}_{x_i}(f^{d(x_j)}(x_j))=f^{d(x_i)+d(x_j)}(x_j)=f^{d(x_j)}(x_j)$, because we know that $n(e)$ divides $d(x_i)$ and $f^{d(x_j)}(x_j)\in Z(e)$.

$$\tilde{f}_{x_j}\circ\tilde{f}_{x_i}(x_j)=\tilde{f}_{x_j}(x_j)=f^{d(x_j)}(x_j).$$

d) For $t\in X$, $t\neq e$, $t\neq x_i$, $t\neq x_j$:

$$\tilde{f}_{x_i}\circ\tilde{f}_{x_j}(t)=\tilde{f}_{x_i}(f_{x_j})(t))=f_{x_i}(f_{x_j}(t))=f_{x_j}(f_{x_i}(t))=\tilde{f}_{x_j}(t))=\tilde{f}_{x_j}\circ\tilde{f}_{x_i}(t),$$

because $f_{x_j}(t)\neq x_i, x_j$, $f_{x_i}(t)\neq x_i, x_j$.

3) $\tilde{f}_x$, $x\in X$, commute with $\tilde{f}_{x_i}$. We already know that

it is true for $x = x_i$; now $\tilde{f}_x$, $x \neq x_i$, $x \neq x_j$ by (20) $\tilde{f}_x = f_x$ (we can suppose that $x \neq e$) .

a) For $t = e$:

$f_x \circ \tilde{f}_{x_j}(e) = f_x(x_j) = f^{d(x)}(x_j) = f^{d(x)+d(x_j)}(e)$,

$\tilde{f}_{x_j} \circ f_x(e) = \tilde{f}_{x_j}(x) = f^{d(x_j)}(x) = f^{d(x_j)+d(x)}(e)$.

b) For $t = x_i$:

$f_x \circ \tilde{f}_{x_j}(x_i) = f_x(x_j) = f^{d(x)}(x_j) = f^{d(x)-1}(f(x_j)) = f^{d(x)+d(x_j)}(x_i)$,

$\tilde{f}_{x_j} \circ f_x(x_i) = \tilde{f}_{x_j}(f^{d(x)}(x_i)) = f^{d(x_j)+d(x)}(x_i)$.

c) For $t \in X$, $t \neq x_i$, $t \neq e$:

$f_x \circ \tilde{f}_{x_j}(t) = f_x(f^{d(x_j)}(t)) = f^{d(x)+d(x_j)}(t) = f^{d(x_j)}(f^{d(x)}(t)) = \tilde{f}_{x_j} \circ f_x(t)$.

4) $\tilde{f}_{x_i}$ commutes with $\tilde{f}_x$ for $x \in X$. This is known for $x = x_j$ and evident for $x = x_i$. Thus $\tilde{f}_x = f_x$ for $x \neq x_j$, $x \neq x_i$ (we can suppose $x \neq e$) .

a) For $t = e$:

$\tilde{f}_{x_i} \circ f_x(e) = \tilde{f}_{x_i}(x) = f^{d(x_i)}(x) = f^{d(x_i)+d(x)}(e)$,

$f_x \circ \tilde{f}_{x_i}(e) = f_x(x_i) = f^{d(x)}(x_i) = f^{d(x)+d(x_i)}(e)$.

b) For $t = x_j$:

$\tilde{f}_{x_i} \circ f_x(x_j) = \tilde{f}_{x_i}(f^{d(x)}(x_j)) = f^{d(x_i)+d(x)}(x_j) = f^{d(x)}(x_j)$, because $f^{d(x)}(x_j) \in Z(e)$ and $\varkappa(e)$ divides $d(x_i)$.

$f_x \circ \tilde{f}_{x_i}(x_j) = f_x(x_j) = f^{d(x)}(x_j)$.

c) For $t = x_i$:

$\tilde{f}_{x_i} \circ f_x(x_i) = \tilde{f}_{x_i}(f^{d(x)}(x_i))$, we know that $f(x_i) \in Z(e)$, hence $f^{d(x)}(x_i) \in Z(e)$ ($d(x) \geq 1$) and $\varkappa(e)$ divides $d(x_i)$ and therefore $\tilde{f}_{x_i}(f^{d(x)}(x_i)) = f^{d(x)}(x_i)$.

$f_x \circ \tilde{f}_{x_i}(x_i) = f_x(x_i) = f^{d(x)}(x_i)$.

d) For $t \in X$, $t \neq e$, $t \neq x_i$, $t \neq x_j$:

$f_x \circ \tilde{f}_{x_i}(t) = f_x(f^{d(x_i)}(t)) = f^{d(x)+d(x_i)}(t) = f^{d(x_i)}(f^{d(x)}(t)) = \tilde{f}_{x_i} \circ f_x(t)$.

The construction 3 has been confirmed.

B) Let us suppose that $x_j \in f^{-1}(f^{d(x_j)+1}(x_i))$ and $\varkappa(e)$ divides $d(x_j)$.

Construction 4: Let $f: X \to X$ be a finite connected transformation with a top element e described above. Then the system $L(M) = \{\tilde{f}_x, x \in X\}$ such that

(23) $\tilde{f}_x = f_x$ for $x \neq x_i, x_j$;

(24) $\tilde{f}_{x_i}(x_j) = x_j$, $\tilde{f}_{x_i}(x_i) = x_j$, $\tilde{f}_{x_i}(t) = f_{x_i}(t)$ for $t \neq x_i, x_j$;

(25) $\tilde{f}_{x_j}(x_j) = x_j$, $\tilde{f}_{x_j}(x_i) = x_j$, $\tilde{f}_{x_j}(t) = f_{x_j}(t)$ for $t \neq x_i, x_j$;

where f_x, $x \in X$ are transformations from the construction 1, is the system of all translations of a commutative monoid M.

Demonstration: By (23) we see that e is a source of $L(M)$. We must show that $L(M) = R(M)$

1) $\tilde{f}_x$ commutes with $\tilde{f}_y$ for $x, y \in X$, $x, y \neq x_i, x_j$. This has been shown when confirming the construction 3.

2) $\tilde{f}_{x_i}$ commutes with $\tilde{f}_{x_j}$.

a) For $t = e$:

$$\tilde{f}_{x_i} \circ \tilde{f}_{x_j}(e) = \tilde{f}_{x_i}(x_j) = x_j, \quad \tilde{f}_{x_j} \circ \tilde{f}_{x_i}(e) = \tilde{f}_{x_j}(x_i) = x_j .$$

b) For $t = x_i$:

$$\tilde{f}_{x_i} \circ \tilde{f}_{x_j}(x_i) = \tilde{f}_{x_i}(x_j) = x_j, \quad \tilde{f}_{x_j} \circ \tilde{f}_{x_i}(x_i) = \tilde{f}_{x_j}(x_j) = x_j .$$

c) For $t = x_j$:

$$\tilde{f}_{x_i} \circ \tilde{f}_{x_j}(x_j) = \tilde{f}_{x_i}(x_j) = x_j, \quad \tilde{f}_{x_j} \circ \tilde{f}_{x_i}(x_j) = \tilde{f}_{x_j}(x_j) = x_j .$$

d) For $t \in X$, $t \neq e$, $t \neq x_i$, $t \neq x_j$:

$$\tilde{f}_{x_i} \circ \tilde{f}_{x_j}(t) = \tilde{f}_{x_i}(f_{x_j}(t)) = f_{x_i}(f_{x_j}(t)) = f_{x_j}(f_{x_i}(t)) = \tilde{f}_{x_j} \circ \tilde{f}_{x_i}(t) .$$

3) $\tilde{f}_{x_i}$ commutes with f_x, $x \in X$, $x \neq x_i$, $x \neq x_j$. We can suppose that $x \neq e$.

a) For $t = e$:

$$\tilde{f}_{x_i} \circ f_x(e) = \tilde{f}_{x_i}(x) = f^{d(x_i)}(x) = f^{d(x_i)+d(x)}(e) ,$$

$$f_x \circ \tilde{f}_{x_i}(e) = f_x(x_i) = f^{d(x)}(x_i) = f^{d(x)+d(x_i)}(e) .$$

b) For $t = x_i$:

$\tilde{f}_{x_i} \circ f_x(x_i) = \tilde{f}_{x_i}(f^{d(x)}(x_i)) = f^{d(x_i)+d(x)}(x_i)$,

$f_x \circ \tilde{f}_{x_i}(x_i) = f_x(x_j) = f^{d(x)}(x_j) = f^{d(x)+d(x_j)}(x_i) = f^{d(x)}(x_i) = f^{d(x)+d(x_i)}(x_i)$,

because $f^{d(x)}(x_j) \in Z(e)$ $(d(x) \geq 1)$ and $\kappa(e)$ divides $d(x_i)$ and $d(x_j)$.

c) For $t = x_j$:

$\tilde{f}_{x_i} \circ f_x(x_j) = \tilde{f}_{x_i}(f^{d(x)}(x_j)) = f^{d(x_i)+d(x)}(x_j) = f^{d(x)}(x_j)$, because $f^{d(x)}(x_j) \in Z(e)$ for $d(x) \geq 1$ and $\kappa(e)$ divides $d(x_i)$.

$$f_x \circ \tilde{f}_{x_i}(x_j) = f_x(x_j) = f^{d(x)}(x_j) .$$

d) For $t \in X$, $t \neq e$, $t \neq x_i$, $t \neq x_j$:

$\tilde{f}_{x_i} \circ f_x(t) = \tilde{f}_{x_i}(f^{d(x)}(t)) = f^{d(x_i)+d(x)}(t) = f^{d(x)}(f^{d(x_i)}(t)) = f_x \circ \tilde{f}_{x_i}(t)$.

4) $\tilde{f}_{x_j}$ commutes with $f_x, x \neq x_i$, $x \neq x_j$. This fact is evident for $x = e$.

a) For $t = e$:

$$\tilde{f}_{x_j} \circ f_x(e) = \tilde{f}_{x_j}(x) = f^{d(x_j)}(x) = f^{d(x_j)+d(x)}(e) ,$$

$$f_x \circ \tilde{f}_{x_j}(e) = f_x(x_j) = f^{d(x)}(x_j) = f^{d(x)+d(x_j)}(e) .$$

b) For $t = x_i$:

$\tilde{f}_{x_j} \circ f_x(x_i) = \tilde{f}_{x_j}(f^{d(x)}(x_i)) = f^{d(x_j)+d(x)}(x_i) = f^{d(x_j)+d(x)+d(x_i)}(e) = f^{d(x)+d(x_i)}(x_j) =$
$= f^{d(x)}(x_j)$, because $f^{d(x)}(x_j) \in Z(e)$ and $\kappa(e) | d(x_i)$.

$$f_x \circ \tilde{f}_{x_j}(x_i) = f_x(x_j) = f^{d(x)}(x_j) .$$

c) For $t = x_j$:

$\tilde{f}_{x_j} \circ f_x(x_j) = \tilde{f}_{x_j}(f^{d(x)}(x_j)) = f^{d(x_j)+d(x)}(x_j) = f^{d(x)}(x_j)$, because $\kappa(e)$ divides $d(x_j)$.

$$f_x \circ \tilde{f}_{x_j}(x_j) = f_x(x_j) = f^{d(x)}(x_j) .$$

d) For $t \in X$, $t \neq e$, $t \neq x_i$, $t \neq x_j$:

$\tilde{f}_{x_j} \circ f_x(t) = \tilde{f}_{x_j}(f^{d(x)}(t)) = f^{d(x_j)+d(x)}(t) = f^{d(x)}(f^{d(x_j)}(t)) = f_x \circ \tilde{f}_{x_j}(t)$.

It has been proved that $L(M) = R(M)$.

We have completely proved the necessity of conditions (i) - (iv). The sufficiency of these conditions is easily proved as follows. From (i) it follows that either $u(e) = 1$ and hence $X = D_f(e)$ or $u(e) > 1$ and then (ii) - (iv) mean that in $\mathcal{C}(f)$ there exist only a few elements with $|\{g \in \mathcal{C}(f) \mid g(e) = t\}| > 1$.

In $\mathcal{C}(f)$ there must exist a transformation g_t such that $g_t(e) = t$ for all t in X. g_t commutes with f, hence for $y \in D_f(e)$

$$g_t(y) = g_t(f^{d(y)}(e)) = f^{d(y)}(g_t(e)) = f^{d(y)}(t).$$

The transformation g_t is determined on $D_f(e)$.

Let $t \neq x_i, x_j$, $y \in T$ (for x_i, x_j applies $x_j \in f^{-1}(f^{d(x_j)+1}(x_i))$, $f(x_i) \notin Z(e)$ and $n(e)$ does not divide $d(x_j)$. Then

$$f(g_t(y)) = g_t(f(y)) = g_t(f^{d(y)+1}(e)) = f^{d(y)+1}(g_t(e)) = f^{d(y)+1}(t).$$

Thus $g_t(y) \in f^{-1}(f^{d(y)+1}(t))$. Conditions (ii) - (iv) mean that $|f^{-1}(f^{d(y)+1}(t))| = 1$; it is evident that $f^{d(y)}(t) \in$ $\in f^{-1}(f^{d(y)+1}(t))$, hence $g_t(y) = f^{d(y)}(t)$. It follows that for $t \neq x_i, x_j$ it is $g_t(y) = f^{d(y)}(t) = f^{d(t)}(y)$ for all y in X.

Transformations g_{x_i}, g_{x_j} are determined for $y \in D_f(e)$ and $y \in T$, $y \neq x_i, x_j$. The proof is the same as for g_t, $t \neq x_i, x_j$.

$$g_{x_i}(f(x_j)) = g_{x_i}(f^{d(x_j)+1}(e)) = f^{d(x_j)+1}(g_{x_i}(e)) = f^{d(x_j)+1}(x_i),$$

hence $g_{x_i}(x_j) \in f^{-1}(f^{d(x_j)+1}(x_i))$.

Thus there exist two transformations from $\mathcal{C}(f)$ such that $\bar{g}_{x_i}(e) = x_i$ and $g_{x_i}(e) = x_i$, where $g_{x_i}(x_j) = f^{d(x_j)}(x_i)$ and

(26) $\bar{g}_{x_i}(x_j) = x_j$.

Transformation g_{x_i} is determined at x_i ($g_{x_i}(x_i) = f^{d(x_i)}(x_i)$).

The transformation g_{x_j} :

$f(g_{x_j}(x_i)) = g_{x_j}(f(x_i)) = g_{x_j}(f^{d(x_i)+1}(e)) = f^{d(x_i)+1}(g_{x_j}(e)) = f^{d(x_i)+1}(x_j)$,

hence there exist two transformations g_{x_j}, $\bar{g}_{x_j}$ from $\mathcal{C}(f)$ such that

(27) $g_{x_j}(x_i) = f^{d(x_i)}(x_j)$ and $\bar{g}_{x_j}(x_i) = x_j$.

If we choose a system $\{g_t, t \in X\}$ we get $R(M_p)$ from the construction 1. Suppose we are able to choose an another system $R = \{g_t', t \in X\}$ from $\mathcal{C}(f)$ such that $R \neq R(M_p)$. We know that $g_{f(e)} = f$ and so the transformations f_y from L (where L contains f) are determined for $y \neq x_i, x_j$.

The transformations $\bar{g}_{x_i}$ and $\bar{g}_{x_j}$ do not commute:

$\bar{g}_{x_i} \circ \bar{g}_{x_j}(x_i) = \bar{g}_{x_i}(x_j) = x_j$, where $x_j \notin D_f(e)$;

$\bar{g}_{x_j} \circ \bar{g}_{x_i}(x_i) = \bar{g}_{x_j}(f^{d(x_i)}(x_i)) = \bar{g}_{x_j}(f^{d(x_i)}(x_i)) = f^{d(x_j)+d(x_i)}(x_i)$,

where $f^{d(x_j)+d(x_i)}(x_i) \in D_f(e)$.

Also $\bar{g}_{x_j}$ does not commute with g_{x_i} :

$\bar{g}_{x_j} \circ g_{x_i}(e) = \bar{g}_{x_j}(x_i) = x_j$, $x_j \notin D_f(e)$ and

$g_{x_i} \circ \bar{g}_{x_j}(e) = g_{x_i}(x_j) = f^{d(x_i)}(x_j)$, $f^{d(x_i)}(x_j) \in D_f(e)$.

Thus every other system $\{g_t', t \in X\}$ different from $R(M_p)$ cannot be a system of all the right translations of a monoid M'. So there exists only one monoid M_p (where p is an identical transformation) such that $L(M_p)$ contains f.

The theorem 1 has been proved.

Now we shall draw our attention to the disconnected transformations.

<u>Theorem 2</u>: A finite disconnected transformation $f: X \to X$ is a determining transformation if and only if

I. f satisfies conditions (i) - (iv);

II. $|Y| = 1$ or f is a disconnected permutation on Y ;

III. for all $x, y \in Y$ such that $x \notin Z(y)$ $\varkappa(x)$ does not divide $\varkappa(y)$;

IV. if $q \neq 1$ is a common divisor of all $\varkappa(x)$, $(\varkappa(x) = |Z(x)|)$, $x \in Y$, then there exists $x_0 \in Y$ such that $\frac{\varkappa(x_0) - q}{q^2}$ is not an integer.

<u>Proof</u>: At first we must prove the necessity of these conditions. Conditions (i) - (iv) have been confirmed in the part dealing with connected transformations.

From the corollary of the construction 1 we know that if f is the determining transformation then p from construction 1 is an identity. Hence f_t for $t \in Y$ are constants thus for $|Y| \neq 1$ M_p cannot be a commutative monoid. So if the restriction $f|Y$ is not disconnected, then f is determining only for $|Y| = 1$.

Let $f|Y$ be not a permutation. This means that there exists a point x in Y which is not cyclic. Let then $\varkappa$ denote the least common multiple of all orders $\varkappa(y), y \in Y$ and define $p: Y \to Y$ by

(28) $$p(y) = f^{\varkappa \cdot u(e)}(y), \quad y \in Y .$$

Clearly p satisfies conditions (12) and (13).

Hence from construction 1 there exists an another monoid M_p such that $L(M_p)$ contains f.

Let $f|Y$ be a disconnected permutation and let exist $y_1, y_2 \in Y$ such that $y_1 \notin Z(y_2)$ and $\varkappa(y_1)$ divides $\varkappa(y_2)$. Then we can define

(29) $p(y) = f^{k}(y_1)$ for $y = f^{k}(y_2)$,

(30) $p(y) = y$ otherwise.

We must show that p from (29) and (30) satisfies conditions (12) and (13).

$p \circ p(y) = p(f^k(y_1)) = f^k(y_1) = p(y)$ for $y = f^k(y_2)$, because $f^k(y_1) \notin Z(y_2)$, $p \circ p(y) = p(y)$ otherwise.

$p \circ f(y) = p(f^{k+1}(y_2)) = f^{k+1}(y_1) = f(f^k(y_1)) = f \circ p(y)$ for $y = f^k(y_2)$,

$p \circ f(y) = f(y) = f \circ p(y)$ otherwise.

Thus there exists another monoid M_p such that $f \in L(M_p)$.

Let $q \neq 1$ be a common divisor for all $\kappa(x)$, $x \in Y$ and let for all $x \in Y$ be $\frac{\kappa(x) - q}{q^2}$ an integer. Then we can construct a monoid M such that $L(M)$ contains f and M is different from M_p.

Let us form a set $\{a_i ; i = 1, \ldots, n\}$ such that $a_i \in Y$ for all i and $Z(a_i) \cap Z(a_j) = 0$ (we know that $E_f(a_i) \cap E_f(a_j) = 0$, because $E_f(a_i) = Z(a_i)$) for $i \neq j$ and for all $x \in Y$ there exists an index i such that $x \in E_f(a_i) = Z(a_i)$.

<u>Construction 5</u>: Let $f : X \to X$ be a finite disconnected transformation with a top element e as described above. Then a family of transformations $\{f_x ; x \in X\}$ such that

(31) $f_x(t) = f^{d(x)}(t)$ for $x \in E_f(e)$, $t \in X$;

(32) $f_x(t) = f^{d(t) \cdot \frac{\kappa(x)}{q}}(x)$ for $x \in Y$, $t \in E_f(e)$;

(33) $f_x(t) = f^{d(t, a_i) \frac{\kappa(x)}{q}}(x)$ for $x \in Y$, $t \in Y$;

where $t \in Z(a_i)$ and $d(t, a_i)$ is a difference of t with regard to a_i, is a system of all left translations of a monoid M. The system of all right translations of M is defined as follows:

(34) $R(M)$: $g_e = 1_X$,

(35) $\quad g_y(t)=f^{d(t)}(y)$ for $t\in E_f(e)$, $y\in X$;

(36) $\quad g_y(t)=f^{d(y)\frac{\varkappa(t)}{q}}(t)$ for $t\in Y$, $y\in E_f(e)$;

(37) $\quad g_y(t)=f^{d(y,a_i)\frac{\varkappa(t)}{q}}(t)$ for $t\in Y$, $y\in Y$.

<u>Demonstration</u>: From (31) and (34) we can see that e is a source of $L(M)$ and $R(M)$. Now we must show that $R(M)$ is a system of all right translations.

1) f_x commutes with g_y for $t\in E_f(e)$, $x\in E_f(e)$, $y\in E_f(e)$, because f_x and g_y for $t\in E_f(e)$ are the same as in the construction 1.

2) For $x\in Y$, $y\in E_f(e)$ we have

a) for $t\in E_f(e)$:

$$f_x\circ g_y(t)=f_x(f^{d(t)}(y))=f^{[d(t)+d(y)]\frac{\varkappa(x)}{q}}(x) \ ,$$

$$g_y\circ f_x(t)=g_y(f^{d(t)\frac{\varkappa(x)}{q}}(x))=g_y(z), \text{ where } z=f^{d(t)\frac{\varkappa(x)}{q}}(x) \ ,$$

$z\in Y$ so $g_y(z)=f^{d(y)\frac{\varkappa(z)}{q}}(z)$, but $\varkappa(z)=\varkappa(x)$ $(z\in Z(x))$,

hence

$$g_y(z)=f^{d(y)\cdot\frac{\varkappa(x)}{q}}(z)=f^{d(y)\cdot\frac{\varkappa(x)}{q}}(f^{d(t)\frac{\varkappa(x)}{q}}(x))=f^{[d(y)+d(t)]\frac{\varkappa(x)}{q}}(x) \ .$$

b) for $t\in Y$:

$$f_x\circ g_y(t)=f_x(f^{d(y)\frac{\varkappa(t)}{q}}(t))=f_x(z)=f^{d(z,a_i)\frac{\varkappa(x)}{q}}(x) \ , \quad \text{where}$$

$z=f^{d(y)\frac{\varkappa(t)}{q}}(x)$. $\quad t,z\in D_f(a_i)=Z(a_i)$ and thus $\varkappa(z)=\varkappa(t)$, hence

$$f^{d(z,a_i)\frac{\varkappa(x)}{q}}(x)=f^{[d(t,a_i)+d(y)\frac{\varkappa(t)}{q}]\frac{\varkappa(x)}{q}}(x) \ .$$

We know that $\dfrac{\varkappa(t)-q}{q^2}=k$, where k is an integer. Hence $\dfrac{\varkappa(t)}{q}=kq+1$.

$$f_x\circ g_y(t)=f^{d(t,a_i)\frac{\varkappa(x)}{q}+d(y)\frac{\varkappa(x)}{q}(kq+1)}(x)=$$

$$=f^{d(t,a_i)\frac{\varkappa(x)}{q}+d(y)\frac{\varkappa(x)}{q}+k\cdot d(y)\varkappa(x)}(x)=f^{d(t,a_i)\frac{\varkappa(x)}{q}+d(y)\frac{\varkappa(x)}{q}}(x) \ ,$$

because $d(y)\cdot k$ is an integer and $f^{k\cdot d(y)\varkappa(x)}(x)=x$.

$$g_y\circ f_x(t)=g_y(f^{d(t,a_i)\frac{\varkappa(x)}{q}}(x))=g_y(z)=f^{d(y)\frac{\varkappa(z)}{q}}(z)$$ and

$\varkappa(z)=\varkappa(x)$, hence $g_y\circ f_x(t)=f^{d(y)\frac{\varkappa(x)}{q}}(f^{d(t,a_i)\frac{\varkappa(x)}{q}}(x))=$
$=f^{d(y)\frac{\varkappa(x)}{q}+d(t,a_i)\frac{\varkappa(x)}{q}}(x)$.

3) For $x\in E_f(e)$, $y\in Y$ we have

a) for $t\in E_f(e)$:

$$f_x\circ g_y(t)=f_x(f^{d(t)}(y))=f^{d(x)+d(t)}(y),$$

$$g_y\circ f_x(t)=g_y(f^{d(x)}(t))=g_y(z)=f^{d(z)}(y)=f^{d(x)+d(t)}(y);$$

b) for $t\in Y$:

$$f_x\circ g_y(t)=f_x(f^{d(y,a_i)\frac{\varkappa(t)}{q}}(t))=f_x(z)=f^{d(x)}(z)=f^{d(x)+d(y,a_i)\frac{\varkappa(t)}{q}}(t),$$

$$g_y\circ f_x(t)=g_y(f^{d(x)}(t))=g_y(z)=f^{d(y,a_i)\frac{\varkappa(z)}{q}}(z)=f^{d(y,a_i)\frac{\varkappa(t)}{q}}(z)=$$
$$=f^{d(y,a_i)\cdot\frac{\varkappa(t)}{q}+d(x)}(t),$$ because $\varkappa(t)=\varkappa(z)$.

4) For $x\in Y$, $y\in Y$ we have

a) for $t\in E_f(e)$:

$$f_x\circ g_y(t)=f_x(f^{d(t)}(y))=f_x(z)=f^{d(z,a_i)\frac{\varkappa(x)}{q}}(x)=f^{[d(y,a_i)+d(t)]\frac{\varkappa(x)}{q}}(x);$$

$$g_y\circ f_x(t)=g_y(f^{d(t)\frac{\varkappa(x)}{q}}(x))=g_y(z)=f^{d(y,a_i)\frac{\varkappa(z)}{q}}(z)=f^{d(y,a_i)\frac{\varkappa(x)}{q}}(z)=$$
$$=f^{d(y,a_i)\frac{\varkappa(x)}{q}+d(t)\frac{\varkappa(x)}{q}}(x)=f^{[d(y,a_i)+d(t)]\frac{\varkappa(x)}{q}}(x),$$ because
$\varkappa(z)=\varkappa(x)$.

b) for $t\in Y$:

$$g_y\circ f_x(t)=g_y(f^{d(t,a_i)\frac{\varkappa(x)}{q}}(x))=g_y(z)=f^{d(y,a_j)\frac{\varkappa(z)}{q}}(z)=f^{d(y,a_j)\frac{\varkappa(x)}{q}}(z)=$$
$$=f^{d(y,a_j)\frac{\varkappa(x)}{q}+d(t,a_i)\frac{\varkappa(x)}{q}}(x).$$

$$f_x\circ g_y(t)=f_x(f^{d(y,a_j)\frac{\varkappa(t)}{q}}(t))=f_x(z)=f^{d(z,a_i)\frac{\varkappa(x)}{q}}(x)=f^{[d(y,a_j)\frac{\varkappa(t)}{q}+d(t,a_i)]\frac{\varkappa(x)}{q}}(x)=$$
$$=f^{d(y,a_j)\frac{\varkappa(t)}{q}\cdot\frac{\varkappa(x)}{q}+d(t,a_i)\frac{\varkappa(x)}{q}}(x)=f^{d(y,a_j)(kq+1)\frac{\varkappa(x)}{q}+d(t,a_i)\frac{\varkappa(x)}{q}}(x)=$$
$$=f^{d(y,a_j)\frac{\varkappa(x)}{q}+d(t,a_i)\frac{\varkappa(x)}{q}+k\cdot d(y,a_j)\varkappa(x)}(x)=f^{d(y,a_j)\frac{\varkappa(x)}{q}+d(t,a_i)\frac{\varkappa(x)}{q}}(x),$$

because $k \cdot d(y, a_j)$ is an integer and $f^{k \cdot d(y,a_j) \varkappa(x)}(x) = x$.

The construction 5 has been confirmed.

Thus we have completely proved the necessity of the conditions given in the theorem 2. Now we shall prove the sufficiency of these conditions.

1) Let $f: X \to X$ be a finite disconnected transformation such that $|Y| = 1$ and f satisfies the conditions (i) - (iv). Therefore all g_t, $t \neq y$ ($\{y\} = Y$), are determined on $E_f(e)$. It is impossible that $g_t(y) \in E_f(e)$, because $f(g_t(y)) = g_t(f(y)) = g_t(y)$ and if $g_t(y) = x \in E_f(e)$, then $f(x) = x$. It means that $E_f(e)$ has a cycle $Z(e)$ such that $|Z(e)| = 1$ and $x \in Z(e)$. But f_y such that $f_y(e) = y$ is a constant; thus f_y must commute with g_t.

$g_t \circ f_y(e) = g_t(y) = x$, $f_y \circ g_t(e) = f_y(t) = y$ and $y \neq x$. So for all t in $E_f(e)$ it is $g_t(y) = y$. And these g_t with $g_y(x) = y$ for all x in X is the system $R(M_n)$ from construction 1. So only M_n is a monoid the $L(M)$ of which contains f.

2) Let f be a finite disconnected transformation with a top element e and let $f|Y$ be a disconnected permutation such that for all $x, y \in Y$, $x \notin Z(y)$ $\varkappa(x)$ does not divide $\varkappa(y)$. We shall show that if $q \neq 1$ is a common divisor for all $\varkappa(x)$, $x \in Y$, then there exists x_0 such that $\frac{\varkappa(x_0) - q}{q^2}$ is not an integer.

Let g_t be a transformation from $\mathcal{C}(f)$ such that $g_t(e) = t$. Since $f | E_f(e)$ is the determining transformation, $g_t(y)$ is determined for $y \in E_f(e)$ and $t \in E_f(e)$.

Thus $g_t(y) = f^{d(y)}(t)$ for $t \in E_f(e)$, $y \in E_f(e)$. From the proof of sufficiency of conditions (i) - (iv) it follows that only $g_t(y) = f^{d(y)}(t)$ for $y \in E_f(e)$, $t \in Y$ can possibly be in $R(M)$ where $R(M)$ is a system of all right translations of a monoid M such that $L(M)$ contains f.

We know that $f|Y$ is a disconnected permutation. For $y \in Y$ it must be

(38) $\qquad f(g_t(y)) = g_t(f(y)) \qquad$ for all $t \in X$.

We shall demonstrate that $g_t | Z(x)$, $x \in Y$, is a permutation. Let $g_t | Z(x)$ not be a permutation. Hence there exist $y_1, y_2 \in Z(x)$ such that $y_1 \neq y_2$ and $g_t(y_1) = g_t(y_2)$. Let $y_1 = f^{d(y_1,y_2)}(y_2)$, then $g_t(y_1) = g_t(f^{d(y_1,y_2)}(y_2)) = f^{d(y_1,y_2)}(g_t(y_2)) = f^{d(y_1,y_2)}(g_t(y_1))$ by (38). Therefore $d(y_1, y_2) = k \cdot \varkappa(y_1)$ and thus $y_1 = y_2$.

Let us suppose that there exist $x_0, y_0 \in Y$ such that $g_t(x_0) = y_0$ where $x_0 \notin D_f(y_0)$. All g_t must commute with f and thus for $x \in D_f(x_0)$

(39) $\qquad g_t(x) = g_t(f^{d(x,x_0)}(x_0)) = f^{d(x,x_0)}(g_t(x_0)) = f^{d(x,x_0)}(y_0)$.

g_t must fulfil a condition (38) for all $x \in D_f(x_0)$, also for x such that $d(x, x_0) = \varkappa(x_0) - 1$.

$$f \circ g_t(x) = f(f^{\varkappa(x_0)-1}(y_0)) = f^{\varkappa(x_0)}(y_0); \; g_t \circ f(x) = g_t(x_0) = y .$$

The condition (38) is fulfilled only for $\varkappa(x_0) = k \cdot \varkappa(y_0)$. Thus $g_t(Z(x)) \subset Z(x)$. And because g_t must commute with f, is $g_t | Z(x) = f^k | Z(x)$, where k is an integer.

The family of transformations $\{g_t, t \in X\}$ must create a system of all right translations of some monoid M. This means that we must be able to construct a system $L = \{f_x, x \in X\}$ such that $f \in L$ and that for all $x \in X$,

$t \in X$, f_x and g_t commute. For every g_t, $t \in Y$ it is $g_t(E_f(e)) \subset D_f(t) = Z(t)$ and $g_t(Z(x)) = Z(x)$. Therefore $f_x(Z(t)) \subset Z(x)$ for all $t \in Y$, $t \notin Z(x)$. Let us suppose that $g_t \mid Z(x) = f^{\ell} \mid Z(x)$ and $g_t \mid Z(t) = f^{k} \mid Z(t)$. To solve our problem we shall use the following property of commutative transformations which has been proved in [2].

Let h be a disconnected permutation with two components Y_1 and Y_2, $|Y_1| = r_1$, $|Y_2| = r_2$. Then there exists a transformation g; $g(y_1) = y_2$ for some $y_1 \in Y$ and $y_2 \in Y$; such that $h \circ g = g \circ h$ if and only if r_2 divides r_1.

In our case this means that ℓ and k cannot be such that $\frac{r(x)}{\ell}$ and $\frac{r(t)}{k}$ are not integers ($r(x)$ does not divide $r(t)$). Therefore $\frac{r(x)}{\ell}$ and $\frac{r(t)}{k}$ must be integers and $\frac{r(x)}{\ell}$ must divide $\frac{r(t)}{k}$. The same applies also for f_t and g_x and that is why also $\frac{r(t)}{k}$ must divide $\frac{r(x)}{\ell}$. Hence

$$\frac{r(t)}{k} = \frac{r(x)}{\ell}. \tag{40}$$

This means that there exists a common divisor of all $r(x)$, $x \in Y$, which is equal to

$$q = \frac{r(x)}{\ell}. \tag{41}$$

If $f: X \to X$ is such that the only common divisor is $q = 1$, then $g_t \mid Z(x) = f^{r(x)} \mid Z(x)$ and thus $g_t(x) = x$ for all x in Y. Hence it is evident that the system $\{g_t ; t \in X\}$ is the system $R(M_p)$ as defined in the construction 1 with $p = 1_Y$.

Let $f: X \to X$ be such that a common divisor q of all $r(x)$, $x \in Y$, is different of 1. We shall define

$g_t \mid Z(x) = f^{\frac{\varkappa(x)}{q}} \mid Z(x)$. Thus we get another system of transformations $R = \{g_t ; t \in X\}$. In order that R is a system of all right translations of some monoid M , there must exist a system $L = \{f_x ; x \in X\}$ such that $f \in L$ and

(42) $f_x \circ g_t = g_t \circ f_x$ for all $x \in X$ and $t \in X$.

From the condition (42) it follows that

(43) $f_x(t) = f^{d(t)\frac{\varkappa(x)}{q}}(x)$, where $x \in Y$ and $t \in E_f(e)$.

We shall use the condition (42)

(44) $f_x \circ g_t(e) = f_x(t)$; $g_t \circ f_x(e) = g_t(x) = f^{\frac{\varkappa(x)}{q}}(x)$, thus $f_x(t) = f^{\frac{\varkappa(x)}{q}}(x)$.

Let y be an element in $D_f(t)$, then there exists an element z in $E_f(e)$ such that $g_t(z) = y = f^{d(z)}(t)$, therefore $d(z) = d(y,t)$. The condition (42) must be fulfilled also for such that z in $E_f(e)$.

$f_x \circ g_t(z) = f_x(y)$; $g_t \circ f_x(z) = g_t(f^{d(z)\frac{\varkappa(x)}{q}}(x)) = f^{d(y,t)\frac{\varkappa(x)}{q}}(x)$; hence

(45) $f_x(y) = f^{d(y,t)\frac{\varkappa(x)}{q}}(x)$ for $y \in D_f(t)$.

The condition (42) must be fulfilled for $y \in D_f(t)$, too.

$$g_t \circ f_x(y) = g_t(f^{d(y,t)\frac{\varkappa(x)}{q}}(x)) = g_t(a) = f^{\frac{\varkappa(a)}{q}}(a) = f^{\frac{\varkappa(x)}{q}}(f^{d(y,t)\frac{\varkappa(x)}{q}}(x)) =$$

$$= f^{\frac{\varkappa(x)}{q} + d(y,t)\frac{\varkappa(x)}{q}}(x) ;$$

$$f_x \circ g_t(y) = f_x(f^{\frac{\varkappa(t)}{q}}(y)) = f_x(a) = f^{d(a,t)\frac{\varkappa(x)}{q}}(x) = f^{[d(y,t) + \frac{\varkappa(t)}{q}]\cdot\frac{\varkappa(x)}{q}}(x) =$$

$$= f^{d(y,t)\frac{\varkappa(x)}{q} + \frac{\varkappa(t)}{q}\cdot\frac{\varkappa(x)}{q}}(x) .$$

Hence $d(y,t)\frac{\varkappa(x)}{q} + \frac{\varkappa(t)}{q}\cdot\frac{\varkappa(x)}{q} - (\frac{\varkappa(x)}{q} + d(y,t)\frac{\varkappa(x)}{q}) = c \cdot \varkappa(x)$,

where c is an integer.

$$\frac{\varkappa(t)}{q}\cdot\frac{\varkappa(x)}{q} - \frac{\varkappa(x)}{q} = c \cdot \varkappa(x) \text{ , hence}$$

(46) $$\frac{\varkappa(t) - q}{q^2} = c .$$

The assertion (46) must be fulfilled for every t in Y.

But we know that in Y there exists x_0 such that $\frac{r(x_0)-q}{q^2}$ is not an integer. Hence there does not exist any monoid M such that R is a system of all right translations of M. Thus only M_p (p is an identity) is a monoid such that $L(M_p)$ contains f.

The sufficiency of the conditions given in Theorem 2 has been proved. Thus the proof of Theorem 2 is complete.

I want to thank to Pavel Goralčík for his kind help and valuable suggestions to me with this paper.

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(Oblatum 28.4.1970)