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ON CONTINUITY OF LINEAR TRANSFORMATIONS COMMUTING WITH GENERALIZED SCALAR OPERATORS

(Preliminary communication)

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1. Let $\mathcal{L}(\mathcal{X})$ be the algebra of all linear continuous operators from a Banach space $\mathcal{X}$ into itself. In papers [4] and [5] the continuity of a linear transformation S commuting with a given $T \in \mathcal{L}(\mathcal{X})$ is investigated. Similarly as in [5] we shall deal with operators having a suitable spectral decomposition.

<u>Definition</u>. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be a decomposable operator if, for each closed subset F of the complex plane $\mathbb{C}$, there is a closed linear subspace $\mathcal{E}(F)$ of $\mathcal{X}$ such that

1^0 $\mathcal{E}(\emptyset) = \{0\}$, $\mathcal{E}(\mathbb{C}) = \mathcal{X}$,

2^0 $\bigcap_{n=1}^{\infty} \mathcal{E}(F_n) = \mathcal{E}(\bigcap_{n=1}^{\infty} F_n)$ where $F_n = \overline{F}_n$;

3^0 if $\{G_j\}_{j=1}^{m}$ is a finite open covering of the complex plane, then $\mathcal{X} = \mathcal{E}(\overline{G}_1) + \ldots + \mathcal{E}(\overline{G}_m)$;

4^0 $T\mathcal{E}(F) \subset \mathcal{E}(F)$ and $\sigma(T \mid \mathcal{E}(F)) \subset F$ for every F closed.

It has been shown in [2] that the definition of the decomposable operator is equivalent to that given in [3] and

(1) $\mathcal{E}(F) = \{x : \sigma_T(x) \subset F\}$ for every F closed.

($\sigma_T(x)$ is the spectrum of x with respect to T.)

Let T be a decomposable operator. Since every $L \in \mathcal{L}(\mathcal{X}), LT = TL$ satisfies $L\mathcal{E}(F) \subset \mathcal{E}(F)$ for $F = \bar{F}$, we shall further suppose that each $\mathcal{E}(F)$ is invariant with respect to our transformation S.

The space $\mathcal{X}$ can be decomposed into a sum of spaces $\mathcal{E}(F)$. We shall, therefore, take into account only the subspaces on which S is discontinuous. Let $\mathcal{E}(F)$ be such that $S|\mathcal{E}(F)$ is not continuous. By the closed graph theorem there is an $x \in \mathcal{E}(F)$ and a sequence $x_n \in \mathcal{E}(F)$ such that $x_n \to 0$ and $Sx_n \to x$. Denote by σ_S the set of all elements $x \in \mathcal{X}$ such that there exists a sequence $x_n \to 0$ with $Sx_n \to x$. Suppose now that $\sigma_S \subset \mathcal{E}(F)$ for some F. We may assume that $F = \sigma(T|\mathcal{E}(F))$. If $\lambda \notin F$, then there is a closed neighbourhood G of λ such that $F \cap G = \emptyset$ and $S|\mathcal{E}(G)$ is continuous by the closed graph theorem. Obviously every λ satisfying the following definition is an element of $F = \sigma(T|\mathcal{E}(F))$.

Definition. We shall call a complex number λ a discontinuity value if the operator $S|\mathcal{E}(F)$ is discontinuous for each closed neighbourhood F of λ.

By (1) the family $\{\mathcal{E}(F)\}_{F=\bar{F}}$ is closed with respect to intersection and we may define the minimal subspace $\mathcal{E}(F_0)$ containing σ_S as the intersection of all subspaces $\mathcal{E}(F)$ for which $\sigma_S \subset \mathcal{E}(F)$.

<u>Lemma</u>. The spectrum $\sigma(T \mid \mathcal{E}(F_0))$ consists of discontinuity values only.

If there is no discontinuity value, then $\sigma_S = \{0\}$ and the transformation S is continuous.

To obtain further properties of the set of discontinuity values we shall reduce our investigation to a subclass of the class of decomposable operators.

2. <u>Definition</u>. Denote by $(C^\infty(R_2), \tau)$ the Fréchet space of all infinitely differentiable complex functions $\varphi(x_1, x_2)$ defined on R_2 with the family of pseudonorms

$$|\varphi|_{K,m} = \sum_{r_1+r_2=0}^{m} \sup_{(x_1,x_2)\in K} \left| \frac{\partial^{r_1+r_2}\varphi(x_1,x_2)}{\partial^{r_1} x_1 \, \partial^{r_2} x_2} \right|$$

for every compact set K and $r_1, r_2, m \geqq 0$.

<u>Definition</u>. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be a generalized scalar operator if there exists a continuous linear mapping $\mathcal{U} : (C^\infty(R_2), \tau) \longrightarrow \mathcal{L}(\mathcal{X})$ such that $\mathcal{U}_{\varphi\psi} = \mathcal{U}_\varphi \mathcal{U}_\psi$ for $\varphi, \psi \in C^\infty(R_2)$,

$$\mathcal{U}_1 = I, \ \mathcal{U}_a = T, \ \text{where } a(\lambda) = \lambda .$$

Every generalized scalar operator is an element of the class of decomposable operators. See [1] and [3]. We shall use the notation $\mathcal{X}_T(F)$ for the subspace $\mathcal{E}(F)$.

<u>Lemma</u>. The set of discontinuity values is empty or it has only a finite number of elements.

Suppose that the set of discontinuity values is nonvoid and consists of the numbers $\lambda_1, \dots, \lambda_k$. We have $\sigma_S \subset \mathfrak{X}_T(\{\lambda_1, \dots, \lambda_k\})$.

<u>Lemma</u>. Let $\{\mu_1, \dots, \mu_n\}$ be a set of complex numbers. Then there is a polynomial $P(\cdot)$ with the roots $\mu_1, \dots, \mu_n$ such that $P(T) | \mathfrak{X}_T(\{\mu_1, \dots, \mu_n\}) = 0$.

From this fact it follows that the operator $P(T)S$ is continuous.

<u>Definition</u>. A complex number λ is said to be a critical eigenvalue of T if λ is an element of the point spectrum of T and the range $R(\lambda I - T)$ is of infinite codimension.

<u>Theorem</u>. Let T be a generalized scalar operator in a Banach space which has no critical eigenvalue. Let S be a linear transformation such that

1° $ST = TS$,

2° $S\mathfrak{X}_T(F) \subset \mathfrak{X}_T(F)$ for every F closed.

Then S is continuous.

Let T have a critical eigenvalue. Then there is a discontinuous S commuting with T and such that $S\mathfrak{X}_T(F) \subset \mathfrak{X}_T(F)$ for $F = \overline{F}$. See also [4]. Indeed, let λ be a critical eigenvalue, let $Ty = \lambda y$ and let f be a discontinuous functional defined on $\mathfrak{X}$ and $f | R(\lambda I - T) \equiv 0$. The transformation $Sx = y \cdot fx$

is discontinuous, $ST = TS$ and each $\mathfrak{X}_T(F)$ is invariant with respect to S.

3. <u>Definition</u>. The subspace $Y \subset \mathfrak{X}$ is called T-divisible if for every λ the equality $(\lambda I - T)Y = Y$ holds.

We can construct the largest T-divisible subspace in $\mathfrak{X}$. There exists a transfinite sequence $Z(\alpha)$ with eventual constant value defined by

1° $Z(0) = \mathfrak{X}$,

2° $Z(\alpha + 1) = \bigcap_{\lambda \in \mathbb{C}} (\lambda I - T) Z(\alpha)$,

3° $Z(\alpha) = \bigcap_{\beta < \alpha} Z(\beta)$ for limit ordinals.

Similarly as in [5] we could prove the theorem under the assumption that $\{0\}$ is the only T-divisible subspace. However, according to the following proposition this assumption is stronger.

<u>Proposition</u>. Let T be a generalized scalar operator for which $\{0\}$ is the only T-divisible subspace.

Then each subspace $\mathfrak{X}_T(F)$ is invariant with respect to any linear transformation commuting with T.

R e f e r e n c e s

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