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M - BARRELLED SPACES

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The aim of our task is to limit a class of topological vector spaces with similar dual properties as those of the second category. It turns out that such spaces (call them the topological vector spaces of barrelled type) may be defined in analogous terms as the barrelled (locally convex) spaces. The notion which lays the ground for the presented elementary theory is that of a multibarrel. By means of it one may attach to every covering $\mathcal{V}$ consisting of bounded subsets of a topological vector space E the $\mathcal{V}$ - M -barrelled modification of the initial topology in E . A topological vector space is said to be $\mathcal{V}$ - M -barrelled if the $\mathcal{V}$ - M -modification of the initial topology coincides with E .

By specification of $\mathcal{V}$ we obtain the M -barrelled spaces and the quasi- M -barrelled spaces. We start our discussion with the first ones; they set an example for the outlined exposition of the remaining cases.

The investigation of spaces of barrelled type was originally stimulated by the endeavour to determine as far as possible a general class of topological vector spaces on which the tensor inductive topology (cf.[10]) coincides with the projective one (cf.[9]).

We refer for all notions as well as for the results presented up to now to [2],[3],[6],[7].

The second part of this tract dealing with generalizations of topological metrizable spaces will be submitted under the title "M-bornological spaces" to CMUC.

All vector spaces considered in this paper will be taken over the same field of scalars (i.e.,over the field of real or complex numbers). By N we always mean the set of all positive integers.

1. MULTIBARRELLED SPACES

In this section we shall enlarge a subcategory of (locally convex) barrelled spaces to a certain category of topological vector spaces including, especially, the class of all vector Baire spaces.

<u>Definition 1.</u> Assume that E is a topological vector space. A subset $B \subseteq E$ will be termed a barrel in E if B is closed, absorbent and balanced in E.

<u>Definition 2</u>. A topological space E will be called multibarrelled (abbreviated M-barrelled) if for any sequence $(B_n ; n \in N)$ of barrels in E the subset

$$\Omega(B_n) = \overline{\bigcup_{n=1}^{\infty} (B_1 + \ldots + B_n)}$$

is a neighborhood of the origin in E.

Remark 1. Suppose that E is p-normable (cf. [6]) with $0 \leq p \leq 1$. If E is M-barrelled, B a barrel in E, then the closed absolutely p-convex envelope $\overline{\Gamma_p B}$ of B is a neighborhood in E.

Proof. For any α with $0 < \alpha \leq 1$ the subset $\alpha \cdot B$ is a barrel in E. Consequently for any $\lambda = (\lambda_n ; n \in N) \in \ell^p$, $0 < p \leq 1$ with $\lambda_n > 0$ and $\| \lambda \| = \sum |\lambda_n|^p < 1$ the subset $\Omega(B_n)$, where $B_n = \lambda_n \cdot B$, is a neighborhood in E.

But $\sum_{i=1}^{m} \lambda_i \cdot B \subseteq \overline{\Gamma_p B}$ for any n, hence $\Omega(B_n) \subseteq \overline{\Gamma_p B}$.

Lemma. Suppose that E is a topological vector space, C a convex and bounded subset in E of the second category (in itself). If B is a barrel in E, then C is absorbed by $B + B$.

Proof. Since B is a barrel in E, we obtain $E = \bigcup(n \cdot B ; n \in N)$. Consequently

$$C = \bigcup_{n \in N} (n \cdot B \cap C) .$$

But $n \cdot B \cap C$ is closed in C, hence we can choose an $n \cdot B \cap C$ with non-empty interior, consequently for some $x_0 \in C$ and for some neighborhood V in E it holds

$$C \cap (x_0 + V) \subseteq n \cdot B \cap C \subseteq n \cdot B .$$

Take m such that $m \cdot V, m \in N$, is absorbing the bounded set $C - x_0$. Then by a direct calculation (cf. [8]) we obtain $C - x_0 \subseteq m\,(C - x_0)$ and

$$C - x_0 \subseteq m\,(C - x_0) \cap m \cdot V \subseteq m\,(n \cdot B - x_0) .$$

From the last relation and from $x_0 \in n \cdot B$ it follows that

$$C \subseteq m \cdot n \cdot B - (m - 1)x_0 \subseteq m \cdot n\,(B + B) .$$

<u>Theorem 1</u>. Suppose that E and F are two topological vector spaces, H is a subset in the space $\mathcal{L}(E\,;F)$ of all linear continuous transformations from E into F. If H is bounded in the topology of pointwise convergence, then H is bounded in the topology of convergence on the class of all convex bounded subsets of the second category in E .

Proof. Let W be a neighborhood in F, V a closed and balanced neighborhood in F with $V + V \subseteq W$. Putting

$$B = \bigcap_{u \in H} u^{-1}(V)$$

we see that B is a barrel in E . According to the precedent lemma for any convex and bounded set of the second category it holds $C \subseteq n\,(B + B)$, consequently

$$u(C) \subseteq n\,(V + V) \subseteq n \cdot W$$

for all $u \in H$. This means that H is bounded on C .

<u>Theorem 2.</u> Let E be a topological vector space of the second category (i.e., a Baire space). Then E is $\mathcal{M}$-barrelled.

Proof. If B is a barrel in E , then from

$E = \bigcup(nB; n \in N)$ we conclude that $n \cdot B$, hence B, possesses an interior point x_o. Thus for a balanced neighborhood U we obtain $x_o + U \subseteq B$, $-x_o + U \subseteq B$, hence $U \subseteq U + U \subseteq B + B$. If $(B_n; n \in N)$ is now a sequence of barrels, then for $B = B_1 \cap B_2$ it holds

$$U \subseteq B_1 + B_2 \subseteq \Omega(B_n) .$$

Recall that a topological vector space E is said to be an F-space if F is metrisable and complete. It should be noticed that in view of Klee theorem (cf.[5]) a metrisable space E is complete under the uniformity induced by the topological vector structure of E if and only if E is complete under a metric defining its topology.

Corollary. (a) Any F-space is M-barrelled.
(b) The Cartesian product $\prod E_\alpha$ of a family of F-spaces is M-barrelled.
(c) Any locally convex M-barrelled space is barrelled in the usual sense.

Proof. The statements (a) and (b) are clear. If B is closed, absorbent and absolutely convex in a locally convex and M-barrelled space E, then for $B_n = 2^{-n} \cdot B$, $n \in N$, we obtain

$$\sum_{n=1}^{k} B_n \subseteq B$$

for any $k \in N$. Hence $\Omega(B_n) \subseteq B$, thus B is a neighborhood in E.

Theorem 3. Let E be an M-barrelled space, F an arbitrary topological vector space. Then any family H of linear continuous mappings from E into F bounded in the topology of pointwise convergence is equicontinuous.

Proof. Take a closed neighborhood W in F and a sequence $(V_n; n \in N)$ of closed and balanced neighborhoods in E such that $V_1 + V_2 + \dots + V_n + V_n \subseteq W$ for any $n \in N$. The subsets

$$B_n = \bigcap_{u \in H} u^{-1}(V_n)$$

$n \in N$, form a sequence of barrels in E. According to the assumption the set $\Omega(B_n)$ in Definition 2 is a neighborhood in E. Since for any $u \in H$ and for any $n \in N$ it holds

$$u\left(\sum_{i=1}^{n} B_i\right) \subseteq V_1 + \dots + V_n \subseteq W ,$$

we obtain $u(\Omega) \subseteq W$ for each $u \in H$. This proves the equicontinuity of H.

As a consequence of Theorem 3 it follows the Banach-Steinhaus

Theorem 4. Suppose that E is an M-barrelled space, F a separated topological vector space. If a sequence $(u_n; n \in N)$ of $\mathcal{L}(E; F)$ (a bounded filter in $\mathcal{L}(E; F)$) is convergent to a mapping $u_0 : E \longrightarrow F$ in the topology of pointwise convergence, then u_0 is continuous on the space E.

Corollary. Let $\mathcal{A}$ be a covering of an M-barrelled space E, F is assumed to be a separated topological vector space. If F is sequentially complete (quasi-complete), then $\mathcal{L}(E; F)$ under the topology of $\mathfrak{A}$ -convergence is sequentially complete (quasi-complete).

To prove the equivalency of the inductive and the projective tensor topologies (cf.[10]) we need

<u>Theorem 5.</u> Let E and F be two metrisable vector spaces, F is assumed to be M -barrelled. Then any separately equicontinuous family H of mappings from $E \times F$ into a separated topological vector space G is equicontinuous on $E \times F$.

The proof of Theorem 5 may be carried out in the same way as the proof of Theorem 3, §3, Chap.III of [2].

<u>Remark 2.</u> If H is a subset in $\mathcal{L}(E; F)$ non-equicontinuous on an M-barrelled space E , then there is a point $x_0 \in E$ such that

$$\sup_{u \in H} |\langle u, x_0 \rangle| = +\infty .$$

In other words, any non-equicontinuous family of linear continuous mappings possesses a singularity in an M -barrelled space.

<u>Remark 3.</u> The definition of a locally convex barrelled space may be formulated as follows: for any barrel B (in our sense) the closed absolutely convex envelope $\overline{\Gamma B}$ is a neighborhood of the origin in E . What we have modified in this definition is this: we replaced the uncountable process of the taking of the absolutely convex envelope by a countable operation

$(B_m ; m \in N) \longrightarrow \overline{U \Sigma B_n}$.

Note that by such a procedure we have obtained in [9] the usual projective tensor topology.

2. FURTHER PROPERTIES OF M-BARRELLED SPACES

Suppose that E is a topological vector space. Let for any sequence $(B_n ; n \in N)$ of barrels the set $\Omega(B_m)$ be defined as in Definition 2. Obviously any such $\Omega(B_m)$ is absorbent and balanced in E . If

$$B'_k = B_{2k-1} \cap B_{2k}$$

for $k = 1, 2, \dots ,$ then evidently

$$\Omega(B'_k) + \Omega(B'_k) \subseteq \Omega(B_k) \quad .$$

Thus the system of all subsets $\Omega(B_m)$ determines a vector topology $T_V = T_V(E)$ on E . It will be termed the associated T_V -topology with E (or, equivalently, the T_V -modification of the initial topology in E).

Remark 4. Let E be a topological vector space. Then the associated T_V -topology possesses the following properties:

(a) It is finer than the initial topology on E .

(b) If H is a subset in $\mathcal{L}(E; F)$ bounded in the topology of pointwise convergence, then H is equicontinuous on E under the topology $T_V(E)$.

(c) If E is separated and complete (sequentially complete) then E under the T_V -topology is complete (sequentially complete).

(d) The space E is M -barrelled if and only if the associated T_V -topology coincides with the initial topology of E .

The proof of these statements is immediate.

If $(\tau_\alpha ; \alpha \in A)$ is the system of all vector topologies on E which have as a base of neighborhoods $(U_\beta^\alpha ; \beta \in B_\alpha)$ closed subsets in E, then the least upper bound topology $T(E)$ of this family, i.e., the topology having as a base of neighborhoods the subsets

$$U_{\beta_1}^{\alpha_1} \cap U_{\beta_2}^{\alpha_2} \cap \dots \cap U_{\beta_n}^{\alpha_n} \quad ,$$

posseeses the same property (it may be called the ultrabarrelled modification of the initial topology of E ; cf.[7]).

Theorem 6. For any topological space E it holds

$$T_V(E) = T(E) \ .$$

Proof. It is clear that $T_V(E) \leq T(E)$. Conversely, if V_n , $n \in N$, is a system of closed in E neighborhoods of the topology $T(E)$ and $V_0 \supseteq$ $\supseteq V_1 + V_1, \dots, V_n \supseteq V_{n+1} + V_{n+1}, n = 1, 2, \dots$, then obviously $\Omega(V_n) \subseteq V_0$. This implies $T(E) \leq T_V(E)$.

Corollary. If u is a linear continuous mapping from E into F , then u is continuous under the topologies $T_V(E)$ and $T_V(F)$.

Proof. The base of neighborhoods of the topology

$T_V(F)$ is formed by closed subsets in F. Consequently, for any such neighborhood V the inverse image $\mu^{-1}(V)$ is closed in E. The rest of the proof is clear.

<u>Remark 5</u>. Let E be a fixed vector space and let $\mathcal{I}(E)$ denote the complete lattice of all vector topologies on E. For any $\tau \in \mathcal{I}(E)$ the taking of the T_V -topology associated with (E, τ) represents an isotonic operator on $\mathcal{I}(E)$. With respect to Remark 4 M -barrelled topologies on E are exactly those elements of $\mathcal{I}(E)$ which are invariant under the operator T_V. If τ_o is now a fixed vector topology on E, then the subset of all vector topologies finer than τ_o in $\mathcal{I}(E)$ is a complete sublattice $\mathcal{I}(E, \tau_o)$ in $\mathcal{I}(E)$. Consequently there exists (cf.[1]) a topology τ' in $\mathcal{I}(E, \tau_o)$ invariant under T_V, hence τ' is an M -barrelled topology. In particular, if τ is the maximal vector topology on E, then τ being invariant under T_V is an M -barrelled topology.

With respect to the subject matter of this section the following question arises: for any $\tau \in \mathcal{I}(E)$ the operator T_V generates a linearly ordered sublattice I in $\mathcal{I}(E)$. What length has I ?

We refer for the notion of the topological direct sum and of the inductive limit to [11].

Theorem 7. Suppose that $(E_n; n \in N)$ is a sequence of topological vector spaces. Then it holds

(a) The topological direct sum ΣE_n is M-barrelled if and only if each E_n; $n \in N$, is an M-barrelled space.

(b) If, moreover, $(E_n; n \in N)$ is a spectrum of M-barrelled spaces, then the inductive limit $\lim\ \mathrm{ind}\ E_n$ is an M-barrelled space.

Proof. Assume that any E_n is M-barrelled. We shall prove that the topological direct sum has the same property. Thus, let $(B_n; n \in N)$ be a sequence of barrels in $E = \Sigma E_n$. Define now a double-sequence $(B_{ik}; i \in N, k \in N)$ by a suitable rearrangement of terms. Put $B_{11} = B_1$, $B_{12} = B_2$, $B_{21} = B_3$, $B_{13} = B_4, \dots$, and, generally, $B_{ik} = B_p$, where $2p = [(i+k)^2 - 3k - i + 2]$. For $m \in N$ the subsets

$$B'_{nm} = E_m \cap B_{nm} \quad ,$$

$n = 1, 2, \dots$, form a sequence of barrels in E_m. According to the assumptions the space E_m is M-barrelled, consequently the set

$$\Omega_m = \overline{\bigcup_{k=1}^{\infty} \sum_{n=1}^{k} B'_{nm}}$$

is a neighborhood in E_m.

What we have to prove is that for any $r \in N$ it holds

$$\sum_{m=1}^{r} \Omega_m \subseteq \overline{\bigcup_{n=1}^{\infty} \sum_{k=1}^{n} B_k} \quad .$$

But from

$$\sum_{m=1}^{k} \Omega_m \subseteq \overline{\sum_{m=1}^{k} \bigcup_{k=1}^{\infty} \sum_{n=1}^{k} B'_{nm}}$$

and from

$$\sum_{n=1}^{k_1} B'_{n1} + \sum_{n=1}^{k_2} B'_{n2} + \dots + \sum_{n=1}^{k_k} B'_{nk} \subseteq \sum_{i+j\le s} B'_{ij} \subseteq \sum_{i+j\le s} B_{ij}$$

where $s = max\,[\,k_1+1,\ k_2+2,\dots,k_k+k]$ we obtain the requested inclusion.

Conversely, if $\sum E_n$ is M-barrelled, then each E_n being topologically isomorphic to a quotient space of $\sum E_n$ (cf.[11]) is M-barrelled.

The proof of the statement (b) is analogous to that of the statement (a).

Remark 6. (a) A closed subspace of an M-barrelled space need not be, in general, M-barrelled. The situation is quite the same as in the theory of locally convex spaces. If, for example, E is a complete locally convex space which is not barrelled (in the sense of locally convex spaces), then E is isomorphic to a closed subspace of the Cartesian product ΠB_α of Banach spaces. On the one hand, ΠB_α is according to Corollary to Theorem 2 M-barrelled, on the other hand, E cannot be M-barrelled, since E is locally convex and non-barrelled.

(b) As to the Cartesian product of a sequence $(E_n; n \in N)$ of M-barrelled spaces our conjecture is that it will be M-barrelled.

(c) The quotient space of an M-barrelled space and the completion of a separated $\mathcal{M}$ -barrelled space is $\mathcal{M}$ -barrelled. An immediate proof is clear (for an another proof compare with [7]).

Remark 7. From the statement (a) and (b), respectively, it follows that an $\mathcal{M}$ -barrelled space can be of the first category (cf.[11]).

3. SPACES OF BARRELLED TYPE

We shall now enlarge the class of $\mathcal{M}$ -barrelled and of $\mathcal{M}$ -bornological spaces (cf.[12]) by the class of quasi- $\mathcal{M}$ -barrelled spaces. Further, we envisage, quite shortly, other topologies under which a vector space possesses properties of a barrelled space.

Definition 3. A topological vector space E will be termed quasi- $\mathcal{M}$ -barrelled (or quasi-multibarrelled) if for any sequence $(B_n;\ n \in N)$, where each B_n is closed, balanced and absorbs any bounded set in E, the set of the form

$$\Omega(B_n) = \overline{\bigcup_{m=1}^{\infty} \sum_{n=1}^{k} B_n}$$

is a neighborhood of the origin in E.

We state without proof the following elementary properties.

(a) Any $\mathcal{M}$ -barrelled space is quasi- $\mathcal{M}$ -barrelled.

(b) Any $\mathcal{M}$ -bornological space is quasi- $\mathcal{M}$ -barrelled.

(c) The completion of any quasi- $\mathcal{M}$ -barrelled (consequently of any $\mathcal{M}$ -bornological) space is quasi- $\mathcal{M}$ - barrelled.

(d) The quotient space of a quasi- $\mathcal{M}$ -barrelled space is also quasi- $\mathcal{M}$ -barrelled.

(e) The topological direct sum ΣE_n of a sequence $(E_n ; n \in N)$ is quasi- $\mathcal{M}$ -barrelled if and only if any E_n has the same property.

(f) The inductive limit of a spectrum $(E_n ; n \in N)$ of quasi- $\mathcal{M}$ -barrelled spaces is quasi- $\mathcal{M}$ -barrelled.

From the definition of a quasi- $\mathcal{M}$ -barrelled space and from Remark 2 of [12] it follows

<u>Theorem 8.</u> If E is a quasi- $\mathcal{M}$ -barrelled space, F an arbitrary topological vector space, H a subset in $\mathcal{L}(E ; F)$ bounded in the strong topology, then H is equicontinuous.

We could associate analogically as in Section 2 with any topological vector space E the T_H -modification of the initial topology in E taking as a base of neighborhoods for the topology $T_H(E)$ the subsets $\Omega(B_n)$ determined in Definition 3. Hence an operator T_H is defined on the lattice of all vector topologies on E. Similarly as in Section 2 we might also formulate for the operator T_H the statements of Remark 4 replacing in (b) of Remark 4 the topology of pointwise convergence by the topology of bounded convergence in $\mathcal{L}(E ; F)$. Especially, the T_H -modification of any vector topology on E is finer (or equal) to the initial topology on E. In addition, the T_H -modification of a vector topology has an absolute character. Namely, take the system $\mathcal{T} = (\tau_\alpha ; \alpha \in A)$ of all vector topologies on a topological vector space E with bases of neighborhoods consisting of closed subsets in E and suppose that all τ_α preserve the same family of bounded subsets in E. The least upper bound T_o of this system is evidently an element of $\mathcal{T}$. Hence $\mathcal{T}$ has the topology T_o as the maximal topology and it obviously holds

Theorem 9. For any topological vector space E the T_H -modification of the initial topology in E coincides with T_o.

We can observe that the M -barrelled and the quasi- M -barrelled spaces may be generalized from a unifying point of view as follows. Let $\mathcal{V}$ be a covering of the topological vector space E consisting of bounded sets in E. The space E is $\mathcal{V}$ - M -barrelled provided that for any sequence $(B_n ; n \in N)$ of barrels, each B_n absorbs any $S \in \mathcal{V}$, the set $\Omega(B_n)$ in Definition 3 is a neighborhood of the origin in E. Thus we obtain topologies of barrelled type depending on the choice of $\mathcal{V}$.

One might expect that in such a way we obtain a scale of different topologies on the category of topological vector spaces. Unfortunately the usual concepts afford only reduced possibilities (see Remark 8). But in any case the extreme topologies of the considered system, $\mathcal{V}$ variable, are the M -barrelled topologies

($\mathcal{T}$ represents the class of all finite subsets in E) and the quasi- M -barrelled topologies ($\mathcal{T}$ stands for the class of all bounded sets in E). The topologies of the just described sort possess similar properties as those discussed in this and in the article [12].

We note two statements only.

Theorem 10. Let H be a subset of $\mathcal{L}(E; F)$ bounded in the topology of $\mathcal{T}$-convergence. If E is $\mathcal{T}$ - M -barrelled, then H is equicontinuous on E.

Theorem 11. Suppose that E is $\mathcal{T}$ - M -barrelled, F separated. If a sequence $(u_n; n \in N)$ is convergent in $\mathcal{L}(E; F)$ to a mapping u_o in the topology of $\mathcal{T}$ -convergence, then u_o is continuous. Similarly, if u_o is a limit of a bounded filter in $\mathcal{L}(E; F)$ under the topology of $\mathcal{T}$-convergence, then u_o is continuous on E.

Remark 8. Denote by $\mathcal{T}_o$ the class of all sequences in E convergent to the origin in E. If B is a barrel (or, more generally, a balanced subset in E), then B is absorbing any bounded set in E if and only if it absorbs any set of $\mathcal{T}_o$. Hence the class of all $\mathcal{T}_o$ - M -barrelled spaces coincides with the class of quasi- M -barrelled spaces.

4. p-BARRELLED SPACES

In this section we shall outline, quite shortly, a straightforward generalization of locally convex barrelled spaces to the case of locally p -convex spaces.

Therefore we state only the basic properties of this category of spaces; its permanence properties represent a formal generalization of the classical case and in view of this analogy they will be omitted.

Recall that under a p -seminorm $0 < p \leq 1$ on a vector space E we understand a function $x \to \|x\|$ on E satisfying

$$0 \leq \|x\| < \infty, \ \|\alpha x\| = |\alpha|^p \|x\|, \ \|x+y\| \leq \|x\| + \|y\| .$$

A topological vector space E will be called locally p -convex if the topological structure of E is determined by a defining system $(\rho_\alpha ; \alpha \in A)$ of p -seminorms. Evidently any such space has a base of neighborhoods consisting of absolutely p -convex and absorbent neighborhoods in E. On the other hand, if a topological space has such a base of neighborhoods, then its topology may be defined by a system of p -seminorms.

Definition 4. A subset B in a topological vector space E will be said a p -barrel $(0 < p \leq 1)$ in E if B is absolutely p -convex, closed and absorbent in E. A space is p -barrelled provided that any p -barrel is a neighborhood of the origin in E.

Remark 9. (a) Any locally p -convex space which is M -barrelled is also p -barrelled.
(b) Any locally convex p -barrelled space is barrelled in the usual sense.

Remark 10. (a) Any complete metrisable locally p -convex space is p -barrelled.

(b) Any Day space (i.e., a complete and p-normable space) is p-barrelled.

Theorem 12. Suppose that E and F are locally p-convex spaces, H a subset in $\mathcal{L}(E; F)$ bounded in the topology of pointwise convergence. If E is p-barrelled, then H is an equicontinuous subset of $\mathcal{L}(E; F)$.

From Theorem 12 we may now conclude the modificated version of the Banach-Steinhaus theorem and its corollary for p-barrelled spaces. In anticipation of the further use we state only

Theorem 13. Let E and F be two metrisable locally p-convex spaces, E p-barrelled. If H is a family of bilinear separately equicontinuous mappings from $E \times F$ into a separated locally p-convex space G, then H is equicontinuous on $E \times F$.

Remark 11. Let $T_V^p(E)$ denote for any locally p-convex space E the modification of the initial topology having as a base of neighborhoods the system of all p-barrels in E. The axiom of additivity of the vector topology follows from

$$B + B \subseteq 2^{\frac{1}{p}} B ,$$

where B is a p-barrel.

The topology $T_V^p(E)$ is identical with the maximal locally p-convex topology having as a base of neighborhoods closed subsets in E.

Remark 12. The results of locally convex spaces concerning the barrelled topologies are included here as a special case for $n = 1$.

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