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A CLASS OF KREISS-TYPE UNIFORMLY BOUNDED SYSTEMS OF OPERATORS

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(Preliminary Communication)

In Kreiss' well-known papers [2,3] certain systems of linear transformations of finite dimensional spaces have been investigated which either form uniformly power-bounded families or which generate uniformly bounded systems of semigroups. H.O. Kreiss gave several equivalent conditions in terms of some spectral properties of the elements of the mentioned systems and their resolvent operators which guarantee that these families have the properties formulated above. His results have been reformulated and some other equivalent conditions have been obtained by several different authors (see [1, 7,8,6,13]) and also the original proofs have been simplified [8]. A nice survey of results concerning the above problems of characterising uniformly power-bounded families and uniformly bounded systems of matrix semigroups besides other problems and results connected with stability of difference operators is contained in Thomée's paper [12]. In Miller's papers [4,5] some estimates of the resolvent operator are given for appropriate classes of finite dimensional operators and some of Kreiss' conditions are shown to be consequences of those estimates.

In this communication a new condition is presented which guarantees the fulfilment of Kreiss' conditions and hence the fulfilment of all other equivalent conditions. A more general problem is considered and it is shown that both problems mentioned in the beginning can be treated in a unified manner. Also a generalized analogue of a very recent result of G. Strang [10] concerning families of matrices generating uniformly bounded systems of holomorphic semigroups is obtained; more precisely, after appropriate specifications, necessary and sufficient conditions equivalent to those of Strang are given which guarantee that a given family of $N \times N$ matrices generates a uniformly bounded system of holomorphic semigroups. Moreover, our results are valid for certain nontrivial classes of generally unbounded linear transformations of complex Banach spaces.

Let X denote a complex Banach space and let $[X]$ denote the space of all bounded linear transformations of X into itself with the usual operator norm. If T is a linear transformation let $\mathcal{D}(T) \subset X$ denote its domain and $\mathcal{R}(T) \subset X$ its range. If T is a closed linear transformation with a domain $\mathcal{D}(T)$ which is dense in X let $\mathcal{U}_\infty(T)$ be the family of all locally analytic functions each of which is piecewise holomorphic on a neighborhood of the extended spectrum $\sigma_e(T) = \sigma(T) \cup \{\infty\}$, where $\sigma(T)$ is the spectrum of T. If $T \in [X]$ let $\mathcal{U}(T)$ be the family of all locally analytic functions each of which is piecewise holomorphic on a neighborhood of the spectrum $\sigma(T)$. For more details concerning $\mathcal{U}_\infty(T)$ and $\mathcal{U}(T)$ the reader is referred to [11, pp.288-292].

Let $\mathcal{M}$ be a family of operators such that each element $T \in \mathcal{M}$ is a closed linear transformation of its domain $\mathcal{D}(T)$ which is dense in X, with $\mathcal{R}(T) \subset X$ and with nonempty resolvent set $\rho(T)$. We shall assume that $\mathcal{M}$ has the following properties:
(a) For each $T \in \mathcal{M}$ its spectrum $\sigma(T)$ is the closure of a collection of isolated poles $\{\lambda_j\}$ of the resolvent operator $R(\lambda, T) = (\lambda I - T)^{-1}$, where I denotes the identity operator, with a possible at most one point of accumulation $\lambda_\infty(T) = \lim_{j \to \infty} \lambda_j$.
(b) If $\lambda \in \sigma(T)$, $\lambda \neq \lambda_\infty(T)$, let $q(\lambda_j) = q(\lambda_j; T)$ be the multiplicity of λ_j as a pole of $R(\lambda, T)$.

It will be assumed that

(1) $$q(\lambda_j; T) \leqq q < +\infty$$

with q independent of $T \in \mathcal{M}$.

It is known that for any $f \in \mathcal{U}_\infty(T)$ (or $f \in \mathcal{U}(T)$ if $T \in [X]$) [11,p.305]

$$f(T) = \sum_{\substack{\lambda \in \sigma(T) \\ d(\lambda,\lambda_\infty) \geqq \rho > 0}} \frac{1}{2\pi i} \int_{C_\lambda} f(\mu) R(\mu,T) d\mu +$$

$$+ \frac{1}{2\pi i} \int_{C'_\rho} f(\mu) R(\mu,T) d\mu + \delta_\infty(T) f(\infty) I ,$$

where $C_\lambda = \{\mu : |\lambda - \mu| = \rho_\lambda,\ \rho_\lambda > 0\}$, $K_\lambda = \{\mu : |\lambda - \mu| \leqq \rho_\lambda\}$ with $K_\lambda \cap \sigma(T) = \{\lambda\}$, $\lambda \in \sigma(T)$, $|\lambda| < \rho \rightarrow |\lambda| < \rho_\infty$, $C'_\rho = \{\mu : d(\mu, \lambda_\infty) = \rho_\infty,\ 0 < \rho_\infty < \rho\}$ with $C'_\rho \cap \sigma(T) = \emptyset$,

$$d(\mu,\lambda_\infty) = \begin{cases} |\mu - \lambda_\infty| & \text{if } \lambda_\infty \neq \infty \\ \dfrac{1}{|\mu|} & \text{if } \lambda_\infty = \infty , \end{cases}$$

$\delta_\infty(T) = 0$ for $T \in [X]$ and $\delta_\infty(T) = 1$ for $T \notin [X]$.

Further

$$\frac{1}{2\pi i} \int_{C_\lambda} f(\mu) R(\mu,T) d\mu = \sum_{k=1}^{q(\lambda)} f^{(k-1)}(\lambda) Z_{\lambda,k} ,$$

where

(2) $$Z_{\lambda,1} = \frac{1}{2\pi i} \int_{C_\lambda} R(\mu,T) d\mu ,$$

$$Z_{\lambda,k+1} = (T - \lambda I) Z_{\lambda,k} ; \quad k = 1, 2, \ldots .$$

A subclass $\mathcal{N} \subset \mathcal{M}$ will be considered. By definition $T \in \mathcal{N}$ if it satisfies:

(c) For every $x \in X$

(3) $$x = \sum_{\substack{\lambda \in \sigma(T) \\ \lambda \neq \lambda_\infty}} Z_{\lambda,1} x ,$$

where $Z_{\lambda,1}$ are defined by formulas (2) and convergence in (3) means the norm convergence in X .

We note that class $\mathcal{M}$ contains all regular operators [9]. The reason for the use of condition (3) is as in [9] - to avoid pathologies in the behavior of invariant subspaces corresponding to λ_∞.

Let $\mathcal{T}$ be a family of closed linear operators and let $\Phi(\mathcal{T})$ be either a subset of $\mathcal{U}_\infty(T)$ for all $T \in \mathcal{T}$ if at least one $T \in \mathcal{T}$ is unbounded or a subset of $\mathcal{U}(T)$ for all $T \in \mathcal{T}$ if all $T \in \in [X]$. It will be assumed that there exists a convex domain $\mathcal{D}$ and an open set Δ such that for every $f \in \Phi(\mathcal{T})$ and its domain $\Delta(f)$, $\Delta \supset \Delta(f) \supset \mathcal{D}$ and such that the following conditions are fulfilled:

(i) For every $\lambda \in \mathcal{D}$ ($\infty \in \mathcal{D}$ if $T \notin [X]$)

$$\sup\{|f(\lambda)| : f \in \Phi(\mathcal{T})\} < \infty ,$$

for $\lambda \in (\Delta - \mathcal{D})$

$$\sup\{|f(\lambda)| : f \in \Phi(\mathcal{T})\} = +\infty$$

and for $\lambda \in \partial\mathcal{D} \cap \mathcal{D}$

$$\sup\{|f^{(k)}(\lambda)| : f \in \Phi(\mathcal{T})\} = +\infty, \quad k \geq 1 ,$$

where $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$ and $f^{(k)}(\lambda)$ denotes the k-th derivative of f at λ. So, $\Phi(\mathcal{T})$ cannot be empty.

(ii) For any fixed $k = 0, 1, \ldots$

$$\sup\{|f^{(k)}(\lambda)|[dist(\lambda, \partial\mathcal{D})]^k : \lambda \in \mathcal{D}, f \in \Phi(\mathcal{T})\} < +\infty ,$$

where

$$dist(\lambda, \partial\mathcal{D}) = \inf\{|\lambda - \mu| : \mu \in \partial\mathcal{D}\} .$$

Note that according to our assumptions $dist(\lambda, \partial\mathcal{D}) < +\infty$ for $\lambda \neq \infty$ and

$dist(\lambda, \partial \mathcal{D}) > 0$ for $\lambda \notin \partial \mathcal{D}$.

Problem. To characterise the class $\mathcal{T} \subset \mathcal{N}$ of operators having the property that

(4) $\quad M = \sup\{\|f(T)\| : f \in \Phi(\mathcal{N}),\ T \in \mathcal{T}\} < +\infty ,$

where $\Phi(\mathcal{N})$ is a family of functions satisfying conditions (i),(ii).

Example 1. Let $\mathcal{N} \subset [X]$ and

$\Phi_1 = \{f_n : f_n(\lambda) = \lambda^n, n = 0,1,\dots\}, \mathcal{D}_1 = \{\lambda : |\lambda| \leqq 1\}$.

The corresponding class $\mathcal{T}_1$ forms a uniformly power-bounded family of operators:

$$M_1 = \sup\{\|T^n\| : T \in \mathcal{T}_1,\ n = 0, 1, \dots\} < +\infty .$$

Example 2. Let $\mathcal{N} \subset [X]$ and $\Phi_2 = \{f_t : f_t(\lambda) = \exp\{t\lambda\}, t > 0\}, \mathcal{D}_2 = \{\lambda : \operatorname{Re} \lambda \leqq 0\}$.

The corresponding class $\mathcal{T}_2$ contains those bounded linear operators which are generators of uniformly bounded semigroups $S(t) = \exp\{tT\}$, $T \in \mathcal{T}_2$, $t > 0$, and the system of these semigroups is uniformly bounded:

$$M_2 = \sup\{\|\exp\{tT\}\| : T \in \mathcal{T}_2,\ t > 0\} < +\infty .$$

Example 3. Let ω be a fixed number with $0 \leqq \omega < \frac{\pi}{2}$. Let $\mathcal{N} \subset [X]$ and

$\Phi_3 = \{f_\tau : f_\tau(\lambda) = \exp(\tau\lambda),\ \tau = te^{i\theta}, t \geqq 0, |\theta| \leqq \omega\}$,

$\mathcal{D}_3 = \{\lambda : \lambda = |\lambda| e^{i\varphi}, |\varphi| \geqq \omega + \frac{\pi}{2}, -\pi < \varphi \leqq \pi\}$.

The corresponding class $\mathcal{T}_3$ contains generators of holomorphic semigroups $S(\tau) = \exp\{\tau T\}$, $\tau =$

$=|\tau|e^{i\theta}$, $|\theta|<\omega<\frac{\pi}{2}$ and the family of these semigroups is uniformly bounded:

$M_3 = \sup\{\|\exp\{\tau T\}\| : T \in \mathcal{T}_3,\ \tau = |\tau|e^{i\theta},\ |\theta| \leqq \omega\} < +\infty$.

Theorem 1. (Main theorem) Let $\mathcal{T}$ and Φ satisfy conditions (a),(b),(c) and (i),(ii) respectively. Let there be a constant $\tau(q)$ such that for all $T \in \mathcal{T}$ and $f \in \Phi$ the relation

$$(5) \qquad \sum_{\lambda \in \sigma(T)} |f^{(k)}(\lambda)|\,[dist\,(\lambda, \partial\mathcal{D})]^{k} \leqq \tau(q)$$

holds for $k = 0, 1, \ldots, q-1$, where q is defined in (1).

Then (4) is equivalent to the following three conditions (α) - (γ):

(α) $\sigma_e(T) \subset \mathcal{D}$, $T \in \mathcal{T}$, $(\sigma(T) \subset \mathcal{D}$, $T \in \mathcal{T}$ if $T \in [X])$,

(β) $\sup\{\|Z_{\lambda,1}(T)\| : T \in \mathcal{T}\} < +\infty$, $Z_{\lambda,k} = \Theta$, $k > 1$
(Θ denotes the zero operator)

for $\lambda \in \sigma(T) \cap \partial\mathcal{D}$ for which $\sup\{|f(\lambda)| : f \in \Phi\} > 0$.

(γ) There exists a constant κ independent of $T \in \mathcal{T}$ such that

$$\|Z_{\lambda,k}(T)\| \leqq \kappa\,[dist\,(\lambda, \partial\mathcal{D})]^{k-1}$$

holds for $\lambda \in \sigma(T) \cap \mathcal{D}$, $\lambda \notin \partial\mathcal{D}$, for which $\sup\{|f^{(k-1)}(\lambda)| : f \in \Phi\} > 0$ and $k = 1, 2, \ldots$.

Since the families Φ_1 , Φ_2, Φ_3 shown in examples 1,2,3 fulfil conditions (i),(ii) and since (a),(b),(c) and (5) are obviously satisfied in the case of $\mathcal{N} = \{N \times N$ complex matrices$\}$ the following three assertions are corollaries of the main theorem.

Theorem 2. A system $\mathcal{T}$ of $N \times N$ complex matrices is uniformly power-bounded if and only if (α)-(γ) hold, i.e. if $|\lambda_j| \leqq 1$ whenever $\lambda_j \in \sigma(T)$ and there is a constant M such that for all $T \in \mathcal{T}$

$$(6) \qquad \| Z_{\lambda_j,1}(T)\| \leqq M, \ Z_{\lambda_j,k} = \Theta, \ k > 1$$

hold for $|\lambda_1| = \ldots = |\lambda_p| = 1 > |\lambda_j|$, $j > p$, $\lambda_j \in \sigma(T)$ and there is a constant κ independent of $T \in \mathcal{T}$ such that

$$\| Z_{\lambda_j,k}(T)\| \leqq \kappa\,(1 - |\lambda_j|)^{k-1}$$

for $\lambda_j \in \sigma(T)$, $|\lambda_j| < 1$, $k = 1, 2, \ldots, q(\lambda_j)$.

Theorem 3. A system $\mathcal{T}$ of $N \times N$ complex matrices generates a uniformly bounded family of uniformly bounded semigroups if and only if $\mathrm{Re}\,\lambda_j \leqq 0$ for $\lambda_j \in \sigma(T)$ and (6) hold for $\lambda_j \in \sigma(T)$, $\mathrm{Re}\,\lambda_j = 0$ and if

$$\| Z_{\lambda_j,k}(T)\| \leqq \kappa\,|\mathrm{Re}\,\lambda_j|^{k-1}$$

for $\lambda_j \in \sigma(T)$, $\mathrm{Re}\,\lambda_j < 0$ and $k = 1, \ldots, q(\lambda_j)$.

Theorem 4. A system $\mathcal{T}$ of $N \times N$ complex matrices generates a uniformly bounded family of holomorphic semigroups $S(\tau)$, $\tau = |\tau| e^{i\theta}$, $|\theta| < \omega$, if and only if $|\arg \lambda_j| \geqq \omega + \frac{\pi}{2}$, $\lambda_j \in \sigma(T)$, if (6) holds for $\lambda_j \in \sigma(T)$ with $|\arg \lambda_j| = \omega + \frac{\pi}{2}$,

$$\| Z_{\lambda_j,k}(T)\| \leqq \varkappa\, |\mathrm{Re}\, \lambda e^{-i\omega}|^{k-1}$$

for $\lambda_j \in \sigma(T)$, $\arg \lambda_j > \omega + \frac{\pi}{2}$, $k = 1, 2, \ldots, q(\lambda_j)$

and

$$\| Z_{\lambda_j,k}(T)\| \leqq \varkappa\, |\mathrm{Re}\, \lambda e^{i\omega}|^{k-1}$$

for $\lambda_j \in \sigma(T)$, $\arg \lambda_j < -\frac{\pi}{2} - \omega$, $k = 1, \ldots, q(\lambda_j)$.

It is easy to see how to modify the statements to solve a pointwise problem (cf. [4,5,6]), i.e. to characterise a class $\mathcal{T}$ of operators T such that for a given family of functions Φ and a fixed element $x \in X$:

(7) $\quad M(x) = \sup\{\| f(T)x\| : T \in \mathcal{T}, f \in \Phi\} < +\infty$.

The corresponding result can be formulated as follows.

<u>Theorem 5</u>. Let $\mathcal{T}$ satisfy conditions (a),(b), (c) and Φ condition (i),(ii) and let x be a fixed element. Let there be a constant $\tau(q)$ such that for all $T \in \mathcal{T}$ and $f \in \Phi$

$$\sum_{\lambda \in \sigma(T)} |f^{(k)}(\lambda)|\,[\mathrm{dist}\,(\lambda, \partial\mathcal{D})]^{k} \leqq \tau(q)$$

$k = 0, 1, \ldots, q-1$,

then (7) is equivalent with the following conditions:

(αx) $\lambda \in \sigma(T), \lambda \neq \lambda_\infty, Z_{\lambda,1}x = \theta \Longrightarrow \lambda \in \mathcal{D}$,

(βx) $\sup\{\| Z_{\lambda,1}(T)x\| : T \in \mathcal{T}\} < +\infty$,

$\quad Z_{\lambda,k}x = 0, \quad k > 1$

for $\lambda \in \sigma(T) \cap \partial \mathcal{D}$ for which $\sup\{|f(\lambda)| : : f \in \Phi\} > 0$.

(γx) There is a constant $\kappa = \kappa(x)$ such that for all $T \in \mathcal{T}$

$$\| Z_{\lambda,k}(T)x \| \leqq \kappa [dist(\lambda, \partial \mathcal{D})]^{k-1}$$

for $\lambda \in \sigma(T) \cap \mathcal{D}$, $\lambda \notin \partial \mathcal{D}$ for which $\sup\{|f^{(k-1)}(\lambda)| : f \in \Phi\} > 0$ and $k = 1, \dots, q(\lambda)$.

A resolvent condition is an easy consequence of Theorem 1.

<u>Theorem 6</u>. Let $\mathcal{T}$ and Φ satisfy conditions (a), (b),(c) and (i),(ii) respectively. Let there be constants $\tau(q)$ and $\nu(q)$ such that for all $T \in \mathcal{T}$ the relation (5) and the following relation hold

$$\sum_{\lambda \in \sigma(T)} \frac{[dist(\lambda, \partial \mathcal{D})]^{k-1}}{|x - \lambda|^{k}} \leqq \frac{\nu(q)}{dist(x, \sigma(T))}$$

for $k = 1, 2, \dots, q$ and $x \notin \mathcal{D}$. Then there exists a constant η independent of $T \in \mathcal{T}$ such that

$$dist(x, \mathcal{D}) \| R(x, T) \| \leqq \eta .$$

Theorem 6 can be used as a mediator for the proof of the equivalence of several other conditions quaranteeing the fulfilment of (4), e.g. the Kreiss-Strang condition concerning the numerical ranges of appropriate transformations of Hilbert spaces [10].

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