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DIFFERENTIABILITY OF CONVEX FUNCTIONALS AND BOUNDEDNESS OF NONLINEAR OPERATORS AND FUNCTIONALS

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Introduction. The first part of this paper concerns the differentiability properties of convex functionals. It is shown that if f is a convex continuous subadditive functional having the first Gâteaux derivative $f'(u)$ and the second Gâteaux differential $V^2 f(u, h, k)$ on some open convex bounded neighbourhood $V(0)$ of 0 of a linear normed space X and if $\|f'(0)\|$ is small, then there exists the Fréchet derivative $f'(u)$ on $V(0)$ and $\|f'(u)\|$ is small on $V(0)$ provided $V^2 f(u, h, k)$ is continuous at $(0, 0)$ uniformly with respect to $u \in V(0)$ (Th.1). Some conditions under which the Gâteaux differential $Vf(u_0, h)$ (or the Gâteaux derivative $f'(u)$ of a convex functional is the Fréchet derivative are established (Theorems 2,4,5,6). Moreover, the subsets of X which consist of the points of X at which a convex functional (under some further conditions) possesses the Fréchet or Hadamard's derivative are described (Theorems 3,7).

The second part of this paper is devoted to the study of boundedness of nonlinear operators and functionals. Each section concludes with a brief note concerning some recent results in these topics. For some further references see [1],

[2], [3] and the references cited here.

1. Differentiability of convex functionals.

The terminology and notations of [1], [2], [3] is used. For Gâteaux, Fréchet differentials and derivatives we use notions and notations given in Vajnberg's book [4, chapt. I]. A functional f is said to be subadditive on a set Q if u_1, $u_2 \in Q$, $u_1 + u_2 \in Q$ imply that $f(u_1+u_2) \leqq f(u_1) + f(u_2)$.

Theorem 1. Let X be a Banach space, f a continuous functional on X, $f(0)=0$. Suppose f has the first and second Gâteaux differentials $Vf(u,h)$, $V^2f(u,h,k)$ on some convex open bounded neighbourhood $V(0)$ of $0 \in X$ such that $|Vf(0,h)| \leqq \varepsilon \|h\|$ for every $h \in X$ and some number $\varepsilon > 0$ and that $V^2 f(u,h,k)$ is continuous at $h = 0$, $k = 0$ uniformly with respect to $u \in$ $\in V(0)$. Assume f is subadditive and convex on $V(0)$.

Then f possesses the Fréchet derivative $f'(u)$ on $V(0)$, $\|f'(u)\| \leqq 3\varepsilon$ for each $u \in V(0)$; f, $f'(u)$ are Lipschitzian on $V(0)$ and f is uniformly differentiable on $V(0)$.

Proof. According to Theorem 1 [5] f has Lipschitzian Fréchet derivative $f'(u)$ on $V(0)$ and f is uniformly differentiable on $V(0)$. By our hypothesis $\|f'(0)\| \leqq \varepsilon$. It remains to prove that $\|f'(u)\| \leqq 3\varepsilon$ for each $u \in V(0)$.

Let $u \in V(0)$ be an arbitrary (but fixed) element of $V(0)$. Choose $t > 0$ sufficiently small such that th, $u \pm th \in V(0)$, where $h \in X$, $\|h\| \leqq 1$.

Then

(1) $f(u+th) = f(u) + \frac{1}{1!} f'(u)th +$

$$+ \frac{1}{2!} t^2 \int_0^1 (1-\tau)\, V^2 f(u+\tau th, h, h)\, d\tau .$$

Since $V^2 f(u, h, k)$ is continuous at $(0,0)$ and homogeneous at h, k we see that there exists a constant $N > 0$ such that $|V^2 f(u, h, h)| \leqq N \|h\|^2$ for each $u \in V(0)$. Using (1) and emplyoing the properties of f we have that

(2) $$f'(u)th \leqq f(th) + \frac{1}{4} t^2 N \|h\|^2 .$$

Since f has the Fréchet derivative $f'(0)$ at 0 and $f(0) = 0$,

(3) $$f(th) = f'(0)th + \omega(0, th) ,$$

where

(4) $$0 \leqq \omega(0, th) \leqq \varepsilon t \|h\|$$

for sufficiently small $t > 0$. From the inequality $f'(0)h \leqq \varepsilon \|h\|$ and the relations (2),(3),(4) it follows that

$$f'(u)h \leqq 2\varepsilon \|h\| + \frac{1}{4} t N \|h\|^2 .$$

Choose $t > 0$ such small that $\frac{1}{4} t N \leqq \varepsilon$. Then $f'(u)h \leqq 3\varepsilon \|h\|$ for each $u \in V(0)$ and $h \in X$, $\|h\| \leqq 1$. On the other side, employing convexity and subadditivity of f on $V(0)$ we obtain

$$f'(u)th \geqq f(u) - f(u-th) -$$

$$- \frac{1}{2} t^2 \int_0^1 (1-\tau)\, V^2 f(u+\tau h, h, h)\, d\tau \geqq$$

$$\geq -f(-th) - \frac{1}{4}t^2 N \|h\|^2 =$$

$$= f'(0)th - \omega(0, t(-h)) - \frac{1}{4}t^2 N\|h\|^2 \geq$$

$$\geq -3\varepsilon t \|h\|$$

for sufficiently small $t > 0$, $\|h\| \leqq 1$ and $u \in V(0)$. Thus we have $|f'(u)h| \leqq 3\varepsilon \|h\|$ for each $u \in V(0)$ and $h \in X$ with $\|h\| \leqq 1$. Hence $\|f'(u)\| \leqq 3\varepsilon$ for each $u \in V(0)$ and this concludes the proof.

Remark 1. We recall a certain assertion which is well-known and useful in real analysis (see for instance [6]): Let g be a twice-differentiable real function of real variable defined on an interval J of the length ℓ. Assume $|g(x)| \leqq \varepsilon$ and $|g''(x)| \leqq k$ for every $x \in J$, where ε, k are some fixed positive numbers. If $4\left(\frac{\varepsilon}{k}\right)^{\frac{1}{2}} \leqq \ell$, then $|g'(x)| \leqq 2(\varepsilon k)^{\frac{1}{2}}$ for every $x \in J$.

Theorem 2. Let X be a reflexive Banach space, f a convex continuous subadditive functional on X having the Gâteaux differential $Vf(u_0, h)$ at $u_0 \in X$. Assume there exists a weakly continuous functional g on the closed ball $D = \{u \in X : \|u\| \leqq 1\}$ such that $f(u) \leqq g(u)$ and $g(-u) \leqq -g(u)$ for each $u \in D$.

Then f possesses the Fréchet derivative $f'(u_0)$ at u_0.

Proof. Continuity and convexity of f imply that $Vf(u_0, h) = f'(u_0)h$, where $f'(u_0)$ denotes the Gâteaux derivative of f at u_0. Suppose that there does not exist the Fréchet derivative $f'(u_0)$ at u_0. From the beginning we proceed as in the proof of Theorem 1 [1]. In the relations (1) - (4) of that proof write u_0 for 0, f for F and the remainder in (1) replace by

$$\omega(u_0, th) = f(u_0 + th) - f(u_0) - f'(u_0)th .$$

Since the one-sided Gâteaux derivative $V_+ f(u_0, h)$ is equal to $f'(u_0)h$ and f is convex, we can deal here only with a sequence $\{t_n\}$ of positive numbers. Let h_0, $\{h_n\}$, $\{t_n\}$ have the same meaning as in the proof of Theorem 1 [1]. Instead of (5) in [1] we write

$$(5)\quad \begin{aligned} f(u_0+t_{n_k}h_{n_k}) - f(u_0) &= f'(u_0)t_{n_k}h_{n_k} + \omega(u_0, t_{n_k}h_{n_k}), \\ f(u_0+t_{n_k}h_0) - f(u_0) &= f'(u_0)t_{n_k}h_0 + \omega(u_0, t_{n_k}h_0) . \end{aligned}$$

Being f convex,

$$(6)\quad 0 \leqq \omega(u_0, t_{n_k}h_{n_k}), \quad 0 \leqq \omega(u_0, t_{n_k}h_0)$$

for every k $(k = 1, 2, \dots)$. From (5) and (6) we have that

$$(7)\quad 0 \leqq \omega(u_0, t_{n_k}h_{n_k}) = f(u_0+t_{n_k}h_{n_k}) - f(u_0) +$$

$$+ f'(u_0)t_{n_k}(h_0 - h_{n_k}) + \omega(u_0, t_{n_k}h_0) + f(u_0) - f(u_0+t_{n_k}h_0) .$$

As $t_{n_k} > 0$, $t_{n_k} \longrightarrow 0$, there exists an index k_0 such that $k \geq k_0 \Longrightarrow 0 < t_{n_k} < 1$. Consider now on only such k for which $k \geqq k_0$. Convexity and subaddivity of f imply

$$(8)\quad f(u_0+t_{n_k}h_{n_k}) - f(u_0) \leqq (1-t_{n_k})f(u_0) + t_{n_k}f(u_0+h_{n_k}) -$$

$-f(u_o) = t_{n_k}[f(u_o + h_{n_k}) - f(u_o)] \leqq t_{n_k} f(h_{n_k})$.

Similarly we obtain that

(9) $f(u_o) - f(u_o + t_{n_k} h_o) \leqq f(u_o - t_{n_k} h_o) - f(u_o) \leqq$

$\leqq t_{n_k}[f(u_o - h_o) - f(u_o)] \leqq t_{n_k} f(-h_o)$.

Since h_o , $h_{n_k} \in D$ for every k $(k = 1, 2, \ldots)$, our hypothesis imply

(10) $f(h_{n_k}) \leqq g(h_{n_k})$, $f(-h_o) \leqq g(-h_o) \leqq -g(h_o)$

for every k $(k = 1, 2, \ldots)$. From (7) - (10) we obtain $(k \geq k_o)$ that

(11) $0 \leqq \frac{1}{t_{n_k}} \omega(u_o, t_{n_k} h_{n_k}) \leqq g(h_{n_k}) - g(h_o) +$

$+ f'(u_o)(h_o - h_{n_k}) + \frac{1}{t_{n_k}} \omega(u_o, t_{n_k} h_o)$.

Since $t_{n_k} \longrightarrow 0_+$ as $k \longrightarrow \infty$ and f possesses the Gâteaux derivative $f'(u_o)$ at u_o , we have that $\frac{1}{t_{n_k}} \omega(u_o, t_{n_k} h_o) \rightarrow 0$. Furthermore, $h_{n_k} \xrightarrow{w} h_o$ implies $g(h_{n_k}) - g(h_o) \longrightarrow 0$ and $f'(u_o)(h_o - h_{n_k}) \rightarrow 0$ as $k \longrightarrow \infty$. Thus $\frac{1}{t_{n_k}} \omega(u_o, t_{n_k} h_{n_k}) \longrightarrow 0$ as $k \rightarrow \infty$ and this is a contradiction (see proof of Th.1 [1]). The theorem is proved.

The result of Th. 2 one may rewrite as follows:

<u>Theorem 2'</u>. Let X be a reflexive Banach space, f a functional on X having the Gâteaux derivative $f'(u_o)$ at $u_o \in X$. Assume f is convex and subadditive on a closed ball $D_1 = \{u \in X : \|u\| \leqq \|u_o\| + 1\}$. Suppose the-

re exists a weakly continuous functional g on a closed ball $D = \{u \in X : \|u\| \leqq 1\}$ such that $f(u) \leqq g(u)$ and $g(-u) \leqq -g(u)$ for each $u \in D$.

Then f possesses the Fréchet derivative $f'(u_0)$ at u_0.

Theorem 3 [3] and Theorem 2 give the following

Theorem 3. Let X be a separable reflexive Banach space, f a convex Lipschitzian subadditive functional on X. Suppose there exists a weakly continuous functional g on $D = \{u \in X : \|u\| \leqq 1\}$ such that $f(u) \leqq g(u)$ and $g(-u) \leqq -g(u)$ for each $u \in D$.

Then the set Z of all $u \in X$ where the Fréchet derivative $f'(u)$ of f at u exists is a $F_{\sigma\delta}$ set of the second category in X and hence it contains a G_δ-set which is dense in X.

Theorem 4. Let X be a reflexive Banach space, f a functional on X having the Gâteaux derivative $f'(u_0)$ at $u_0 \in X$. Suppose f is convex on some convex open neighbourhood $V(u_0)$ of u_0. Assume there exists a functional g on $V(u_0)$ such that $f(u_0) = g(u_0)$, $f(u) \leqq g(u)$ for each $u \in V(u_0)$ and that g possesses the Fréchet derivative $g'(u_0)$ at u_0.

Then f possesses the Fréchet derivative $f'(u_0)$ at u_0.

Proof. Suppose that the Fréchet derivative $f'(u_0)$ does not exist at $u_0 \in X$. Let $\{h_{n_k}\}$, $\{t_{n_k}\}$ have the similar meaning as in the proof of Theorem 1 [1] (see also the proof of Th.2). In view of the existence of the Gâteaux derivative $f'(u_0)$ at u_0, we may restrict our consideration only for $\{t_{n_k}\}$ with $t_{n_k} \to 0_+$ as $k \to \infty$. Since $h_{n_k} \xrightarrow{w} h_0$ and $\| h_{n_k} \| = 1$ $(k = 1, 2, \dots)$, $\| h_0 \| \leqq 1$. As $t_{n_k} \longrightarrow 0_+$ whenever $k \to \infty$, there exists an integer k_0 such that $k \geqq$ $\geqq k_0 \Longrightarrow u_0 + t_{n_k} h_{n_k}$, $u_0 - t_{n_k} h_0 \in V(u_0)$. Moreover, $0 \leqq \omega(u_0, t_{n_k} h_{n_k})$ for each $k \geqq k_0$, where $\omega(u_0, t_{n_k} h_{n_k}) = f(u_0 + t_{n_k} h_{n_k}) - f(u_0) - f'(u_0) t_{n_k} h_{n_k}$. In fact, convexity of f on $V(u_0)$ implies $(0 < \alpha < 1$, $k \geqq k_0)$ that

$$f(u_0 + \alpha t_{n_k} h_{n_k}) \leqq (1-\alpha) f(u_0) + \alpha f(u_0 + t_{n_k} h_{n_k}) .$$

Hence

$$\frac{1}{\alpha} [f(u_0 + \alpha t_{n_k} h_{n_k}) - f(u_0)] \leqq$$

$$\leqq f(u_0 + t_{n_k} h_{n_k}) - f(u_0) .$$

Since $\lim\limits_{\alpha \to 0_+} \frac{1}{\alpha} [f(u_0 + \alpha t_{n_k} h_{n_k}) - f(u_0)] = V_+ f(u_0, t_{n_k} h_{n_k})$ and $V_+ f(u_0, t_{n_k} h_{n_k}) = f'(u_0) t_{n_k} h_{n_k}$, we obtain the desired conclusion at once from this fact and the last inequality. Now we proceed as in the proof of Theorem 2. For the first difference on the right side in (7) we have that $(k \geqq k_0)$

$$(12)\quad f(u_0 + t_{n_k} h_{n_k}) - f(u_0) \leqq g(u_0 + t_{n_k} h_{n_k}) - g(u_0)$$

by our hypothesis. Since g possesses the Fréchet derivative $g'(u_0)$ at u_0,

$$(13)\quad g(u_0 + t_{n_k} h_{n_k}) - g(u_0) = g'(u_0) t_{n_k} h_{n_k} + \omega_1(u_0, t_{n_k} h_{n_k}),$$

where

$$(14) \qquad \frac{1}{t_{n_k}}\,\omega_1(u_o, t_{n_k} h_{n_k}) \to 0$$

as $k \to \infty$, for $t_{n_k} \to 0_+$ and $\|h_{n_k}\| = 1$. Further by convexity of f on $V(u_o)$ and according to our hypothesis $(k \geq k_o)$

$$(15) \qquad \begin{aligned} f(u_o) - f(u_o + t_{n_k} h_o) &\leqq f(u_o - t_{n_k} h_o) - f(u_o) \leqq \\ &\leqq g(u_o - t_{n_k} h_o) - g(u_o) = \\ &= -g'(u_o)\, t_{n_k} h_o + \omega_1(u_o, t_{n_k}(-h_o)) , \end{aligned}$$

where

$$(16) \qquad \frac{1}{t_{n_k}}\,\omega_1(u_o, t_{n_k}(-h_o)) \to 0, \quad k \to \infty .$$

From (7),(12),(13),(15) it follows that

$$(17) \qquad \begin{aligned} 0 \leqq \frac{1}{t_{n_k}}\,\omega(u_o, t_{n_k} h_{n_k}) &\leqq g'(u_o)(h_{n_k} - h_o) + \\ + \frac{1}{t_{n_k}}(\omega_1(u_o, t_{n_k} h_{n_k}) &+ \omega_1(u_o, -t_{n_k} h_o) + \omega(u_o, t_{n_k} h_o)) + \\ &+ f'(u_o)(h_o - h_{n_k}) \end{aligned}$$

for each $k \geq k_o$. Since f has the Gâteaux derivative at $u_o \in X$, $t_{n_k}^{-1}\,\omega(u_o, t_{n_k} h_o) \longrightarrow 0$ as $k \to \infty$. As $h_{n_k} \xrightarrow{w} h_o$, we have that $g'(u_o)(h_{n_k} - h_o) \to 0$, $f'(u_o)(h_o - h_{n_k}) \to 0$. These facts and (14),(16),(17) imply that $\frac{1}{t_{n_k}}\,\omega(u_o, t_{n_k} h_{n_k}) \longrightarrow 0$ as $k \to \infty$, which is a contradiction (see the beginning of the proof of Th.1 [1]). Hence f possesses the Fréchet derivative $f'(u_o)$ at $u_o \in X$. Theorem is proved.

One may proved the following

Theorem 5. Let X be a reflexive Banach space, f a convex functional on some convex open neighbourhood $V(u_0)$ of $u_0 \in X$ and having the Gâteaux derivative $f'(u_0)$ at u_0. Suppose there exists a subadditive functional g on an open ball $B_R = \{u \in X : \|u\| < R\}$ containing u_0 such that $f(u_0) = g(u_0)$ and $f(u) \leqq g(u)$ for each u of some convex open neighbourhood $V_1(u_0)$ of u_0. Assume g possesses the Fréchet derivative $g'(0)$ at 0.

Then f possesses the Fréchet derivative $f'(u_0)$ at u_0.

Theorem 6. Let X be a reflexive Banach space, f a convex functional on the closed ball $D = \{u \in X : \|u\| \leqq 1\}$, $f(0) = 0$. Suppose f is weakly continuous on D, $f(-u) \leqq -f(u)$ for each $u \in D$ and that there exists the Gâteaux derivative $f'(0)$ at 0.

Then f possesses the Fréchet derivative $f'(0)$ at 0.

We shall use a notion of the Hadamard's derivative [7,chapt.VIII,p.150-151],[8],[9],[10],[12],[19,Theorem 3.3]. Let F be a continuous mapping of an open set Ω of a Banach space X into Banach space Y. A mapping F is said to have a Hadamard's differential at $u_0 \in \Omega$ if there exists a linear mapping A_{u_0} of X into Y having the following property: for any continuous mapping g of $J = \langle 0, 1 \rangle$ into Ω such that $g(0) = u_0$ and that the derivative $g'(0)$ of g at 0 (with respect to J) exists, then $t \to F(g(t))$ has at the point $t = 0$ a derivative (with respect to J)

equal to $A_{u_0} g'(0)$. The linear map A_{u_0} is called a Hadamard's derivative of F at u_0.

One may prove that A_{u_0} is a continuous mapping from X into Y. Moreover, if F is Lipschitzian on Ω and there exists a linear Gâteaux differential $DF(u_0, h)$ at $u_0 \in \Omega$, then F possesses the Hadamard's derivative A_{u_0} at $u_0 \in \Omega$ [7]. This result together with Theorem 3 [3] give the following

__Theorem 7__. Let X be a separable Banach space, f a convex Lipschitzian functional on X. Then the set Z of all $u \in X$ where the Hadamard's derivative A_u exists is a $F_{\sigma\delta}$-set of the second category in X.

__Remarks__. The properties of the one-sided Gâteaux differentials and derivatives of convex functionals are also studied in [14,§ 3]. The Fréchet and Gâteaux differentiability of convex functionals hass been recently investigated by E. Asplund [15]. One of his interesting results is as follows: If X is a Banach space which admits an equivalent norm such that the corresponding dual norm in X^* is locally uniformly rotund, then the set W of all $x \in X$, where a continuous convex functional f is Fréchet-differentiable is a G_δ-set which is dense in X. In particular, each Banach space X such that X^* is separable and each reflexive Banach space which admits an equivalent Fréchet-differentiable norm has the above property.

2. __Boundedness of nonlinear operators and functionals__. Let X, Y be linear normed spaces, $F: X \to Y$ a

mapping of X into Y . A mapping F is said to have the Baire property in $M \subseteq X$ if there exists a set $N \subset M$ of the 1. category in M such that $F/_{M-N}$ is continuous. A set $A \subset X$ is called a Baire set in X if there exists an open set G in X such that $G-A$, $A-G$ are both the sets of the 1. category in X (see [16],chapt.I,§ 11; [17] § 22C). Each closed and each open set is a Baire set. It is known [16,chapt.I] that $M \subset X$ is a Baire set $\Longleftrightarrow M = G - P$, where G is a F_{σ}-set and P is a set of the first category in X . In particular a set $Z = G - R$, where G is open in X and R is a set of the first category in X is a Baire set in X . If $F: X \to Y$ is a mapping having the Baire property in X , then for each open (or closed) subset $G \subset Y$ the set $F^{-1}(G) \subset X$ is a Baire set in X . Conversely: if Y is a separable space and for each $G \subset Y$ open (or closed) in Y , the set $F^{-1}(G)$ is a Baire set in X , then F has the Baire property in X . If M is a Baire set of the second category in a topological group Q , then the set $\{x y^{-1}: x \in M, y \in M\}$ is a neighbourhood of the unit element of Q [17,§ 22C]. In particular: if X is a linear normed space and $M \subset X$ is a Baire set of the second category in X , then the set W of all differences $w = u - v$, where u, $v \in M$ is a neighbourhood of zero in X [23]. A mapping $F: X \to Y$ is said to be a function of the 1. Baire class if it is a point-limit of the sequence $\{F_n(u)\}$ of continuous mappings F_n $(n = 1, 2, \dots)$ of X into Y . A mapping

$F: X \to Y$ (a functional f on X) is called bounded (upper-bounded) in X if for each bounded set $M \subset X$, $F(M)$ is bounded in Y ($f(M)$ is upper-bounded). Henceforth E_1 denotes the set of all real numbers.

All theorems of this section are stated for mappings or functionals which are defined on a linear normed space X of the 2. category in itself (in particular for mappings which are defined on Banach spaces).

Theorem 9. Let X, Y be linear normed spaces, X of the 2. category in itself, $F: X \to Y$ a mapping of X into Y. Suppose the following conditions are fulfilled:

(a) $\| F(\lambda u) \| = |\lambda|^{\gamma} \| F(u) \|$ for every $u \in X$, $\lambda \in E_1$, where γ is some positive number.

(b) There exist an open subset $M \neq \emptyset$ in X and a mapping $G: M \to Y$ of M into Y having the Baire property in M and is such that $\| F(u) \| \leq \| G(u) \|$ for each $u \in M$.

(c) There exists a constant $K > 0$ such that
$$\| F(u - v) \| \leq K \max(\| F(u) \|, \| F(v) \|)$$
for each $u, v \in M$.

Then F is bounded mapping in X.

Proof. Since G has the Baire property in M, there exists a set $A \subset M$ of the 1. category in M such that $G/_{M-A}$ is continuous. Being M open and nonvoid in the space X of the second category in itself, M is a set of the 2. category in X. Furthermore, A is a set of the 1. category in X and hence $M - A \neq \emptyset$.

Therefore there exists $u_0 \in M - A$ such that $G/_{M-A}$ is continuous at u_0. Thus for $\varepsilon_0 > 0$ there exists an open subset $N \subset M$ such that $u_0 \in N$ and $u \in N - A \Longrightarrow \| G(u) - G(u_0) \| \leqq \varepsilon_0$.
But $Z = N - A$ is a Baire set of the second category in X and hence the set W of all differences $w = u - v$, where $u, v \in Z$, is a neighbourhood of 0 in X. According to (a),(b) for $w \in W$ (i.e. $w = u - v$, $u, v \in Z$) we have

$$\| F(w) \| = \| F(u-v) \| \leqq$$
$$\leqq K \max (\| F(u) \|, \| F(v) \|).$$

Since $Z \subset M$ and $u \in Z \Longrightarrow \| F(u) \| \leqq \| G(u) \| \leqq \leqq \varepsilon_0 + \| G(u_0) \|$, we have that $\| F(w) \| \leqq K(\varepsilon_0 + \| G(u_0) \|)$ for each $w \in W$. Hence F is bounded in some neighbourhood of 0 and in view of (a) of Th.9, F is bounded on each bounded ball of X. This concludes the proof.

<u>Corollary 1</u>. Let X, Y be linear normed spaces, of the 2. category in itself, $F: X \longrightarrow Y$ a mapping of X into Y. Suppose the following conditions are fulfilled:

(a) $\| F(\lambda u) \| = |\lambda|^{\gamma} \| F(u) \|$ for each $u \in X$, $\lambda \in E_1$, where γ is some positive number.

(b) F is continuous at some point $u_0 \in X$ and there exists a constant $K > 0$ such that

$$\| F(u-v) \| \leqq K \max (\| F(u) \|, \| F(v) \|)$$

holds for each u, v of some open neighbourhood $V(u_0)$ of u_0.

Then F is bounded in X.

Theorem 9'. Assume X, Y are the same as in Theorem 9. Suppose the assumption (b) of Th.9 is fulfilled and that F satisfies the condition $\| F(u+v) \| \leqq$ $\leqq K \max(\|F(u)\|, \|F(v)\|)$ for every $u, v \in X$, where K is some positive constant. If $F(-u) = -F(u)$ for each $u \in M$, then F is bounded in X.

Proof. Using the similar arguments as in the proof of Th.9 we conclude that F is bounded on some open neighbourhood W of 0. Hence there exist the numbers $\sigma > 0$, $C > 0$ such that $\| u \| \leqq \sigma \Longrightarrow u \in W$ and that $\| u \| \leqq \sigma \Longrightarrow \| F(u) \| \leqq C$. Let D_R be a closed ball centered about 0 and with radius $R > 0$, v its arbitrary element. There exists an integer n_0 such that $R n_0^{-1} \leqq \sigma$. By our hypothesis

$$\|F(v)\| = \|F(\tfrac{v}{n_0} \cdot n_0)\| \leqq K \max(\|F(\tfrac{v}{n_0})\|,$$

$$\|F(\tfrac{v}{n_0}(n_0-1))\|) \leqq \ldots \leqq K^{n_0} \|F(\tfrac{v}{n_0})\| \leqq K^{n_0} C .$$

Hence F is bounded on D_R and being D_R arbitrary, this proves the boundedness of F in X.

Corollary 2. Let X, Y be linear normed spaces, X of the second category in itself, $F: X \to Y$ a mapping of X into Y such that F satisfies the condition $\| F(u+v) \| \leqq K \max(\|F(u)\|, \|F(v)\|)$ for every $u, v \in X$ (K is some positive constant) and that F is continuous at some point $u_0 \in X$. If $F(-u) = -F(u)$ for each u of some open neighbourhood $V(u_0)$ of u_0, then F is bounded in X.

Remark 2. We recall the result of S. Banach [18,p.79] concerning the continuity of linear operators: If $A : X \to X$ is an additive operator from Banach space X into X and such that $\|A(u)\| \leq \|G(u)\|$ for every $u \in X$, where G is a (nonlinear) operator from X into X having the Baire property in X, then A is continuous (and hence homogeneous, i.e. $A(\lambda u) = \lambda A(u)$ for every $u \in X$ and $\lambda \in E_1$) on X.

Theorem 10. Let X be a linear normed space of the second category in itself, f a subadditive functional on X such that f is lower-semicontinuous at 0. Suppose there exist an open subset $M \neq \emptyset$ of X, a functional g defined on M such that $f(u) - f(v) \leq g(u) - g(v)$ for each $u, v \in M$. Assume g possesses the Baire property in M and $f(-u) \leq -f(u)$ for each $u \in M$.

Then f is continuous in X and upper-bounded on each closed ball $D_R = \{u \in X : \|u\| \leq R\}$ of X.

Proof. First of all $f(0) = 0$. Indeed, for some $u \in M$ we have that
$f(0) = f(u-u) \leq f(u) + f(-u) \leq f(u) - f(u) = 0$.
On the other hand $f(0) \leq 2f(0)$ implies $f(0) \geq 0$ and hence $f(0) = 0$. By our hypothesis there exists $u_0 \in M - A$, where A is a set of the 1. category in M, such that the restriction $g/_{M-A}$ of g to $M - A$ is continuous at u_0. Thus for $\varepsilon_0 > 0$ there exists an open subset $N \subset M$ such that $u_0 \in N$

and $u \in N-A \Longrightarrow |g(u)-g(u_0)| \leqq \frac{\varepsilon_0}{2}$.

The set W of all differences $w = u - v$, where u, $v \in N-A$, is a neighbourhood of 0 in X . Hence there exists $\delta_0 > 0$ such that $\|w\| < \delta_0 \Longrightarrow w \in W$. For any $w \in W$ with $\|w\| < \delta_0$ we have

$$f(w) = f(u-v) \leqq f(u) + f(-v) \leqq f(u) - f(v) \leqq$$

$$\leqq g(u) - g(v) \leqq |g(u) - g(u_0)| + |g(u_0) - g(v)| \leqq \varepsilon_0 .$$

On the other side f is lower-semicontinuous at 0 . Therefore there exists $\delta_1 > 0$ such that $\|w\| < \delta_1 \Longrightarrow f(w) \geqq f(0) - \varepsilon_0 = -\varepsilon_0$. Set $\delta = \min(\delta_0, \delta_1)$, then $\|w\| < \delta \Longrightarrow |f(w)| < \varepsilon_0$. This denotes that f is continuous at 0 . Hence f is continuous on X (see [19,Th.25.2]). Continuity of f at 0 and subadditivity of f imply that f is upper-bounded on each closed ball D_R . This concludes the proof.

<u>Theorem 11</u>. Let X be a linear normed space of the second category in itself, f a seminorm (i.e. f is subadditive and $f(\alpha u) = |\alpha| f(u)$ for every $u \in X$ and $\alpha \in E_1$) on X . Suppose there exist an open subset $M \neq \emptyset$ of X and a functional g defined on M having the Baire property in M such that $f(u) \leqq g(u)$ for each $u \in M$.

Then f is continuous and hence bounded in X .

<u>Remark 3</u>. The above theorems can be used to investigation of boundedness and continuity of Gâteaux differentials. One may also apply them to investigation of the exis-

tence of the bounded differential [20]. The following fact is well-known: If f is a functional having the Baire property in the space X of the 2. category in itself and if there exists a linear Gâteaux differential $Df(u_0, h)$ at $u_0 \in X$, then f possesses the Gâteaux derivative $f'(u_0)$ at u_0.

In the case when f is continuous on X, this fact can be obtained without using the Baire's theorems as follows: Denote $f_n(h) = n(f(u_0 + n^{-1}h) - f(u_0))$, $n = 1, 2, \dots$, $h \in X$. Then $f_n(h)$ are continuous on X and $\lim_{n\to\infty} f_n(h) = Df(u_0, h)$ for every $h \in X$. Thus $Df(u_0, h)$ is a function of the 1. Baire class. Then the sets $P = \{h \in X : Df(u_0, h) \leqq 0\}$, $Q = \{h \in X : Df(u_0, h) \geqq 0\}$ are G_δ-sets in X ([21,Th.14.3.1]). Hence $N = P \cap Q$ is a G_δ-set in X and $N = \{h \in X : Df(u_0, h) = 0\}$. Since N is linear and G_δ-set in the space X of the second category in itself, by Mazur-Sternbach Theorem [22,§ 3] it is closed in X. Now it is sufficient to use the following assertion ([23], for functionals see also [24], chapt.I,cor.2): Let X, X_1 be linear normed spaces, $\dim X_1 < \infty$, $\mathcal{U} : X \to X_1$ a linear (i.e. additive and homogeneous) mapping of X into X_1. If the set $\mathcal{U}^{-1}(0)$ is closed in X, then $\mathcal{U}$ is continuous in X.

In sequel we shall use a property of subset of a linear normed space X which has been introduced by S. Mazur and W. Orlicz in [25]. A subset M of a linear normed space X over the field Φ of real or complex numbers is said to be a Mazur-Orlicz set if the space X is not the union $\bigcup_{i=1}^{\infty} M_i$ of a sequence of sets $M_i = \alpha_i M + u_i$, where $\alpha_i \in \Phi$, $u_i \in X$ $(i = 1, 2, \dots)$. The following nostions have been introduced by M. Zorn [26]. A subset D of X is linearly open if for $u, h \in X$ the elements α of Φ for which $u + \alpha h \in D$ form an open subset of Φ. A mapping G defined on a linearly open set $D \subset X$ with values in Y is called linearly continuous if for arbitrary (but fixed) $u, v \in X$ the function $G(u + \xi v)$ is continuous in ξ (i.e. in ξ for which $u + \xi v \in D$). The following result is due to M. Zorn [26] : Let F be a mapping defined on a linearly open set $D \subset X$ with values in Y. If F is linearly continuous and if there exists a Mazur-Orlicz set $P \subset X$ such that F is bounded on $D - P$, then F is bounded on D. Using this result we prove the following

Theorem 12. Let X, Y be linear normed spaces, X of the 2. category in itself, $F : X \to Y$ a mapping of X into Y. Suppose there exist an open subset $D \neq \emptyset$ of X, a linearly continuous mapping G from D into Y such that $\| F(u) \| \leq \| G(u) \|$ and

(18) $\| F(u - v) \| \leq K \max (\| F(u) \|, \| F(v) \|)$

for each $u, v \in D$, where K is some positive number. If there exists a Mazur-Orlicz set $P \subset X$ such that G is bounded on $D - P$, then F is bounded in some neighbourhood of $0 \in X$. Moreover, if F satisfies the condition (a) of Theorem 9, then F is bounded in X.

<u>Proof</u>. By Zorn's result G is bounded on D. Hence F is also bounded on D. Since D is a Baire set of the second category in X, the set $W = \{w : w = u - v ; u, v \in D\}$ is a neighbourhood of 0. Using (18) we see that F is bounded on W. The second assertion is obvious.

In next $B(u, r)$ will denote the open ball centered about point u and with radius $r > 0$. Using the properties of Baire sets and Baire functions one is able to prove the following

<u>Proposition 1</u>. Let X, Y be separable linear normed spaces, $F: X \to Y$ a mapping of X into Y, ε a positive number. Suppose that for every point $u \in X$ there exist $r^{(u)} > 0$ and a mapping $G^{(u)}$ defined on an open ball $B(u, r^{(u)})$ and with values in Y having the Baire property in $B(u, r^{(u)})$ such that $\| F(v) - G^{(u)}(v) \| < \varepsilon$ for each $v \in B(u, r^{(u)})$.

Then there exists a mapping $G : X \to Y$ of X into Y having the Baire property in X and $\| G(u) - F(u) \| < \varepsilon$ for every $u \in X$.

The last assertion is an extension of the well-known corresponding result [21,Th.16.6.1] which was proved for real function of the first Baire class (i.e. for function which is a point-limit of a sequence of continuous functions).

<u>Remarks</u>: Recall that for nonlinear operators the notions of boundedness and continuity are not equivalent

[4,chapt.I]. However, if F is uniformly continuous on the closed ball $D_R = \{u \in X: \|u\| \leqq R\}$, then F is bounded [4,p.30]. The connections between linear boundedness and boundedness of nonlinear operators have been studied by S. Yamamuro [27] (see also M. Šragin: Ref.Žurn. 1964,85 # 520). Boundedness of convex functionals was investigated in [28,Th.4], [3,corol.1]. For some results concerning the boundedness of nonlinear operators see [2,Th. 3,4]. Theorem 9' generalizes the result of Th. 4 [2]. The assumption (a) of Th. 3 [2] is redundant, thus read the Theorem 3 [2] as follows:

Let X, Y be linear normed spaces, X of the second category in itself, $F: X \to Y$ a mapping of X into Y such that the following conditions are fulfilled:

(1) $\|F(u+v)\| \leqq M \max(\|F(u)\|, \|F(v)\|)$

for every $u, v \in X$, where M is some positive constant;

(2) $u_n \in X,\ u \in X,\ u_n \to u \implies$
$\implies \|F(u)\| \leqq \overline{\lim_{n\to\infty}} \|F(u_n)\|$.

Then F is bounded in X.

Indeed, denote $X_n = \{u \in X : \|F(u)\| \leqq n\}$. Then X_n $(n = 1, 2, \ldots)$ are closed in X and $X = \bigcup_{n=1}^{\infty} X_n$.

By Baire category theorem at least one of X_n, say X_{n_0}, must contain a closed ball $D(u_0, r) = \{u \in X : \|u - u_0\| \leqq r\}$. Then for $v \in X$ with $\|v\| \leqq r$ we have $v + u_0 \in D(u_0, r)$ and

$\|F(v)\| = \|F((v+u_0)-u_0)\| \leqq M \max(\|F(v+u_0)\|,$

$$\| F(-u_0) \| \leqq M \max (n_0, \| F(-u_0) \|) .$$

Thus F is bounded on the closed ball centered about origin and with radius $r > 0$. This fact and the condition (1) imply that F is bounded in X (see the end of the proof of Th.9').

Some results concerning the uniform boundedness principle for nonlinear operators and related topics will be published later.

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