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ON THE CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF CONVEX FUNCTIONALS

Josef DANEŠ, Josef KOLOMÝ, Praha

Introduction. This paper is a continuation of our investigations [1,2] concerning the continuity and differentiability properties of nonlinear functionals (in particular convex functionals).

Section 2 concerns with the continuity and boundedness property of such functionals, while § 3 is devoted to the differentiability of convex functionals. Among others there is shown that a convex functional f defined on a linear normed space X possesses the Fréchet derivative $f'(x_0)$ at $x_0 \in X$ if and only if f is smooth and continuous at x_0. The case of continuous Fréchet derivative is also considered.

Theorems 2,3,5,6 contain some answers to an open question C) by M.Z. Nashed [3, p.75] concerning the Gâteaux and Fréchet differentiability of convex functionals. Some attention is also paid to study of critical points and the existence of the Gâteaux derivative of directionally smooth functionals. This paper concludes with some important examples of convex functionals and their properties. For the recent results in these topics see the bibliography cited in [1.2].

1. Notations and definitions. Let X be a real linear normed space, X^* its dual, E_n the euclidean n-space, $\langle x, e^* \rangle$ the pairing between $e^* \in X^*$ and $x \in X$. A functional f defined on a convex set $M \subseteq X$ is called convex (strictly convex) if

(1) $f(\lambda x + (1-\lambda)y) \leqq \lambda f(x) + (1-\lambda) f(y)$ for each $x, y \in M$ and $\lambda \in \langle 0, 1 \rangle$ (if the sign $<$ holds in (1) for each $x, y \in M$ and $\lambda \in (0,1)$). We shall use the symbols " $\longrightarrow$ ", " $\xrightarrow{w}$ " to denote the strong and weak convergence in X. A functional f is said to be weakly lower-semicontinuous at $x_0 \in X$ if $x_n \xrightarrow{w} x_0 \Longrightarrow f(x_0) \leqq \varliminf_{n \to \infty} f(x_n)$. We shall say that a functional f possesses the Gâteaux differential $Vf(x_0, h)$ at $x_0 \in X$ there exists

(2) $\lim_{t \to 0} \frac{1}{t} [f(x_0 + th) - f(x_0)] = Vf(x_0, h)$

for every $h \in X$. Thus $Vf(x_0, h)$ is in general nonlinear (and not continuous) mapping on X. If $Vf(x_0, h)$ is linear in $h \in X$, we denote this differential by $Df(x_0, h)$. A functional f is said to have the Gâteaux derivative $f'(x_0)$ at x_0 if there exists $Df(x_0, h)$ at x_0 and this mapping is bounded on X. The one-sided Gâteaux derivative $V_+ f(x_0, h)$ of f at $x_0 \in X$ is defined by (2) for $t \to 0_+$. By $df(x_0, h)$ we shall understand the Fréchet differential of f at $x_0 \in X$ (cf. [4], chapt. I). If $df(x_0, h)$ is continuous on X, then we shall say that f possesses the Fréchet derivative $f'(x_0)$ at x_0. Through this paper in theorems and propositions we shall assume that functionals f, $V_+ f(x_0, h)$ are finite.

2. Continuity properties of convex functionals

Theorem 1. Let X be a linear normed space, f a convex functional on X. Suppose there exists a constant $M > 0$ such that $f(x_0+h)+f(x_0-h)-2f(x_0) \leqq M\|h\|$ for each $h \in X$ with $\|h\| = R > 0$, where x_0 is a fixed element of X. If either a) $V_+ f(x_0, h)$ is upper bounded on some open convex subset $B \neq \emptyset$ of X (in particular $V_+ f(x_0, h)$ is upper-semicontinuous at some $h = h_0 \in \in X$) or b) X is complete and $V_+ f(x_0, h)$ is lower-semicontinuous on X, or c) f is continuous at x_0, then

(3) $$|f(x_0+h) - f(x_0)| \leqq N\|h\|$$

for each $h \in X$ with $\|h\| \leqq R$, where the constant $N > 0$ does not depend on h.

Proof. Since f is convex, the one-sided Gâteaux differential $V_+ f(x_0, h)$ exists, is positive homogeneous and subadditive in $h \in X$ [5]. Hence $V_+ f(x_0, h)$ is convex on X. Assuming a) (b)) and using theorem 2 [6,II, § 5] (the Gelfand lemma[7,chapt.I]) we see that $V_+ f(x_0, h)$ is continuous (bounded) on X. But continuity of this mapping at h = 0 implies the boundedness of $V_+ f(x_0, h)$ in some neighborhood of 0. From the positive homogeneity of $V_+ f(x_0, h)$ it follows that there exists a constant $C > 0$ such that

(4) $$|V_+ f(x_0, h)| \leqq C\|h\| .$$

Assume c), then (4) is satisfied by Theorem 8a) [1]. By lemma 2 [1] we have that

(5) $$-C\|h\| \leqq V_+ f(x_0, h) \leqq f(x_0+h) - f(x_0) .$$

On the other hand, let h be an arbitrary element of X with $\|h\| < R$. Then $f(x_0+h)+f(x_0-h)-2f(x_0) =$
$= f(x_0 + \frac{\|h\|}{R}\frac{Rh}{\|h\|}) + f(x_0 - \frac{\|h\|}{R}\frac{Rh}{\|h\|}) - 2f(x_0)$.

Employing convexity of f and aware that $\|h\| R^{-1} < 1$, $\left\| \frac{h}{\|h\|} R \right\| = R$ we obtain

$$f(x_0 + h) + f(x_0 - h) - 2f(x_0) \leqq (1 - \|h\| R^{-1}) f(x_0) + \|h\| R^{-1} f(x_0 + \frac{Rh}{\|h\|}) + (1 - \|h\| R^{-1}) f(x_0) + \|h\| R^{-1} f(x_0 - \frac{Rh}{\|h\|}) - 2f(x_0) =$$

$$= \|h\| \cdot R^{-1} [f(x_0 + Rh\|h\|^{-1}) + f(x_0 - Rh\|h\|^{-1}) - 2f(x_0)] \leqq \frac{\|h\|}{R} \cdot M \cdot \left\| \frac{Rh}{\|h\|} \right\| = M \|h\|$$

by our assumption. Hence for each $h \in X$ with $\|h\| \leqq R$ there is

(6) $$f(x_0 + h) + f(x_0 - h) - 2f(x_0) \leqq M \|h\| .$$

According to lemma 2 [1] and (6),(4),

$$\begin{aligned} f(x_0 + h) - f(x_0) &= f(x_0 + h) + f(x_0 - h) - 2f(x_0) \\ &\quad + f(x_0) - f(x_0 - h) \leqq \\ &\leqq M\|h\| + V_+ f(x_0, h) \leqq \\ &\leqq (M + C)\|h\| = N \|h\| \end{aligned}$$

for each $h \in X$ with $\|h\| \leqq R$, where $N = M + C$. This inequality together with (5) give (3). This completes the proof.

Corollary 1. Let X be a linear normed space, f a convex functional on X. Suppose there exists a constant $M > 0$ such that $f(h) + f(-h) - 2f(0) \leqq M\|h\|$ for each $h \in X$ with $\|h\| = R$. Furthermore, let one of the following three conditions be fulfilled: a) $V_+ f(0, h)$ is upper bounded on some open convex subset $B \neq \emptyset$ of X (in particular, $V_+ f(0, h)$ is upper-semicontinuous at some $h = h_0 \in X$); b) X is complete and $V_+ f(0, h)$ is lower-semicontinuous on X ; c) f is lower-semicontinuous at 0 .

Then f is bounded on the closed ball $D_R (\| h \| \leq R)$.

Proof. This assertion follows at once from Theorem 1 and Theorem 4 [2].

3. Differentiability of convex functionals. The following assertion is true [cf. [1], Th.5 see also Correction]: If X is a linear normed space, Y a Banach space, $F: X \to Y$ a demicontinuous mapping of X into Y, then the set Z of all $x \in X$ where the Gâteaux differential $VF(x, h)$ exists for any (but fixed) $h \in X$ is a $F_{\sigma\delta}$-set. In particular, if f is convex continuous functional on X, then the set Z of all $x \in X$ where the linear Gâteaux differential $Df(x, h)$ exists for any (but fixed) $h \in X$ is a $F_{\sigma\delta}$-set. Moreover, the following result has been established in [2]: Let X be a separable linear normed space, f a convex finite functional on X . Suppose that there exists an open convex subset $U \neq \emptyset$ of X such that f is upper bounded on U (in particular, assume that f is upper-semicontinuous at some point $x_0 \in X$). Then the set Z of all $x \in X$ where the Gâteaux derivative $f'(x)$ of f exists is a G_δ-set.

Now we prove the following

Theorem 2. Suppose X is a linear separable normed space, f a convex continuous functional on X . Then the set Z of all $x \in X$ where the Gâteaux derivative $f'(x)$ of f exists is a $F_{\sigma\delta}$-set.

__Proof.__ According to lemma 2 [1] we have that

$$f(x)-f(x-h) \leqq V_+ f(x,h) \leqq f(x+h)-f(x)$$

for each x and $h \in X$. Since f is continuous on X, $V_+ f(x,h)$ is continuous at $h = 0$ for each $x \in X$. Being $V_+ f(x,h)$ subadditive in $h \in X$ and $V_+ f(x,0) = 0$ for each $x \in X$, $V_+ f(x,h)$ is continuous in $h \in X$ for every $x \in X$. Let $h_1, h_2, \dots$ be a countable and dense subset of X and denote by Z_n the set of all $x \in X$ where the Gâteaux differential $Vf(x,h_n)$ exists for fixed h_n $(n = 1, 2, \dots)$; i.e.

$$Z_n = \{x \in X / V_+ f(x,h_n) = -V_+ f(x,-h_n)\} .$$

According to above mentioned theorem [1,Th.5,], Z_n is a $F_{\sigma\delta}$-set for each n $(n = 1, 2, \dots)$. As $x_0 \in Z \Longleftrightarrow$ $\Longleftrightarrow V_+ f(x_0,h_n) = -V_+ f(x_0,-h_n)$ in view of continuity of $V_+ f(x_0,h)$ in $h \in X$ and separability of X, we have that $x_0 \in Z_n$ $(n = 1, 2, \dots)$ and $Z = \bigcap_{n=1}^{\infty} Z_n$ (cf.[8]). Since Z is an intersection of $F_{\sigma\delta}$-sets Z_n $(n = 1, 2, \dots)$, Z is also a $F_{\sigma\delta}$-set. By Proposition 6 [2] for each $x \in Z$ we have $Vf(x,h) = f'(x)h$, where $f'(x)$ denotes the Gâteaux derivative of f at x. This concludes the proof.

__Corollary 2.__ Suppose the assumptions of Theorem 2 are satisfied. Then the set P of all $x \in X$ where the Gâteaux derivative of f does not exist is a $G_{\delta\sigma}$-set.

Similarly as previously we can improve the result of Theorem 6 [1] as follows:

__Theorem 3.__ Suppose X is a separable Banach space, f a convex Lipschitzian functional on X. Then the set Z of all $x \in X$ where the Gâteaux derivative $f'(x)$ of f exists is a $F_{\sigma\delta}$-set of the second category and hence it contains a G_δ-set which is dense in X .

Remark 1. For each convex functional f we have that

(7) $\lim_{t \to 0_+} [f(x_0+th)+f(x_0-th)-2f(x_0)] = 0$,

where x_0, h are arbitrary (but fixed) elements of the convex open domain of f . In fact from convexity of f $(0 \leqq t \leqq 1)$ we obtain at once that

$$0 \leqq f(x_0+th)+f(x_0-th)-2f(x_0) \leqq$$

$$\leqq (1-t)f(x_0)+tf(x_0+h)+(1-t)f(x_0)+$$

$$+tf(x_0-h)-2f(x_0) =$$

$$= t[f(x_0+h)+f(x_0-h)-2f(x_0)] .$$

As $t \to 0_+$, we have (7). To investigation of convex functionals we introduce the following

Definition 1. Let X,Y be linear normed spaces. A mapping $F: X \to Y$ is said to be directionally smooth at $x_0 \in X$ if for each (fixed) $h \in X$

(8) $\lim_{t \to 0} \| \frac{1}{t}[F(x_0+th)+F(x_0-th)-2F(x_0)] \| = 0$.

We shall say that $F: X \to Y$ is uniformly directionally smooth at $x_0 \in X$ with respect to $h \in X$ with $\|h\| = 1$ if (8) holds uniformly with respect to $h \in X$, $\|h\| = 1$.

Remark 2. Smooth functions in E_1; i.e. functions which satisfy $f(x_0+h)+f(x_0-h)-2f(x_0) = o(h)$ have been introduced by Riemann and have been largely studied by A.Zygmund [9],[10] in connection with trigonometric series.

Analogously, a mapping $F: X \to Y$ is said to be smooth at $x_0 \in X$ if

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \| F(x_0+h)+F(x_0-h)-2F(x_0) \| = 0 .$$

It is easily seen that if $F: X \to Y$ possesses the Gâteaux differential $VF(x_0, h)$ (the Fréchet differential $dF(x_0, h)$) at $x_0 \in X$, then F is directionally smooth (smooth) at x_0. The converse is not true in general.

<u>Proposition 1</u>. Let f be a convex functional defined on a convex open subset M of X (X is a linear normed space). Suppose that f is directionally smooth at $x_0 \in M$. Then f possesses a linear Gâteaux differential $Df(x_0, h)$ at x_0. Moreover, if f is continuous at x_0, then $Df(x_0, h) = f'(x_0)h$, where $f'(x_0)$ denotes the Gâteaux derivative of f at x_0.

<u>Proof</u>. Since

$$\frac{1}{t}[f(x_0+th)+f(x_0-th)-2f(x_0)] \to 0$$

whenever $t \to 0$, we have

$$(9) \quad \frac{1}{t}[f(x_0+th)-f(x_0)]-\frac{1}{t}[f(x_0)-f(x_0-th)] \to 0$$

as $t \to 0_+$. In view of convexity of f, the one-sided Gâteaux differential $V_+f(x, h)$ exists at $x_0 \in M$ for $h \in X$. From (9) it follows that

$$V_+f(x_0, h) = -V_+f(x_0, -h) .$$

Hence f possesses the Gâteaux differential $Vf(x_0, h)$ at x_0. Convexity of f implies that $Vf(x_0, h) = Df(x_0, h)$. Suppose f is continuous at x_0. By lemma 2 [1] we see that $Df(x_0, h)$ is continuous at h = 0 and hence bounded in $h \in X$. This completes the proof.

Corollary 3. Let X be a linear normed space, f a convex functional defined on a convex open subset M of X . Assume f is directionally smooth and continuous at $x_0 \in M$. If x_0 is an extremal point of f , then x_0 is critical point of f , i.e. $f'(x_0) = 0$.

Let E be a topological vector space over the real numbers E_1 with dual E^* and suppose that f is a proper convex functional on E , i.e. f is an everywhere-defined functional with values in $(-\infty, +\infty)$ not identically $+\infty$. A subgradient [11,12] of f at $x \in E$ is an $x^* \in E^*$ such that

$$f(y) \geq f(x) + \langle y - x, x^* \rangle$$

for all $y \in E$, where $\langle y-x, x^* \rangle$ denotes the value of x^* at the point y - x . Denote by $\partial f(x)$ the set of all subgradients of f at x . If $\partial f(x) \neq \emptyset$, f is said to be subdifferentiable at x (see the above cited papers). Thus ∂f is a multivalued mapping from E to E^* assigning to each $x \in E$ all its subgradients.

Recently J.J.Moreau [11,13] has proved the following assertion: If f is convex, finite continuous at $x_0 \in E$ and the subdifferential $\partial f(x_0)$ of f at x_0 consists of a single point, then f possesses the Gâteaux differential $Vf(x_0, h)$ at x_0 . The subdifferential $\partial f(x_0)$ of f at x_0 consists of a single point if f is strictly convex.

It is easy to construct the functionals which are directionally smooth at x_0 and being not strictly convex.

The existence of subgradients has been investigated by G.J.Minty [12], J.J.Moreau [14], A.Brøndsted, R.T.Rockaffelar [15]. It is known [12],[14] that a convex functional f is subdifferentiable whenever it is finite and continuous. If E is a Banach space and f is lower-semicontinuous on E , then the set of points where f is subdifferentiable is dense in the effective domain of f (which is the convex set of all $x \in E$ such that $f(x) < +\infty$) [15]. The relation between convex subdifferentiable functionals and supportably convex ones has been obtained by M.Z.Nashed [16].

Proposition 2. Let X be a linear normed space, f a convex functional on X such that either a) $V_+ f(x_0, h)$ is upper bounded on some convex open subset $D \neq \emptyset$ of X (in particular, $V_+ f(x_0, h)$ is upper-semicontinuous at some $h = h_0 \in X$), where x_0 is a fixed element of X, or b) X is complete, $V_+ f(x_0, h)$ is lower-semicontinuous on X . Then f is subdifferentiable at x_0 .Moreover, if f is strictly convex, then f possesses the linear Gâteaux differential $Df(x_0, h)$.

Proof. Assuming a) or b) $V_+ f(x_0, h)$ is continuous on X (cf. the proof of Th.1). According to lemma 2 [1]

(10) $-V_+ f(x_0, -h) \leqq V_+ f(x_0, h) \leqq f(x_0 + h) - f(x_0)$

for each $h \in X$. Since $V_+ f(x_0, h)$ is continuous subadditive and positive homogeneous using the consequence of the Hahn-Banach Theorem [17,Th.1′,§ 6,chapt.IV] we have that there exists an element $x^* \in X^*$ such that

(11) $-V_+ f(x_0, -h) \leqq \langle h, x^* \rangle \leqq V_+ f(x_0, h)$

for every $h \in X$. Hence the inequalities (10),(11) imply

$f(x_0 + h) - f(x_0) \geqq \langle h, x^* \rangle$ for each $h \in X$,

i.e. f is subdifferentiable at x_0 . This result together with the above mentioned Moreau's one give at once the second assertion of our proposition.

Remark 3. The assertion that a convex functional f continuous at $x_0 \in X$ is subdiferentiable at x_0 we obtaion also at once as follows. By lemma 2 [1]

$$f(x_0) - f(x_0 - h) \leq V_+ f(x_0, h) \leq f(x_0 + h) - f(x_0)$$

for every $h \in X$. In view of continuity of f at x_0 it follows that $V_+ f(x_0, h)$ is continuous at h = 0. $V_+ f(x_0, h)$ being subadditive and positive homogeneous, $V_+ f(x_0, h)$ is convex and by Th.[6,chapt.II,§ 5] it is continuous on X. Now we proceed as above.

If the conditions a) or b) of Proposition are satisfied at each point $x \in X$, then f is subdifferentiable everywhere on X.

Proposition 3. Let X be a Banach space, f a continuous functional on X. Suppose that f is directionally smooth at some point $x_0 \in X$ and that x_0 is an extremal point of f. If $\nabla^2_{th_1, th_2} f(x_0) = O(t)$ for arbitrary $h_1, h_2 \in X$, where $\nabla^2_{h_1,h_2} f(x_0) = f(x_0 + h_1 + h_2) + f(x_0 - h_1) + f(x_0 - h_2) - 3f(x_0)$, then $f'(x_0) = 0$, i.e. x_0 is a critical point of f.

Proof. Since f is directionally smooth at x_0 and x_0 is an extremal point of f, the Gâteaux differential $Vf(x_0, h)$ of f exists at x_0. For arbitrary $h \in X$, $h_1, h_2 \in X$ we set

$$\varphi(x_0, t, h) = f(x_0 + th) + f(x_0 - th) - 2f(x_0),$$

$$\Delta^2_{h_1,h_2} f(x_0) = f(x_0 + h_1 + h_2) - f(x_0 + h_1) - f(x_0 + h_2) + f(x_0).$$

Then

$\Delta^2_{th_1,th_2} f(x_0) = \nabla^2_{th_1,th_2} f(x_0) - \varphi(x_0,t,h_1) - \varphi(x_0,t,h_2)$.

As f is directionally smooth at x_0 ,we have that $\varphi(x_0,t,h_i) = o(t)$, $(i = 1,2)$ and $\nabla^2_{th_1,th_2} f(x_0) = O(t)$ by our assumption. Hence $\Delta^2_{th_1,th_2} f(x_0) = O(t)$ which implies that $Vf(x_0,h) = Df(x_0,h)$ [cf. 4 , chapt.I,§ 3]. Since X is complete and f is continuous on X , using the Baire's theorems, we have that $Df(x_0,h) = f'(x_0)h$, where $f'(x_0)$ denotes the Gâteaux derivative of f at x_0 . As x_0 is an extremal point of f , $f'(x_0) = 0$ which concludes the proof.

Theorem 4. Let X be a linear normed space, f a convex functional on X. Suppose there exists the Gâteaux differential $Vf(x,h)$ of f at $x_0 \in X$. Let one of the following three conditions be fulfilled: a/ f is continuous at x_0 ; b/ $V_+ f(x_0,h)$ is upper bounded on some open convex subset $M \neq \emptyset$ of X ; c/ X is complete and $V_+ f(x_0,h)$ is lower-semicontinuous on X.

Then f possesses the Gâteaux derivative $f'(x_0)$ at x_0 .

Proof. The case a/ is the assertion of Proposition 6 [2]. Assuming b/,c/ we have that $V_+ f(x_0,h)$ is bounded on X . By convexity of f

$$Vf(x_0,h) = V_+ f(x_0,h) = Df(x_0,h) .$$

Hence $Df(x_0,h) = f'(x_0)h$ which concludes the proof.

Remark 4. Theorem 4 and Proposition 1 imply the validity of the following assertion: Let f be a convex functional on X directionally smooth at $x_0 \in X$. Assume that one of the three conditions a/,b/,c/ of Th.4 is satisfied. Then f possesses the Gâteaux derivative $f'(x_0)$ at x_0.

Now we shall study the Fréchet differentiability of convex functionals in linear normed spaces. Some general theorems concerning the Gâteaux and Fréchet differentiability of operators have been obtained in [18,19,20].

Theorem 5. Let X be a linear normed space, f a convex functional on X. Then f possesses the Fréchet derivative $f'(x_0)$ at $x_0 \in X$ if and only if f is smooth and continuous at x_0.

Proof. The first part of our Theorem is obvious. Suppose f is smooth and continuous at x_0. Then f is directionally smooth at x_0 and hence by Proposition 1 there exists the Gâteaux differential $\nabla f(x_0, h)$. Using Proposition 6 [2] we see that $\nabla f(x_0, h) = f'(x_0)h$, where $f'(x_0)$ denotes the Gâteaux derivative of f at x_0.

Set

12 $$u(x_0, h) = f(x_0 + h) + f(x_0 - h) - 2f(x_0).$$

Since f is convex, for an arbitrary $h \in X$ we have $u(x_0, h) \geq 0$ and

$$f(x_0 + h) - f(x_0) - f'(x_0)h = \omega(x_0, h) \geq 0,$$

$$f(x_0 - h) - f(x_0) + f'(x_0)h = \omega(x_0, -h) \geq 0.$$

Hence $(h \neq 0)$

$$0 \leq \|h\|^{-1}\omega(x_0, h) \leq \|h\|^{-1}(\omega(x_0, h) + \omega(x_0, -h)) =$$

$$= \|h\|^{-1} u(x_0, h) .$$

Being f smooth at x_0,

$$\|h\|^{-1}\omega(x_0, h) \longrightarrow 0$$

as $\|h\| \to 0$. Thus f has the Fréchet derivative $f'(x_0)$ at x_0 and this concludes the proof.

Theorem 5. Let X be a linear normed space / a Banach space /, f a convex functional on X. Then f possesses the Fréchet derivative $f'(x_0)$ at $x_0 \in X \Longleftrightarrow f$ is smooth at x_0 and $V_+ f(x_0, h)$ is upper bounded on some open convex subset $M \neq \emptyset$ of X / $V_+ f(x_0, h)$ is lower-semicontinuous on X /.

Let X, Y be linear normed spaces, $f : X \to Y$ a mapping of X into Y. For $x_0 \in X$, $B(x_0, r)$ will denote the open spherical ball centered at x_0 with radius r and $u(x_0, h)$ the expression given by (12).

A mapping f is said to be locally uniformly smooth on an open subset $M \subset X$ if for each $\varepsilon > 0$ and an arbitrary $x_0 \in M$ there exist positive numbers $\delta(x_0, \varepsilon)$ and $r(x_0, \varepsilon)$ such that

(13) $$\|u(x, h)\| < \varepsilon \|h\|$$

if $0 < \|h\| < \delta$ and $x \in B(x_0, r) \cap M$.

Similarly, f is said to be uniformly smooth on M if for any positive number $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \| h \| < \delta$, then (13) holds for each $x \in M$.

A mapping f is said to be locally uniformly differentiable / uniformly differentiable/ [4, chapt.I.] on M if f has the Fréchet derivative $f'(x)$ everywhere on M and the remainder

$$\omega(x, h) = f(x+h) - f(x) - f'(x)h$$

is locally uniformly bounded / uniformly bounded/: i.e. for each $\varepsilon > 0$ and an arbitrary $x_0 \in M$ there exist $\delta(x_0, \varepsilon) > 0$, $r(x_0, \varepsilon) > 0$ such that

$$\| \omega(x,h) \| < \varepsilon \| h \| \tag{14}$$

if $0 < \| h \| < \delta$ and $x \in B(x_0, r) \cap M$ / i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \| h \| < \delta$, then (14) holds for each $x \in M$ /.

The following theorem explains the connections between the above notions.

Theorem 6. Let X be a linear normed space, f a convex functional on X, $M \subset X$ an open subset of X. Then f is locally uniformly / uniformly / differentiable on M if and only if f is continuous and locally uniformly / uniformly / smooth on M.

Theorems 2,3,5,6 contain some answers to an open problem C/ by M.Z. Nashed [3, p.75] concerning the

Gateaux and Fréchet differentiability of convex functionals.

Analysing the proof of Theorems 4.1, 4.2 [4] and using the above result we obtain the following

Theorem 7. Let X be a linear normed space, f a convex functional on X. Then f has a continuous Fréchet derivative $f'(x)$ on the open ball $B_R (\|x\| < R)$ if and only if f is locally uniformly smooth and locally uniformly continuous on B_R.

Theorem 8. Let X be a linear normed space, f a convex functional on X. Suppose f is uniformly continuous on the open ball $B_{R+\alpha}$. Then f possesses an uniformly continuous Fréchet derivative $f'(x)$ on $B_R \Longleftrightarrow f$ is uniformly smooth on B_R $(R > 0,\ \alpha > 0)$.

Remark 5. One may obtain analogous necessary and sufficient conditions for continuity /uniform continuity/ of the Gateaux differential $D f(x, h)$ in variable x under an arbitrary /but fixed/ direction $h \in X$ of a convex functional f using the similar notions of locall uniform/uniform/ directionall smoothness and the results in [18,pp.324-328]. We leave the discussion of these facts to the reader. In Theorems 7,8 one may replace the open ball B_R by an open convex bounded subset D of X.

Remark 6. Let f be a convex functional on X. Suppose that f is smooth at $x_0 \in X$. If x_0 is an extremal point of f, then $d f(x_0, h) = 0$ for every $h \in X$. If a functional f

/ not necessary convex / defined on X is smooth at an extremal point $x_0 \in X$ and $\nabla^2_{th_1, th_2} f(x_0) = o(t)$ for arbitrary h_1, $h_2 \in X$, where

$$\nabla^2_{th_1, th_2} f(x_0) = f(x_0 + h_1 + h_2) + f(x_0 - h_1) + f(x_0 - h_2) - 3f(x_0),$$

then $d f(x_0, h) = 0$ for every $h \in X$.

Remark 7. A mapping $f : X \to Y$ is smooth at $x_0 \in X \iff$ for each two sequences $\{h_n\} \in X$ with $\| h_n \| = 1$ / $n = 1,2,\ldots$ / and $\{t_n\}$ of positive numbers t_n /$n = 1,2,\ldots$/ with $\lim_{n \to \infty} t_n = 0$ there is

(15) $$\lim_{n \to \infty} t_n^{-1} \| u(x_0, t_n h_n) \| = 0.$$

Indeed, suppose f is smooth at $x_0 \in X$ and $\{t_n\}$, $\{h_n\}$ be arbitrary sequences with the above properties. Then $t_n \| h_n \| = t_n \to 0$ as $n \to \infty$ and the condition of smoothness of f at x_0 implies at once (15). Conversely, assume (15) is satisfied and f is not smooth at x_0. Then there exist $\varepsilon_0 > 0$ and the sequence $\{\bar{h}_n\} \in X$ such that $\| \bar{h}_n \| < n^{-1}$ and

(16) $$\| \bar{h}_n \|^{-1} \| u(x_0, \bar{h}_n) \| > \varepsilon_0 .$$

Set

$$h_n = \bar{h}_n \| \bar{h}_n \|^{-1}, \quad t_n = \| \bar{h}_n \| .$$

Then $\bar{h}_n = t_n h_n$ with $t_n \to 0$ as $n \to \infty$ and $\| h_n \| = 1$ / $n = 1,2,\ldots$ /. Hence by (16) we have that

$$t_n^{-1} \| u(x_0, t_n h_n) \| > \varepsilon_0$$

which contradicts to our hypothesis.

Another equivalent condition of smoothness is the following:

a mapping f is smooth at $x_0 \in X \Longleftrightarrow$ the limit

$$\lim_{t \to 0} \| t^{-1} u(x_0, th) \| = 0$$

is uniform with respect to $h \in X$, $\| h \| = 1$.

3. <u>Examples of convex functionals</u>.

A. Let $g(u,x)$ ($u \in (-\infty, +\infty)$, $x \in G$, here G denotes a measurable subset of an euclidean n-space E_n) be a N-function [4,chapt.VI.], $h(u) = g(u(x),x): L_p \to L_q$ ($p^{-1} + q^{-1} = 1$) an operator of Nemyckij from L_p into L_q. Suppose $g(u,x)$ is monotone function in $u \in (-\infty, +\infty)$ for almost all $x \in G$. Then the functional

$$(17) \qquad f(u) = \int_G dx \int_0^{u(x)} g(v,x)\, dv$$

is convex, continuous, weakly lower-semicontinuous, bounded on L_p and Lipschitzian on each bounded closed ball D_R ($\|x\| \leqq R$) of L_p. Moreover, $f(u)$ satisfies the condition of smoothness at every $u \in L_p$.

<u>Proof</u>. Since $h(u)$ is the Fréchet gradient [21] of $f(u)$, f is continuous on L_p. In view of monotonicity of $g(u,x)$ in $u \in (-\infty, +\infty)$ for almost all $x \in G$, $h(u)$ is a monotone operator on L_p and hence f is convex [22]. Then for each real constant c the set $E(c) = \{ x \in X \mid f(x) \leqq c \}$ is convex closed set and hence it is weakly closed. By Proposition 1 [1] f is weakly lower-semicontinuous on L_p. According to mean-value theorem

$$|f(u)| = |f(u) - f(0)| = |\langle f'(\theta u), u \rangle| = |\langle h(\theta u), u \rangle| \leqq$$

$$\leqq \| h(\theta u) \|_{L_q} \cdot \| u \|_{L_p}, \quad \theta \in (0,1).$$

Since $h: L_p \to L_q$, h is bounded and continuous [21, I.]. The boundedness of f on L_p follows at once from this fact and the above inequality. Again, using the mean-value theorem and employing boundedness of h, we see that f is Lipschitzian on each closed bounded ball D_R of L_p. Since f is Fréchet-differentiable on L_p, f satisfies the condition of smoothness at every $u \in L_p$.

Remark. The fact that the functional $f(u)$ defined by (17) is weakly lower-semicontinuous has been firstly observed by M.M.Vajnberg [23]. But his proof depends on another arguments.

B. Under the assumptions of the example A suppose that K is a linear continuous operator from L_q into L_p which admits a splitting $K = AA^*$, where $A: L_2 \to L_p$ is linear and continuous (so that $A^*: L_q \to L_2$). Furthermore, assume h is such that

$$(h\varphi_1 - h\varphi_2, \varphi_1 - \varphi_2) \leqq \|\varphi_1 - \varphi_2\|^2$$

for every $\varphi_1, \varphi_2 \in L_p$. Then the functional

(18) $$\phi(u) = \frac{1}{2}\|u\|^2 - f(Au),$$

where f is defined by (17), is convex, continuous, weakly lower-semicontinuous, bounded on L_2 and Lipschitzian on each closed bounded ball D_R ($\|u\| \leqq R$) of L_2.

Indeed, $\phi(u)$ has the Gâteaux derivative $\phi'(u)$ on L_2 and

$$\phi'(u) = u - A^*h(Au)$$

Since

$$\langle \phi'(u_1) - \phi'(u_2), u_1 - u_2 \rangle = \|u_1 - u_2\|^2 - \langle A^*h(Au_1) - A^*hAu_2, u_1 - u_2 \rangle = \|u_1 - u_2\|^2 - \langle h(Au_1) - h(Au_2), Au_1 - Au_2 \rangle$$

for every $u_1, u_2 \in L_2$, using our hypothesis

$(Au_1, Au_2 \in L_p)$ we have that

$$\langle \Phi'(u_1) - \Phi'(u_2), u_1 - u_2 \rangle \geq 0 .$$

Thus Φ is convex on L_2. In view of continuity of A on L_2 and f on L_p, Φ is continuous and hence weakly lower-semicontinuous on L_2. The boundedness of Φ follows at once from the boundedness of f and A. The property that Φ is Lipschitzian on each closed bounded ball D_R of L_2 is obvious.

The functionals defined by (17) and (18) play an important rôle in variational methods of solutions of nonlinear equations.

Suppose the assumptions of the example B are fulfilled. Consider the equation

(19) $$\varphi - K h \varphi = 0$$

in the space L_p. Then this equation investigated in L_p is equivalent to the one

(20) $$u - A^* h(Au) = 0$$

in L_2 in the following sense: If u_0 is a solution of (20) in L_2, then $\varphi_0 = Au_0$ is a solution of (19) in L_p. Conversely: if φ_0 is a solution of (19) in L_p, then $u_0 = A^* h(\varphi_0)$ is a solution of (20) in L_2. For solving the equation (20) it is sufficient to assume for instance that the functional $\Phi(u)$ is such that $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$. Then Φ has at least one critical point in L_2 and thus the equation (20) has at least one solution $u_0 \in L_2$. Hence $\varphi_0 = Au_0$ is a solution of (19).

References

[1] J. KOLOMÝ: On the differentiability of mappings and convex functionals.Comment.Math.Univ.Carolinae 8(1967),735-752.A correction,this Journal 9 (1968),197.

[2] J. KOLOMÝ: Remarks on nonlinear functionals.Comment.Math. Univ.Carolinae 9(1969),145-155.

[3] M.Z. NASHED: Some remarks on variations and differentials. Am.Math.Montly Vol.73 1966,No 4,63-76.

[4] M.M. VAJNBERG: Variacionnyje metody issledovanija nelinejnych operatorov.Moskva 1956.

[5] F.A. VALENTINE: Convex sets.New York 1964.

[6] N. BURBAKI: Topologičeskije vektornyje prostranstva.Moskva 1959.

[7] N.I. ACHIEZER, I.M. GLAZMAN: Teorija linejnych operatorov v Gil´bertovom prostranstve.Moskva 1966.

[8] S. MAZUR: Über konvexe Mengen in linearen normierten Räumen.Studia Math.IV(1933),70-84.

[9] A. ZYGMUND: Trigonometric series I.Cambridge 1959.

[10] A. ZYGMUND: Smooth functions.Duke Math.J.12(1945),47-76.

[11] J.J. MOREAU: Sur la fonction polaire d´une fonction semicontinue supérieurement.C.R.Acad.Sci.Paris, 258(1964),1128-1131.

[12] G.J. MINTY: On the monotonicity of the gradient of a convex function.Pac.J.of Math.14(1964),243-247.

[13] J.J. MOREAU: Sous-differentiabilité.Proc.Colloq.Convexity,Copenhagen 1965,Copenhagen 1967,185-201.

[14] J.J. MOREAU: Fonctionelles sous-differentiables.C.R.Acad. Sci.257(1963),4117-4119.

[15] A.BRØNDSTED, R.T. ROCKAFFELAR: On the subdifferentiability of convex functions.Bull.Amer.Math.Soc. 16(1965),605-611.

[16] M.Z. NASHED: Supportably and weakly convex functions with application to approximation theory and nonlinear programming.Journal Math.Anal.and Appl.18(1967),No 3,504-521.

[17] K. YOSIDA: Functional Analysis.Springer 1965.

[18] J. KOLOMÝ: On the differentiability of mappings in functional spaces.Comment.Math.Univ.Carolinae 8 (1967),315-330.

[19] V. ZIZLER: On the differentiability of mappings in Banach spaces.Commet.Math.Univ.Carolinae 8(1967), 415-430.

[20] J. KOLOMÝ, V. ZIZLER: Remarks on the differentiability of mappings in linear normed spaces.Comment. Math.Univ.Carolinae 8(1967),691-704.

[21] M.A. KRASNOSELSKIJ: Topologičeskije metody v teorii nelinejnych integralnych uravnenij.Moskva 1956.

[22] R.I. KAČUROVSKIJ: O monotonnych operatorach i vypuklych funcionalach.Uspechi mat.nauk 15(1960),No 4, 213-215.
Učen.zap.Moskov.obl.ped.inst.N.K.Krupskoj,t. 110(1962),vyp.7,231-243.

[23] M.M. VAJNBERG: Novyje teoremy dlja nelinejnych operatorov i uravnenij.Uč.zap.Moskov.obl.ped.inst.im. N.K.Krupskoj t.77,vyp.5,131-143.

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