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LYAPUNOV'S DIRECT METHOD IN ABSTRACT LOCAL SEMI-FLOWS

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In [1] a very interesting idea of stability and boundedness analysis in differential equations theory is described. In this paper this idea will be used in a little more abstract form to study several properties of abstract local semi-flows in an abstract set.
(The notion of an abstract local semi-flow was introduced by O. Hájek in his seminar at the Mathematical Institute of the Caroline University at 1965, see also [2].)

First several notions and notations will be introduced.

1. <u>Notation</u> Throughout the paper, P will denote an arbitrary abstract set, R the one-dimensional Euclidean space, R^+ its subspace $\langle 0, +\infty)$. In what follows, a map $g: P \times R \rightarrow R^+$ will be given. This map, about which we suppose only to be defined on the whole set $P \times R$, will play a very important role. (From the context it will be clear that nontrivial results may be obtained only in the case of g such that the set $\{(x, \theta) \in P \times R : g(x, \theta) = 0\}$ is nonvoid. The following two cases are of special interest: there are given a metric φ on $P \times R$, a nonvoid set $K \subset P \times R$, and a map g_1 such that $g_1(x, \theta) = \inf\{\varphi((x, \theta), (y, \xi)) : (y, \xi) \in K\}$, or, if $K_\theta = \{x \in P : (x, \theta) \in K\}$ is nonvoid for all $\theta \in R$, g_2 is such

that $g_2(x,\theta) = \inf\{\rho((x,\theta),(y,\theta)) : (y,\theta) \in K\}$. Clearly, the functions g_1 and g_2 are, in some sense, distances between a point and a given subset in $P \times R$. The set K and the function g_2 are very similar to the set M and the distance d in the Yoshizawa's study of the M-stability and M-boundedness [3].)

In this paper we shall occupy ourselves with partial maps $t : R \times P \times R \to P$, and we shall use the following notation: domain t will denote the set $\{(\theta, x, \alpha) \in R \times P \times R : t(\theta, x, \alpha)$ is defined$\}$. The value of the map t at a point $(\beta, x, \alpha) \in$ $\in$ domain t will be denoted by ${}_\beta t_\alpha x$. To every pair $\beta, \alpha \in R$, $\beta \geq \alpha$, there is assigned a partial map

$${}_\beta t_\alpha : P \to P : {}_\beta t_\alpha(x) = {}_\beta t_\alpha x .$$

If α, β, γ are reals, $\alpha \leq \beta \leq \gamma$, then ${}_\gamma t_\beta \circ {}_\beta t_\alpha$ denotes the composition of the maps ${}_\gamma t_\beta$ and ${}_\beta t_\alpha$. Finally, let D denote the set $\{(x,\alpha) \in P \times R : (\alpha, x, \alpha) \in \mathit{domain}\ t\}$, and define a map

$$\varepsilon : D \to R \cup \{+\infty\} : \varepsilon(x,\alpha) = \sup\{\theta \in R : (\theta, x, \alpha) \in \mathit{domain}\ t\}.$$

The notation just introduced will be used in the formulation of the following definition.

2. <u>Definition</u> A partial map $t : R \times P \times R \to P$ will be called an <u>abstract local semi-flow</u> on P iff it has the following properties:

(i) ${}_\alpha t_\alpha x = x$ holds for each $(x,\alpha) \in D$;

(ii) $\varepsilon(x,\alpha) > \alpha$ holds for each $(x,\alpha) \in D$;

(iii) ${}_\gamma t_\beta \circ {}_\beta t_\alpha = {}_\gamma t_\alpha$ holds whenever $\alpha \leq \beta \leq \gamma$ and at least one side of this equality is defined.

An abstract local semi-flow will be called global iff $\varepsilon(x, \alpha) = +\infty$ holds for each $(x, \alpha) \in D$.

3. Remark From 2(iii) there follows directly the following simple proposition: if $(\gamma, x, \alpha) \in domain\ t$ and $\gamma > \alpha$, then $(\theta, x, \alpha) \in domain\ t$ for each $\theta \in \langle \alpha, \gamma \rangle$. Hence and from 2(ii) we then obtain the following assertion: corresponding to each $(x, \alpha) \in D$, there exists $\beta > \alpha$ such that $(\theta, x, \alpha) \in domain\ t$ holds for each $\theta \in \langle \alpha, \beta \rangle$.

4. Definition A partial map $s : R \to P$ will be called a solution of an abstract local semi-flow t iff the following conditions are satisfied:

(i) domain s is a nondegenerate interval in R ;

(ii) $s(\beta) = {}_\beta t_\alpha\, s(\alpha)$ holds for each pair $\alpha, \beta \in domain\ s$, $\alpha \leq \beta$.

5. Conditions Let there be given an abstract global semi-flow t° on R^+, maps $V : P \times R \to R^+$, $\mu : R \to (\sigma, +\infty)$, $0 < \sigma \in R$, $\gamma : (\tau, +\infty) \to R$, $0 < \tau \in R$, $u : R^+ \to R^+$, $v : R^+ \to R^+$, such that $\limsup_{\theta \to +\infty} u(\theta) = +\infty$, $\liminf_{\theta \to 0_+} v(\theta) = 0$, u, v strictly increasing. Finally, t will denote an abstract local semi-flow on P. We shall formulate the following two conditions.

(i) $\mu(\theta) . V({}_\theta t_\alpha x, \theta) \leq {}_\theta t^\circ_\alpha\, r$ for each $r \in R^+$ and $(x, \alpha) \in D$ such that $\mu(\alpha) . V(x, \alpha) \leq r$, and each $\theta \in \langle \alpha, \varepsilon(x, \alpha))$; $u(q(x, \theta)) \leq V(x, \theta) \leq \gamma(\theta) . v(q(x, \theta))$ holds for each $(x, \theta) \in D$.

(ii) $V({}_\theta t_\alpha x, \theta) \leq {}_\theta t^\circ_\alpha\, r$ holds for each $r \in R^+$ and $(x, \alpha) \in D$ such that $V(x, \alpha) \leq r$, and each $\theta \in \langle \alpha, \varepsilon(x, \alpha))$; $u(q(x, \theta)) \leq V(x, \theta) \leq v(q(x, \theta))$

holds for each $(x, \theta) \in \mathcal{D}$.

6. Properties We shall say that the abstract local semi-flow t has one of properties 6(i) - 6(iv) iff the correspondingly numbered condition of following is satisfied:

(i) there is a positive function $\omega(\alpha, \xi)$ $(\alpha \in R,\ \xi > 0)$ such that $q({}_\theta t_\alpha x, \theta) < \xi$ holds whenever $(x, \alpha) \in \mathcal{D}$, $q(x, \alpha) \leq \omega(\alpha, \xi)$, $\theta \in \langle \alpha, \varepsilon(x, \alpha))$;

(ii) the function $\omega(\alpha, \xi)$ in (i) does not depend on α;

(iii) there is a positive function $\beta(\alpha, \omega)$ $(\alpha \in R,\ \omega > 0)$ such that $q({}_\theta t_\alpha x, \theta) < \beta(\alpha, \omega)$ holds whenever $(x, \alpha) \in \mathcal{D}$, $q(x, \alpha) \leq \omega$ and $\theta \in \langle \alpha, \varepsilon(x, \alpha))$;

(iv) the function $\beta(\alpha, \omega)$ in (iii) does not depend on α.

Similarly, we shall say that an abstract global semi-flow t has one of properties 6(v) - 6(viii) iff the correspondingly numbered condition of following is satisfied:

(v) there is a positive function $\Theta(\alpha, \eta, \omega)$ $(\alpha \in R,\ \eta > 0,\ \omega > 0)$ such that $q({}_\theta t_\alpha x, \theta) < \eta$ holds whenever

$$(x, \alpha) \in \mathcal{D},\ q(x, \alpha) \leq \omega,\ \theta \geq \alpha + \Theta(\alpha, \eta, \omega);$$

(vi) the function $\Theta(\alpha, \eta, \omega)$ in (v) does not depend on α;

(vii) there are positive functions $\Theta(\alpha, \omega)$ and $\sigma(\alpha)$ $(\alpha \in R,\ \omega > 0)$ such that $q({}_\theta t_\alpha x, \theta) < \sigma(\alpha)$ holds whenever $(x, \alpha) \in \mathcal{D}$, $q(x, \alpha) \leq \omega$, $\theta \geq \alpha + \Theta(\alpha, \omega)$;

(viii) the functions $\Theta(\alpha, \omega)$ and $\sigma(\alpha)$ in (vii) do not depend on α.

In the same way, for the abstract global semi-flow t° on

R^+ and for a special choice of a map $g: R^+ \times R \to R^+ : g(\kappa, \theta) = \kappa$, we shall say that t^o has property $6(i)^o$ iff

$(i)^o$ there is a positive function $\omega(\alpha, \xi)$ such that ${}_\theta t_\alpha \kappa < \xi$ for each $\kappa \leq \omega(\alpha, \xi)$ and $\theta \geq \alpha$;

and, analogously, for properties $6(ii)^o$ - $6(viii)^o$.

7. Note It is easily to see that properties (i),(iii),(v) and (vii) correspond to those of stability, equi-M-boundedness, quasi-equi-asymptotic stability and equi-ultimate-M-boundedness, as they are defined e.g. in [3], and the remaining properties are their corresponding θ-uniform modifications.

8. Theorem Let t be an abstract local semi-flow on P , t^o an abstract global semi-flow on R^+.

(i) Let condition 5(i) be satisfied. If t^o has property $6(i)^o$, then t has property 6(i).

(ii) Let condition 5(ii) be satisfied. If t^o has property $6(i)^o$ or $6(ii)^o$, then t has the corresponding property 6(i) or 6(ii).

(iii) Let condition 5(i) be satisfied. If t^o has property $6(iii)^o$, then t has property 6(iii).

(iv) Let condition 5(ii) be satisfied. If t^o has property $6(iii)^o$ or $6(iv)^o$, then t has the corresponding property 6(iii) or 6(iv).

Proof. Ad (i): According to the assumption, there is a positive function $\omega^o(\alpha, \xi^o)$ such that ${}_\theta t^o_\alpha \omega^o(\alpha, \xi^o) < \xi^o$ for each $\theta \geq \alpha$; and for each $(x, \alpha) \in D$ such that $\mu(\alpha) \cdot \gamma(\alpha) \cdot v(g(x,\alpha)) \leq \omega^o(\alpha, \xi^o)$ there holds $\mu(\alpha) \cdot V(x,\alpha) \leq \omega^o(\alpha, \xi^o)$. Hence there follows

$$\mu(\theta) . V({}_\theta t_\alpha x, \theta) \leq {}_\theta t_\alpha^o \, \omega^o(\alpha, \xi^o) < \xi^o ,$$

so that

$$(1) \quad \sigma . \mu(q({}_\theta t_\alpha x, \theta)) \leq \mu(\theta) . \mu(q({}_\theta t_\alpha x, \theta)) \leq \leq \mu(\theta) . V({}_\theta t_\alpha x, \theta) \leq {}_\theta t_\alpha^o \, \omega^o(\alpha, \xi^o) < \xi^o$$

holds whenever

$$(2) \quad (x, \alpha) \in D, \ \mu(\alpha) . \gamma(\alpha) . v(q(x, \alpha)) \leq \omega^o(\alpha, \xi^o), \quad \theta \in \langle \alpha, \ \varepsilon(x, \alpha)) .$$

Now, given any $\xi > 0$ and $\alpha \in R$, choose ξ^o in (1) and (2) so that $\xi^o = \sigma . \mu(\xi)$ and define $\omega(\alpha, \xi)$ so that the relation $\omega^o(\alpha, \sigma . \mu(\xi)) \geq \mu(\alpha) . \gamma(\alpha) . v(\omega(\alpha, \xi)) > 0$ is fulfilled. Then, from (1) and (2) we obtain that

$$q({}_\theta t_\alpha x, \theta) \leq \xi \quad \text{holds whenever} \quad (x, \alpha) \in D ,$$

$$q(x, \alpha) \leq \omega(\alpha, \xi), \quad \theta \in \langle \alpha, \ \varepsilon(x, \alpha)) ,$$

i.e. t has property 6(i).

Ad (ii): The first part of this assertion follows directly from the preceding assertion (i). If t^o has property $6(ii)^o$, then the function $\omega^o(\alpha, \xi^o)$ in the proof of 6(i) does not depend on α, so that it is also possible to define $\omega(\alpha, \xi)$ independently of α.

Hence t has property 6(ii).

Ad (iii): According to the assumption, there is a positive function $\beta^o(\alpha, \omega^o), \alpha \in R, \omega^o > 0$ such that ${}_\theta t_\alpha^o \, \omega^o < \beta^o(\alpha, \omega^o)$ holds for each $\theta \geq \alpha$. Let $\mu(\alpha) . \gamma(\alpha) . v(q(x, \alpha)) \leq \omega^o$. Then there holds

$$\mu(\alpha) . V(x, \alpha) \leq \mu(\alpha) . \gamma(\alpha) . v(q(x, \alpha)) \leq \omega^o ,$$

hence the relation

$$(3) \quad \sigma . \mu(q({}_\theta t_\alpha x, \theta)) \leq \mu(\theta) . V({}_\theta t_\alpha x, \theta) \leq {}_\theta t_\alpha^o \, \omega^o < \beta^o(\alpha, \omega^o)$$

follows whenever

(4) $(x,\alpha)\in D,\ \mu(\alpha)\cdot\gamma(\alpha)\cdot v(g(x,\alpha))\le\omega^{o},$
$\theta\in\langle\alpha,\ \varepsilon(x,\alpha))$.

Now, given any $\alpha\in R$, $\omega>0$, choose ω^{o} in (3) and (4) so that $\omega^{o}=\mu(\alpha)\cdot\gamma(\alpha)\cdot v(\omega)>0$ and define $\beta(\alpha,\omega)$ so that $\beta^{o}(\alpha,\mu(\alpha)\cdot\gamma(\alpha)\cdot v(\omega))\le 6\cdot\mu(\beta(\alpha,\omega))$ is satisfied. Then from (3) and (4) we obtain that

$g({}_{\theta}t_{\alpha}x,\theta)\le\beta(\alpha,\omega)$ holds whenever $(x,\alpha)\in D$,
$g(x,\alpha)\le\omega$, $\theta\in\langle\alpha,\ \varepsilon(x,\alpha))$,

i.e. t has property 6(iii).

Ad (iv): The first part of the assertion follows directly from 8(iii). Since the function $\beta^{o}(\alpha,\ \omega^{o})$ in the proof of 8(iii) can be chosen independently on α, the second part of the assertion follows easily.

This completes the proof of theorem 8.

9. __Theorem__ Let t be an abstract global semi-flow on P, t^{o} an abstract global semi-flow on R^{+}.

(i) Let condition 5(i) be satisfied. If t^{o} has property $6(\mathrm{v})^{o}$, then t has property 6(v).

(ii) Let condition 5(ii) be satisfied. If t^{o} has property $6(\mathrm{v})^{o}$ or $6(\mathrm{vi})^{o}$, then t has the corresponding property 6(v) or 6(vi).

(iii) Let condition 5(i) be satisfied. If t^{o} has property $6(\mathrm{vii})^{o}$, then t has property 6(vii).

(iv) Let condition 5(ii) be satisfied. If t^{o} has property $6(\mathrm{vii})^{o}$ or $6(\mathrm{viii})^{o}$, then t has the corresponding property 6(vii) or 6(viii).

(v) Let condition 5(i) be satisfied and let $\mu(\theta)\to+\infty$

for $\theta \to +\infty$. If t^{o} has property $6(i)^{o}$ or $6(iii)^{o}$, then t has property 6(v).

Proof. Ad (i): According to the assumption, there is a function $\theta^{o}(\alpha, \eta^{o}, \omega^{o})$ such that ${}_{\theta}t^{o}_{\alpha}\, \omega^{o} < \eta^{o}$ for each $\theta \geq \alpha + \theta^{o}(\alpha, \eta^{o}, \omega^{o})$, hence

$$(5)\quad \sigma \cdot u(g({}_{\theta}t_{\alpha}x, \theta)) \leq \mu(\theta) \cdot V({}_{\theta}t_{\alpha}x, \theta) \leq {}_{\theta}t^{o}_{\alpha}\, \omega^{o} < \eta^{o}$$

holds whenever

$$(6)\quad \mu(\alpha) \cdot \gamma(\alpha) \cdot v(g(x, \alpha)) \leq \omega^{o}, \ (x, \alpha) \in D,$$
$$\theta \geq \alpha + \theta^{o}(\alpha, \eta^{o}, \omega^{o}).$$

Let there be given $\alpha \in R$, $\eta > 0$, $\omega > 0$. Choose ω^{o} and η^{o} in (5) and (6) so that $\eta^{o} = \sigma \cdot u(\eta)$, $\omega^{o} = \mu(\alpha) \cdot \gamma(\alpha) \cdot v(\omega)$ and define $\theta(\alpha, \eta, \omega) = \theta^{o}(\alpha, \sigma \cdot u(\eta), \mu(\alpha) \cdot \gamma(\alpha) \cdot v(\omega))$. Then from (5) and (6) there follows

$$g({}_{\theta}t_{\alpha}x, \theta) < \eta \quad \text{whenever } (x, \alpha) \in D,\ g(x, \alpha) \leq \omega(\alpha, \xi),$$
$$\theta \geq \alpha + \theta(\alpha, \eta, \omega),$$

i.e. t has property 6(v).

Ad (ii): The first part of the assertion follows directly from 9(i). To prove the second part, it suffices to observe that from $6(vi)^{o}$ it follows that $\theta^{o}(\alpha, \eta^{o}, \omega^{o})$ in the proof of 6(i) does not depend on α ; hence the existence of the function θ with the required properties follows easily.

Ad (iii): According to the assumption there are functions $\theta^{o}(\alpha, \omega^{o})$ and $\delta^{o}(\alpha)$ such that ${}_{\theta}t_{\alpha}\, \omega^{o} < \delta^{o}(\alpha)$ for each $\theta \geq \alpha + \theta^{o}(\alpha, \omega^{o})$. Hence it follows that

$$(7)\quad \sigma \cdot u(g({}_{\theta}t_{\alpha}x, \theta)) \leq \mu(\theta) \cdot V({}_{\theta}t_{\alpha}x, \theta) \leq {}_{\theta}t^{o}_{\alpha}\, \omega^{o} < \delta^{o}(\alpha)$$

holds whenever

$$(8)\quad (x, \alpha) \in D, \ \mu(\alpha) \cdot \gamma(\alpha) \cdot v(g(x, \alpha)) \leq \omega^{o}, \ \theta \geq \alpha + \theta^{o}(\alpha, \omega^{o}).$$

Let there be given $\alpha \in R$, $\omega > 0$. Take ω^{o} in (7) and (8) so that $\omega^{o} = \mu(\alpha) \cdot \gamma(\alpha) \cdot v(\omega)$ and define the functions

$\Theta(\alpha, \omega)$ and $\sigma(\alpha)$ so that the relations $\Theta(\alpha, \omega) = \Theta^{o}(\alpha, \mu(\alpha) . \gamma(\alpha) . \nu(\alpha))$ and $\sigma^{o}(\alpha) \geq \sigma . \mu(\sigma(\alpha))$ are satisfied. Hence and from (7) and (8) it follows that

$$g({}_{\theta}t_{\alpha} x, \theta) \leq \sigma(\alpha) \text{ holds whenever } (x, \alpha) \in \mathcal{D},$$
$$g(x, \alpha) \leq \omega, \quad \theta \geq \alpha + \Theta(\alpha, \omega),$$

i.e. t has property 6(vii).

Ad (iv): The proof follows easily from that of 9(iii).

Ad (v): First suppose that t^{o} has property $6(i)^{o}$. Let there be given $\alpha \in R$, $\eta > 0$ and $\omega > 0$. Let ξ^{o} in (1) and (2) be such that $0 < \omega^{o}(\alpha, \xi^{o}) \leq \mu(\alpha) . \gamma(\alpha) . \nu(\omega)$. Then

$$\mu(g({}_{\theta}t_{\alpha} x, \theta)) \leq \frac{\xi^{o}}{\mu(\theta)},$$

whenever $(x, \alpha) \in \mathcal{D}$, $g(x, \alpha) \leq \omega$, $\theta \geq \alpha$. According to the assumption $\frac{\xi^{o}}{\mu(\theta)} \rightarrow 0$ for $\theta \rightarrow +\infty$, hence there exists $\theta^{o}(\xi^{o}, \eta)$ such that $g({}_{\theta}t_{\alpha} x, \theta) < \eta$ for $\theta \geq \theta^{o}(\xi^{o}, \eta)$. On defining $\Theta(\alpha, \eta, \omega) = \theta^{o}(\xi^{o}, \eta) - \alpha$ (ξ^{o} depends on α and ω), we obtain that

$$g({}_{\theta}t_{\alpha} x, \theta) < \eta \text{ holds whenever } (x, \alpha) \in \mathcal{D}, g(x, \alpha) \leq \omega,$$
$$\theta \geq \alpha + \Theta(\alpha, \eta, \omega),$$

i.e. t has property 6(v).

Now let t^{o} have property $6(iii)^{o}$ and let there be given $\alpha \in R$, $\eta > 0$, and $\omega > 0$. Choose ω^{o} in (3) and (4) so that $\omega^{o} = \mu(\alpha) . \gamma(\alpha) . \nu(\omega)$. Then we have

$$\mu(g({}_{\theta}t_{\alpha} x, \theta)) \leq \frac{\beta^{o}(\alpha, \omega^{o})}{\mu(\theta)}, \text{ whenever } (x, \alpha) \in \mathcal{D},$$
$$g(x, \alpha) \leq \omega, \quad \theta \geq \alpha.$$

According to the assumption, $\frac{\beta^{o}(\alpha, \omega^{o})}{\mu(\theta)} \rightarrow 0$ for $\theta \rightarrow +\infty$, and hence there exists $\theta'(\alpha, \omega^{o}, \eta)$ such that

$\frac{\beta^{\circ}(\alpha, \omega^{\circ})}{\mu(\theta)} < \mu(\eta)$ for $\theta \geq \theta'(\alpha, \omega^{\circ}, \eta)$. Setting $\theta(\alpha, \eta, \omega) = \theta'(\alpha, \omega^{\circ}, \eta) - \alpha$, we obtain that

$$g({}_{\theta}t_{\alpha} x, \theta) \leq \eta \quad \text{holds whenever } (x, \alpha) \in D, g(x, \alpha) \leq \omega,$$
$$\theta \geq \alpha + \theta(\alpha, \eta, \omega),$$

i.e. t has property 6(v); this completes the proof of theorem 8.

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