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MEASURABLE CARDINALS IN SOME GÖDELIAN SET THEORIES

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There will be proved some consistency theorems on measurable cardinals in the Gödel-Bernays set theory (see [1]), using the method of construction of syntactic models (see [2]). All the results were presented in the Seminar on set theory at the Caroline University in Prague, in December 1965. I am obliged to P. Vopěnka for his valuable suggestions and guidance.

We shall use results and notations from [1],[2] with slight changes without any further reference. If $\mathfrak{A}(\underline{X})$ is an operation introduced by the formula $\varphi(Z,\underline{X})$ and if $T \vdash (\forall \underline{Y} \vartheta(\underline{Y}))(\forall \underline{X})\ (\exists ! Z)\varphi^{*\underline{Y}}(Z,\underline{X})$, we denote by $\mathfrak{A}^{*\underline{Y}}(\underline{X})$ the operation introduced by the formula $\varphi^{*\underline{Y}}(Z,\underline{X})$. Analogously for predicates, special classes, etc.

The cardinal $m \neq \omega_0$ is said to be <u>measurable</u> iff there exists a function $\mu \in \{0,1\}^{\mathcal{P}(m)}$ such that $\mu(\emptyset) = 0, \mu(m) = 1, (\forall x)(x \in m \rightarrow \mu(\{x\}) = 0)$ and $(\forall t)[(t \subseteq \mathcal{P}(m) \,\&\, \mathrm{card}\, t < m \,\&\, (\forall x,y)(x,y \in t \,\&\, x \neq y \rightarrow x \cap y = \emptyset)) \rightarrow \mu(\bigcup t) = \sum_{x \in t} \mu(x)]$ (μ is a non-trivial two-valued measure $\aleph_\alpha$-additive for every $\aleph_\alpha < m$).

Obviously, $m \neq \omega_0$ is measurable if and only if m carries a non-trivial ultrafilter $\mathcal{J}$ such that $(\forall t)(t \subseteq \mathcal{J} \,\&\, \mathrm{card}\, t < m \rightarrow \bigcap t \in \mathcal{J})$. It is known that the first cardinal which carries any non-trivial ultrafilter $\mathcal{J}$ with the property $(\forall t)(t \subseteq \mathcal{J})$ & $card\ t \leq \aleph_0 \rightarrow \bigcap t \in \mathcal{J})$ (σ -multiplicative ultrafilter) is (the first) measurable.

We denote by MC the statement that "there exists a measurable cardinal". Assuming that axioms A to D and MC are compatible, we shall prove the consistency of these axioms with the axiom of choice E (in § 2), with the axiom $\neg E$ (in § 5) and with some theorems concerning the continuum-hypothesis (in §§ 3,4,6).

§ 1.

1.1. Metadefinition. A model $\mathfrak{M} \equiv (\vartheta(\underline{Y}), \chi(X, \underline{Y}), \psi^{\in}(X_1, X_2, \underline{Y}))$ of a Gödelian set theory $\Sigma^{(1)}$ in a Gödelian set theory $\Sigma^{(2)}$ is called perfect iff it is normal and there is a normal operation $\mathcal{L}(\underline{Y})$ such that the following statements are provable in $\Sigma^{(2)}$:

(i) $(\forall \underline{Y}\, \vartheta(\underline{Y}))$ ($\mathcal{L}(\underline{Y})$ is complete, closed with respect to $\mathcal{F}_1, \ldots, \mathcal{F}_8$, and is an almost universal class),

(ii) $(\forall \underline{Y} \vartheta(\underline{Y}))(\chi(X,\underline{Y}) \equiv X \subseteq \mathcal{L}(\underline{Y})\ \&\ (\forall x)(x \in \mathcal{L}(\underline{Y}) \rightarrow X \cap x \in \mathcal{L}(\underline{Y})))$,

(iii) $(\forall \underline{Y} \vartheta(\underline{Y}))(\psi^{\in}(X_1, X_2, \underline{Y}) \equiv X_1 \in X_2)$

(in the case of a non-parametric model, "a normal operation" must be replaced by "a normal constant").

Conversely, if (i) is provable in a Gödelian set theory T (the axioms of which are A to D and some $\in$ -formulas) for a normal predicate ϑ and a normal operation $\mathcal{L}$, then $(\vartheta, \mathcal{L})$ determines, in an evident way, a perfect model of the theory Σ in T .

An operation $\mathfrak{U}$ is said to be absolute in $\mathfrak{M}$ iff

$\Sigma^{(2)} \vdash (\forall \underline{Y} \vartheta(\underline{Y}))(\chi(X_1,\underline{Y}) \& \ldots \& \chi(X_\ell,\underline{Y}) \rightarrow \mathcal{U}^{*\underline{Y}}(\underline{X}) = \mathcal{U}(\underline{X}))$.

Absoluteness for predicates, special classes, etc., is defined analogously. The concepts $\{\}$, $<>$, $\mathcal{U}n$, $\mathcal{D}$, $\mathcal{C}onv$, $\mathcal{O}rd$, $\emptyset$, ω_0, $\mathcal{F}nc$, F, L, ... are absolute in every perfect model (see [1]).

1.2. Let L_K be a normal operation such that the class L_K is constructed in complete analogy with Gödel's class L, except that the operation $\mathcal{F}_9(X,Y) = K \cap X$ is added to $\mathcal{F}_1, \ldots, \mathcal{F}_8$. Let $\vartheta(K)$ be a normal predicate such that $T \vdash (\exists K)\, \vartheta(K)$. We denote by $\overline{\mathcal{M}}$ the perfect model determined by $(\vartheta(K), L_K)$. This is, essentially, the model constructed by Lévy in [4]. The normal operations F_K, Od_K, As_K,... are defined with respect to L_K analogously to the definition of F, Od, As... for L. The proofs of the following assertions were given in [4]:

$(\forall K)(Cls^{*K}(K \cap L_K))$, $(\forall K)(\mathcal{M}(K) \rightarrow K \cap L_K \in L_K)$, $(\forall K)(L_{K \cap L_K} = L_K)$.

The axiom E holds in the model $\overline{\mathcal{M}}$. Defining $K_{(x)} = K''\{x\}$, we obtain

$(\forall K)(\forall x)(x \in L_K \rightarrow Cls^{*K}(K_{(x)} \cap L_K))$

and (for proofs see [3],[7])

(A) $(\forall k)(V = L_K \rightarrow (\forall_\alpha)(\aleph_\alpha \geq card\ Od'_k\, k \rightarrow 2^{\aleph_\alpha} = \aleph_{\alpha+1})$,

(B) $(\forall k)$(if $V = L_K$, $card\ Od'_k\, k = \aleph_\alpha$,

the cardinal $\aleph_\alpha$ is regular and $k \subseteq F''_k\, \aleph_\alpha$, then $(\forall \beta)(\beta < \alpha \rightarrow 2^{\aleph_\beta} \leq \aleph_\alpha)$).

§ 2.

2.1. Metatheorem. Let $\mathcal{M} \equiv (\vartheta, \mathcal{L})$ be a perfect model in the theory T, let the following statement be

provable in T:
$(\forall \underline{Y}\, \vartheta(\underline{Y}))(\exists m)(\exists \mathcal{J})[m \in \mathcal{L}(\underline{Y})\ \&\ \mathcal{J} \cap \mathcal{L}(\underline{Y}) \in \mathcal{L}(\underline{Y})\ \&\ m$ is cardinal $\&\ \mathcal{J}$ is a non-trivial σ-multiplicative ultrafilter on $\mathcal{J}]$.

Then the axiom MC holds in the model $\mathfrak{M}$.

<u>Hint</u>. If $\underline{Y}$ satisfies $\vartheta(\underline{Y})$, and $m, \mathcal{J}$ are the corresponding cardinal and ultrafilter, we obtain $m \in^{*\underline{Y}} N^{*\underline{Y}}$ (N is the class of all cardinals), $m^{*\underline{Y}}(\mathcal{J} \cap \mathcal{L}(\underline{Y}))$.
Then $m \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y}), \neq^{*\underline{Y}} \emptyset^{*\underline{Y}}$. Obviously, $\emptyset^{*\underline{Y}} \notin^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y}), \{x\}^{*\underline{Y}} \notin^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$, and if $x \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y}), y \geq^{*\underline{Y}} x$, then $y \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$.
From $y \cup^{*\underline{Y}} z =^{*\underline{Y}} x \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$ we have $x, y, z \in \mathcal{L}(\underline{Y})$, $x = y \cup z$, $x \in \mathcal{J}$ (using the absoluteness of $\in, =, \cup$), hence $y \in \mathcal{J}$ or $z \in \mathcal{J}$.
This implies $y \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$ or $z \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$. Finally, let i satisfy $\mathrm{card}^{*\underline{Y}} i =^{*\underline{Y}} \omega_0^{*\underline{Y}}$ and $i \subseteq^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$. The absoluteness of $=, \subseteq, \omega_0$ and $\bigcap$ and the σ-multiplicativity of $\mathcal{J}$ implies $\mathrm{card}\, i = \omega_0$, $i \subseteq \mathcal{J}$, $\bigcap^{*\underline{Y}} i = \bigcap i$. Hence $\bigcap^{*\underline{Y}} i \in^{*\underline{Y}} \mathcal{J} \cap \mathcal{L}(\underline{Y})$.
This proves that $(\forall \underline{Y}\, \vartheta(\underline{Y}))$ ($\mathcal{J} \cap \mathcal{L}(\underline{Y})$ is a $*\underline{Y}$-non-trivial $*\underline{Y}$-σ-multiplicative $*\underline{Y}$-ultrafilter on m).

2.2. Let T be the set theory with the axioms A to D and MC. We define: $\vartheta(\mathcal{J}) \equiv \mathcal{J}$ is a non-trivial σ-multiplicative ultrafilter on the first measurable cardinal.
Let $\overline{\mathfrak{M}}_0$ be the model determined by $(\vartheta(\mathcal{J}), L_{\mathcal{J}})$. The axioms A to E hold in $\overline{\mathfrak{M}}_0$. From 1.2 we obtain $T \vdash (\forall \mathcal{J}\, \vartheta(\mathcal{J}))(\mathcal{J} \cap L_{\mathcal{J}} \in L_{\mathcal{J}})$.
By 2.1, the axiom MC holds in the model $\overline{\mathfrak{M}}_0$.

§ 3.

In §§ 3 - 5, we assume familiarity with the method of construction of ∇-models of set theory Σ^* which is explained in [9] to [12].
The terminology and notations introduced there will be used. All the following considerations concern the set theory $\Sigma^* + (MC)$.

Denote by ϑ the first measurable cardinal, $\mathcal{J}$ a non-trivial σ-multiplicative ultrafilter on ϑ. Let μ be the non-trivial two-valued σ-additive measure corresponding to $\mathcal{J}$. Let $\nabla(ind, G, \langle c, t\rangle, r, j)$ be the ∇-model of the theory Σ^*. The concepts of this model will be denoted asterisks.

Using ∇-models we shall prove some results concerning the continuum-hypothesis for cardinals less than ϑ. Lévy in [5] and Solovay in [8] obtained analogous results for cardinals in the Zermelo-Fraenkel set theory.

3.1. Definition. Let $\langle c, t\rangle$ be a topological space. We define

(a) $b < t \equiv b \subseteq t \,\&\, (\forall u)(\exists v)(\emptyset \neq u \in t \rightarrow v \in b \,\&\, v \neq \emptyset \,\&\, v \subseteq u)$,

(b) $\chi_t(c,t) = \min\{\aleph_\xi ; (\exists b)$ (b some basis $\langle c,t\rangle \,\&\, \mathrm{card}\, b = \aleph_\xi)\}$.

3.2. We define the functions $\underline{\mu}$, $\bar{\mu}$ in the model ∇ thus:

$$\mathcal{D}^*(\underline{\mu}) =^* \mathcal{P}^*(k_\vartheta),\ \mathcal{D}^*(\bar{\mu}) =^* \mathcal{P}^*(k_\vartheta),\ \mathcal{W}^*(\underline{\mu}) =^* \mathcal{W}^*(\bar{\mu}) =^* \{0^*, 1^*\}^*$$

and

$$\underline{\mu}(f) =^* 1^* \equiv (\exists g)[g \in^* k_{\mathcal{J}} \,\&\, g \subseteq^* f],$$

$$\bar{\mu}(f) =^* 0^* \equiv (\exists g)[g \in^* k_{\mathcal{P}(\vartheta)-\mathcal{J}} \,\&\, g \supseteq^* f].$$

Obviously $\underline{\mu} \leq^* \bar{\mu}$.

3.3. **Lemma.** If $\chi_t(c,t) < \vartheta$, then $\mu^* =^* \underline{\mu} =^* \bar{\mu}$ is a non-trivial σ-additive two-valued measure on k_ϑ in the model ∇.

Proof I. First prove $\underline{\mu} =^* \bar{\mu}$. By 3.2, one should show that $\bar{\mu}(f) =^* 1^* \rightarrow \underline{\mu}(f) =^* 1^*$. Choose some basis b of the space $\langle c,t \rangle$ with $card\, b = \chi_t(c,t)$. For $f \subseteq^* k_\vartheta$ define

$u \in b_f \equiv u \in b \ \&\ (\forall y)(y \in u \rightarrow f(y) = f^u \ \&\ u \subseteq F^{\ulcorner} f^u \subseteq \vartheta^{\urcorner})$.

Evidently $\bigcup b_f \in j$. For $u \in b_f$ define

$u_f^+ = \{x;\, x \in \vartheta \ \&\ u \subseteq F^{\ulcorner} x \in f^{u \urcorner}\}$, $u_f^- = \{x;\, x \in \vartheta \ \&\ u \subseteq F^{\ulcorner} x \notin f^{u \urcorner}\}$.

Let $v \in b_f$, so that $\bigcup_{u \subseteq v,\, u \in b_f} (u_f^+ \cup u_f^-) = \vartheta$. But $card\, b_f \leq$ $\leq \chi_t(c,t) < \vartheta$, so that there exists a $u_v \subseteq v$ (determined uniquely with the axiom of choice), and either $u_{vf}^+ \in \mathcal{J}$ or $u_{vf}^- \in \mathcal{J}$. There is $u_{vf}^+ \cap u_{vf}^- = \emptyset$. Also $\{u_v;\, v \in b_f\} < b_f$ and $\bigcup b_f \in j$. Hence there exists a disjoint system $\{u_v;\, v \in b'\}$, $b' \subseteq b_f$, whose union belongs to j.

Now define $g^+(y) = u_{vf}^+$, $g^-(y) = u_{vf}^-$ for $y \in u_v$, $v \in b'$. Obviously, $g^+ \subseteq^* f$, $g^- \subseteq^* k_\vartheta -^* f$ and $g^+ \in^* k_{\mathcal{P}(\vartheta)}$, $g^- \in^* k_{\mathcal{P}(\vartheta)}$. Because $k_\vartheta -^* g^- \supseteq^* f$ and $\bar{\mu}(f) =^* 1^*$, there is $k_\vartheta -^* g \in^* k_{\mathcal{J}}$, hence $g^- \in^* k_{\mathcal{P}(\vartheta) - \mathcal{J}}$. This implies $g^+ \in^* k_{\mathcal{J}}$. Since $g^+ \subseteq^* f$, we obtain $\mu(f) =^* 1^*$.

II. Put $\mu^* =^* \underline{\mu} =^* \bar{\mu}$. Easily, $\mu^*(\emptyset^*) =^* 0^*$, $\mu^*(k_\vartheta) =^* 1^*$, $f \in^* k_\vartheta \rightarrow \mu^*(\{f\}^*) =^* 0^*$.

Let $a \subseteq^* \mathcal{P}^*(k_\vartheta)$, $card^* a \leq^* \aleph_0^*$, $x, y \in^* a \ \&\ x \neq^* y \rightarrow x \cap^* y =^* 0^*$,

(a) Let $x \in^* a$ and $\mu^*(x) =^* 1^*$.

If $y \in^* a$, $y \neq^* x$, then $\mu^*(y) =^* 0^*$ holds. For if not, then there exist $g, h \in^* k_{\mathcal{J}}$, $g \subseteq^* x$, $h \subseteq^* y$, hence

$g \cap^* h =^* \emptyset^*$, and also $g =^* \bar{g}$, $h =^* \bar{h}$, so that $\{x; \bar{g}(x), \bar{h}(x) \in \mathcal{J} \ \& \ \bar{g}(x) \cap \bar{h}(x) = \emptyset\} \in j$, a contradiction. Since $U^* a \supseteq^* x$, we have $\mu^*(U^* a) =^* 1^* =^* \sum^*_{y \in^* a} \mu^*(y)$.

(b) Let $(\forall x)(x \in^* a \rightarrow \mu^*(x) =^* 0^*)$.

There exists a set d such that $(\forall x)(x \in^* a \rightarrow (\exists y)(y =^* =^* x \ \& \ y \in d))$, $(\forall y_1, y_2)(y_1, y_2 \in d \ \& \ y_1 =^* y_2 \rightarrow y_1 = y_2)$.

There is $\mu(c,t) \leq \chi_t(c,t) < \vartheta$, and the relative cardinal numbers of the model, greater than or equal to $k_{\mu(c,t)}$ are the cardinal numbers of the model (see Th.4 from [11]). Hence, easily, $card\ d < \vartheta$ (using $card^* a \leq^* \aleph^*_0$). By the assumption, for every $x \in d$ there exists a $g_x \in^* \in^* k_{\mathcal{P}(\vartheta)-\mathcal{J}}$ such that $x \subseteq^* g_x$ and $g^u_x \in \mathcal{P}(\vartheta) - \mathcal{J}$ for every $u \in b_{g_x}$ (g_x is chosen using the axiom of choice). Put $z = \bigcup_{u \in b_{g_x}, x \in d} g^u_x$. Obviously $U^* g \subseteq^* k_z$. From $card\ b_{g_x} \leq \chi_t(c,t) < \vartheta$ and $card\ d < \vartheta$ we obtain $z \in \mathcal{P}(\vartheta) - \mathcal{J}$.

Hence $\mu^*(U^* a) =^* 0^*$.

3.4. Let $\chi_t(c,t) < \vartheta$ (e.g. let $card\ c < \vartheta$). Then the axiom MC holds in the model ∇.

3.5. The results mentioned in this section were proved in [6].

<u>Definition</u>. Let $\langle c, t \rangle$ be a topological space, $b < t$.

$$\sigma(u, \aleph_\alpha, b) \equiv (\forall a)[(\emptyset \notin a \subseteq b \cap \mathcal{P}_{(u)} \ \& \ (\forall v_1, v_2)(v_1, v_2 \in a \rightarrow v_1 \subseteq v_2 \vee v_2 \subseteq v_1) \ \& \ card\ a \leq \aleph_\alpha) \rightarrow \emptyset \neq \bigcap a \in t],$$

$$\aleph_{\sigma_b} = \min\{\aleph_\alpha; \ \bigcup\{u; \sigma(u, \aleph_\alpha, b)\} \notin j\}.$$

<u>Lemma</u>. Let $f \in \widetilde{\Pi(0)}$, $g \subseteq^* f$, $card^* g \leq^* \aleph^*_\alpha <^* k_{\aleph_{\sigma_b}}$. Then $g \in \widetilde{\Pi(0)}$, $g =^* \bar{g}$, $\bar{g} \in \overline{\Pi(0)}$ and $\{x; card\ \bar{g}(x) \leq \aleph^*_\alpha(x)\} \in j$, $\{x; \bar{g}(x) \in \Pi(0)\} \in j$

(one may assume that $\{x;\ \aleph^*_\alpha(x)$ is a cardinal number$\}\in$ $\in \mathfrak{j}$).

Let $\vartheta < \aleph_{\sigma_\ell}$. Then k_ϑ is the first measurable cardinal number of the model ∇. Therefore, the axiom MC holds in the model ∇.

§ 4.

4.1. Definition. Let h be a cardinal number of ∇. Denote by h^+ the first cardinal number of ∇ greater than h. Denote by $c(h)$ the cardinality of $\mathcal{P}^*(h)$ in the model ∇.

4.2. Metadefinition. A model ∇ is said to be of type $[MC \mathbin{\&} 2^{\aleph_\alpha} = \aleph_{\alpha+1}]$ (or $[MC \mathbin{\&} 2^{\aleph_\alpha} \neq \aleph_{\alpha+1}]$ respectively) iff its parameters depend on $\aleph_\alpha$ in such a manner that the following is provable in the set theory $\Sigma^* + (MC)$:

(a) every relatively cardinal number h of ∇, $h \leq^* k_{\omega_\alpha}$, is a cardinal number of ∇,

(b) $c(k_{\omega_\alpha}) =^* k^+_{\omega_\alpha}$ (or $c(k_{\omega_\alpha}) \neq^* k^+_{\omega_\alpha}$ respectively),

(c) in the model ∇ there exists a measurable cardinal number.

4.3. Construction. I. Let $\aleph_\alpha$ be a cardinal number. Using the method from [12] one can construct the model $\nabla[\omega_{\alpha+1} \to \omega_{\beta+1}]$ for $\aleph_\beta = 2^{\aleph_\alpha}$. This model is of type $[MC \mathbin{\&} 2^{\aleph_\alpha} = \aleph_{\alpha+1}]$. To see this, observe that $\aleph_{\alpha+1} < \nu(c,t)$, hence condition (a) from 4.2 is fulfilled, $\aleph_{\alpha+1} \leq \aleph_{\sigma_\ell}$ (ℓ is the basis described in def.4 from [12]), hence (by 3.5) the condition (b) is also fulfilled. $\aleph_\alpha < \vartheta$ implies $\chi_\ell(c,t) < \vartheta$, $\vartheta \leq \aleph_\alpha$ implies $\vartheta < \aleph_{\sigma_\ell}$. Hence by 3.4 or 3.5, the axiom MC holds in this model.

4.4. Construction II. Let $\aleph_\alpha$ be a regular cardinal number, $\aleph_\alpha \neq \vartheta$. We may suppose $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Using [12], the model $\nabla[2^{\aleph_\alpha} = \aleph_{\alpha+3}]$ can be constructed. This model is of type $[MC \,\&\, 2^{\aleph_\alpha} \neq \aleph_{\alpha+1}]$. Condition (a) from 4.2 follows from $\aleph_\alpha \leq \nu(c,t)$. Since $\mu(c,t) \leq \aleph_{\alpha+2}$, $k_{\omega_{\alpha+2}}$, $k_{\omega_{\alpha+3}}$ are cardinals of ∇ and $c(k_{\omega_\alpha}) \geq^* k_{\omega_{\alpha+3}}$.
The validity of condition (b) in 4.2 follows. Finally, $\aleph_\alpha < \vartheta$ implies $\chi_t(c,t) < \vartheta$, and $\aleph_\alpha > \vartheta$ implies $\aleph_{\sigma_\ell} > \vartheta$. So, by 3.4 or 3.5, the axiom MC holds in this model.

4.5. Note. A more detailed discussion of various possible cases of relations between cardinals and the cardinalities of their power sets in the model ∇ with the axiom MC may be performed analogously to [12]. Of course, one cannot assume the generalized continuum-hypothesis in the set theory.

§ 5.

In this section, the method of construction of permutation submodels of ∇-model explained in [13] will be used.

5.1. Metatheorem. Let $\chi_t(c,t) < \vartheta$ or $\aleph_{\sigma_\ell} > \vartheta$ be provable in the set theory. Then the axiom MC holds in the permutation submodel of the model ∇.

Hint. a) Let $\chi_t(c,t) \leq \vartheta$. $\vartheta, \mathcal{J} \in \Pi(0)$; then $k_\vartheta, k_{\mathcal{J}}$ are sets of the model ∇_p (see lemma 11 in [13]). In ∇_p, define $f \in^p \mathcal{J}_1 \equiv f \subseteq^p k_\vartheta \,\&\, (\exists g)(g \in^p k_{\mathcal{J}} \,\&\, f =^p g)$;

hence $\mathcal{J}_1 \in^* \tilde{P}$. By 3.3, $\kappa_t(c,t) < \vartheta$ implies that $\mu^* =^* =^* \underline{\mu} =^* \bar{\mu}$ is a non-trivial σ-additive two-valued measure on k_ϑ in the model ∇; denote by $\mathcal{J}_2$ the corresponding ultrafilter in ∇. By lemma 16 in [13], $\tilde{P}$ is a complete class in ∇, therefore

$$f \in^* \tilde{P} \cap^* \mathcal{J}_2 \equiv f \in^* \tilde{P} \,\&\, f \subseteq^* k_\vartheta \,\&\, (\exists g)(g \in^* k_\gamma \,\&\, g \subseteq^* f) \equiv$$

$$\equiv f \subseteq^P k_\vartheta \,\&\, (\exists g)(g \in^P k_\gamma \,\&\, g \subseteq^P f) \equiv f \in^P \mathcal{J}_1 \equiv f \in^* \mathcal{J}_1 .$$

Thus $\tilde{P} \cap^* \mathcal{J}_2 =^* \mathcal{J}_1 \in^* \tilde{P}$. By 2.1, the axiom MC holds in the model ∇_P.

b) Let $\kappa_{\sigma_\vartheta} > \vartheta$. By 3.5, k_ϑ is a σ-measurable cardinal of ∇ and k_γ is a non-trivial σ-multiplicative ultrafilter on k_ϑ in the model ∇. But $\mathcal{J} \in \Pi(0)$, hence k_γ is a set of ∇_P.

Consequently, $k_\gamma \cap^* \tilde{P} =^* k_\gamma \in^* \tilde{P}$. By 2.1, the axiom MC holds in the model ∇_P.

5.2. Example. If $\omega_\beta \neq \vartheta$, then the assumptions of theorem 5.1 are satisfied for the permutation model constructed in Example 1 in [14] (for $\omega_\beta < \vartheta$ use 3.4, for $\omega_\beta > \vartheta$ use 3.5), so that the axiom MC holds in it. In this model, the power set of k_{ω_β} cannot be well-ordered. Choosing e.g. $\beta = 0$ we obtain: it is consistent to suppose that the power set of the first measurable cardinal cannot be well-ordered. In particular, the negation of the axiom of choice is consistent with MC.

§ 6.

Now the relative consistency of the axiom MC with the "weakly generalized" continuum-hypothesis will be demonstrated. The corresponding perfect model will be constructed by the method from [3] and [7]. We use the results from Lévy's generalizing paper [4] which were mentioned in 1.2. We continue our considerations in the theory $\Sigma^* + (MC)$.

6.1. <u>Definition</u>. Let ϑ be the first measurable cardinal. Define:

$$\mathcal{A}_1(k, \omega_\psi) \equiv k \,\mathcal{F}n\, \omega_\psi \,\&\, [(\forall_\alpha)(\alpha \in \omega_\psi \to (\mathcal{D}(k'\alpha) = \alpha \,\&$$
$$\& \,\mathcal{W}(k'\alpha) = \bar{\mathcal{Z}} \,\&\, \mathcal{U}n_2(k'\alpha) \,\&\, \mathcal{R}el(k'\alpha)))],$$

$$\mathcal{A}_2(k) \equiv k \,\mathcal{F}n\, \{0\} \times 2^\vartheta \,\&\, \mathcal{W}(k) = \mathcal{P}(\vartheta),$$

$$\mathcal{A}_3(k, \omega_\varphi, \omega_\psi) \equiv k \,\mathcal{F}n\, \{1\} \times \omega_\psi \,\&\, \mathcal{U}n_2(k) \,\&\, \mathcal{W}(k) \subseteq \mathcal{P}(\omega_\varphi),$$

$\mathcal{A}_4(k, \mathcal{J}) \equiv k = \{<\mathcal{J}, \{2\}>\}$ and $\mathcal{J}$ is a non-trivial σ-multiplicative ultrafilter on ϑ,

$$\mathcal{A}_5(k) \equiv (\exists k_1, \dots, k_4)(\exists \omega_\varphi, \omega_\psi, \mathcal{J})[k = k_1 \cup \dots \cup k_4 \,\&\, \mathcal{A}_1(k_1, \omega_\psi) \,\&$$
$$\& \dots \& \,\mathcal{A}_4(k_4, \mathcal{J})],$$

$$\mathcal{A}(k) \equiv (\exists l)[\mathcal{A}_5(l) \,\&\, (<y\, x> \in k \equiv (\exists z)(<z\, x> \in l \,\&\, y \in z))] \,\&$$
$$\& \,\mathcal{R}el(k).$$

k_1, k_2, k_3, k_4 are determined uniquely for every k. Then also $\omega_\varphi, \omega_\psi, \mathcal{J}$ are determined uniquely for every k (and can be constructed from k using the operations $\mathcal{F}_1$ to $\mathcal{F}_8$). The existence of a set k corresponding to ω_φ, ω_ψ for every ω_φ, ω_ψ is guaranteed by the axiom of choice.

6.2. Define:

$\vartheta_0(k) \equiv \mathcal{A}(k)$ and the following holds for the cardinals ω_φ, ω_ψ corresponding to k: $\vartheta \leq \omega_\varphi < \omega_\psi$, ω_ψ is regular and

(1) if $\vartheta < \omega_\varphi$ then $2^\vartheta \leq \omega_\psi \leq 2^{\aleph_\varphi}$,

(2) if $\vartheta = \omega_\varphi$ then $2^{\aleph_\varphi} = \omega_\psi$.

Let M be the perfect model determined by $(\vartheta(k), L_k)$ where $\vartheta(k) \to \vartheta_0(k)$. The axioms A to E hold in the model M. In the following, we shall write φ_k instead of φ^{*k}, etc.

6.3. Metatheorem. (a) $\vdash (\forall k\, \vartheta(k))$ $(\aleph'_k\alpha = \aleph'_\alpha$ if $\alpha \leq \psi$)

$$(b)\ \vdash (\forall k\, \vartheta(k))((2^{\aleph'_k\alpha})_k = 2^{\aleph'_k\alpha} \text{ if } \alpha \leq \vartheta\ )$$

$$(c)\ \vdash (\forall k\, \vartheta(k))((2^{\aleph'_k\alpha})_k \leq \aleph'_k\psi \text{ if } \alpha \leq \varphi\ )$$

$$(d)\ \vdash (\forall k\, \vartheta(k))((2^{\aleph'_k\alpha})_k = \aleph'_k\psi \text{ if } \varphi \leq \alpha < \psi)$$

$$(e)\ \vdash (\forall k\, \vartheta(k))((2^{\aleph'_k\alpha})_k = \aleph'_k\alpha+1 \text{ if } \psi \leq \alpha\ )$$

(f) ϑ is the first measurable cardinal of the model $\mathbb{M}$.

Corollary. Suppose $2^{\vartheta} = \aleph_{\vartheta+1}$ holds in the set theory (this assumption is consistent by 4.3). Let $\mathbb{M} \equiv (\vartheta_1(k), L_k)$ where $\vartheta_1(k) \to \vartheta_0(k)$, and $\vartheta_1(k)$ implies that the ω_φ, ω_ψ corresponding to k fulfil $\omega_\varphi = \vartheta$, $\omega_\psi = \omega_{\varphi+1}$. By (e), $\vdash (\forall k\, \vartheta_1(k))((2^{\aleph'_k\alpha})_k = \aleph'_k\ \alpha + 1$ if $\alpha \geq \vartheta$). Thus the statement "the continuum hypothesis holds for all cardinals greater than or equal to the first measurable cardinal" is consistent with the axioms A to E and MC.

Note. In particular, the axioms MC and $(\exists k)(V = L_k)$ hold in the model $\mathbb{M}$.

Hint. Let k have the property ϑ.

I. In the following, we use some estimates of $Od'_k\, \mathcal{F}(x, y)$ on the basis of estimates of $Od'_k\, x$, $Od'_k\, y$, where the operations $\mathcal{F}$ are composed from $\mathcal{F}_1$ to $\mathcal{F}_9$. Their proofs are left to the reader (see [1], [3], [4]).

α) Let $\gamma \in \omega_\psi$. Then $Od'_k \gamma \in \omega_\psi$, $Od'_k \bar{\bar{\gamma}} \in \omega_\psi$, hence $Od'_k((\bar{\bar{\gamma}} \times \gamma) \times \{\gamma\}) \in \omega_\psi$. Since $k_{(\gamma)} = \mathcal{W}(k \cap ((\bar{\bar{\gamma}} \times \gamma) \times \{\gamma\}) > \in L_k$, we obtain $Od'_k k_{(\gamma)} \in \omega_\psi$.

β) Let $\nu \in 2^\vartheta$. Then $k_{(\langle 0 \nu \rangle)} = \mathcal{W}(k \cap (\vartheta \times \{\langle 0 \nu \rangle\})) \in L_k$. Since $Od'_k \vartheta \in \omega_{\vartheta+1} \subseteq \omega_\psi$, $Od'_k \nu \in 2^\vartheta \subseteq \omega_\psi$, we obtain $Od'_k k_{(\langle 0 \nu \rangle)} \in \omega_\psi$.

γ) Let $\nu \in \omega_\psi$. Then $k_{(\langle 1 \nu \rangle)} = \mathcal{W}(k \cap (\omega_\varphi \times \{\langle 1 \nu \rangle\})) \in L_k$. Since $Od'_k \omega_\varphi \in \omega_\psi$, $Od'_k \nu \in \omega_\psi$, we obtain $Od'_k k_{(\langle 1 \nu \rangle)} \in \omega_\psi$.

δ) By the definition of k, for every $u \leq \vartheta$ there exists an $\iota \in 2^\vartheta$ such that $u = k_{(\langle 0 \nu \rangle)}$. β) implies $Od'_k u \in \omega_\psi$, hence $\mathcal{P}(\vartheta) \subseteq F''_k \omega_\psi = F'_k \omega_\psi$. Therefore $k_{(\{2\})} = \mathcal{J} = \mathcal{J} \cap F'_k \omega_\psi = \mathcal{W}(k \cap (F'_k \omega_\psi \times \{\{2\}\})) \in L_k$, $Od'_k k_{(\{2\})} \in \omega_{\psi+1}$.

II. Obviously,

$$k = \bigcup_{\gamma \in \omega_\psi} (k_{(\gamma)} \times \{\gamma\}) \cup \bigcup_{\nu \in 2^\vartheta} (k_{(\langle 0 \nu \rangle)} \times \{\langle 0 \nu \rangle\}) \cup \bigcup_{\nu \in \omega_\psi} (k_{(\langle 1 \nu \rangle)} \times \{\langle 1 \nu \rangle\}) \cup \{k_{(\{2\})} \times \{\{2\}\}\}$$

(see α) to δ) and the definition of k). By α) to γ), $Od'_k(\ ) \in \omega_\psi$ for every term contained in brackets following the sign $\bigcup$. By δ), there is $k_{(\{2\})} \times \{\{2\}\} \subseteq F'_k \omega_\psi$. Hence $k \subseteq F'_k \omega_\psi$, $k = k \cap F'_k \omega_\psi = F'_k J'_k < 9\, \omega_\psi\, 0 > \in L_k$. This implies $Od'_k k \in \omega_{\psi+1}$, $card_k\, Od'_k k = \omega_\psi$.

III. (a) Let $\gamma < \omega_\psi$. By α), $k_{(\gamma)} \in L_k$. But $\mathfrak{D}(k_{(\gamma)}) = \gamma \,\&\, \mathcal{W}(k_{(\gamma)}) = \bar{\bar{\gamma}} \,\&\, \mathcal{U}n_2(k_{(\gamma)}) \,\&\, \mathcal{R}el(k_{(\gamma)})$, hence $card_k \gamma = card\, \gamma$. Then (a) follows easily (see [3]) ;

(b) - (e) By II, $k \in L_k$. The absoluteness of the operation L_k implies $V_k = L_k = (L_k)_k$. Now, $card_k\, Od'_k k = \omega_\psi$, $k \approx_k F''_k \omega_\psi$ and ω_ψ is a k-regular k-cardinal (the latter statement is implied by the absoluteness of cardinals, see (a)). Therefore, (A) and (B) from 1.2 hold in the model M. Hence we have proved that

$(\forall\alpha)(\alpha \geq \psi \rightarrow (2^{\aleph'_k \alpha})_k = \aleph'_k\, \alpha + 1)$ – this is statement (e),

$(\forall\beta)(\beta < \psi \rightarrow (2^{\aleph'_k \beta})_k \leq_k \aleph'_k\, \psi$.

(c) and the inequality $\leq_k$ in (d) are immediate corollaries of the second proposition. Suppose $card_k(2^{\aleph'_k \varphi})_k <_k$ $<_k \aleph'_k\, \psi$. Then the absoluteness of the cardinals (see (a)) implies $(2^{\aleph'_k \varphi})_k < \aleph'_k\, \psi$; but $k_{(\langle 0\iota\rangle)} \subseteq \aleph'_k\, \varphi$, $k_{(\langle 0\iota\rangle)} \in L_k$ for every $\iota \in \omega_\psi$ and the cardinality of the set $\{k_{(\langle 0\iota\rangle)};\ \iota \in \omega_\psi\}$ is not less than ω_ψ . This proves (d).

By β) from I, there is $\mathcal{P}_k(\vartheta) = \mathcal{P}(\vartheta) \cap L_k = \mathcal{P}(\vartheta)$. Hence $\mathcal{P}_k(\omega_\alpha) = \mathcal{P}(\omega_\alpha)$ holds for every $\omega_\alpha \in \vartheta$. Let f be a k-one-to-one k-mapping of $\mathcal{P}(\omega_\alpha)$ on the k-cardinal ω_β , hence $(2^{\aleph'\alpha})_k = \omega_\beta$. Then f is also a one-to-one mapping of $\mathcal{P}(\omega_\alpha)$ on ω_β in the set theory. Therefore $card\ \omega_\beta = 2^{\aleph'\alpha}$. By (c), $\omega_\beta = (2^{\aleph'\alpha})_k \leq \omega_\psi$, hence (a) implies $card\ \omega_\beta = card_k\ \omega_\beta$, and consequently $(2^{\aleph'\alpha})_k =$ $= 2^{\aleph'\alpha}$ (use $\omega_\beta = card_k\ \omega_\beta$). This proves (b).

(f) We know that $k_{(\langle 2\rangle)} = k''\{\{2\}\} = \mathcal{J}$, hence $\mathcal{J} \cap L_k =$ $= k_{(\langle 2\rangle)} \cap L_k \in L_k$. Therefore by 2.1 (see the hint), in the model M there exists a non-trivial σ-multiplicative ultrafilter on the cardinal ϑ . Let $\omega_\alpha < \vartheta$ be some measurable cardinal in the model M . Then for our k there exists a k-non-trivial k-σ-multiplicative k-ultrafilter j on ω_α .

There is $\mathcal{P}_k(\omega_\alpha) = \mathcal{P}(\omega_\alpha)$. By (b), $(2^{\aleph'\alpha})_k = 2^{\aleph'\alpha} \in \vartheta$, hence $\mathcal{P}_k(\mathcal{P}_k(\omega_\alpha)) = \mathcal{P}\mathcal{P}(\omega_\alpha)$. This result and the absoluteness of the ω_1 (see (a)) implies that j is a non-trivial σ-multiplicative ultrafilter on ω_α in the set theory. But this is in contradiction with $\omega_\alpha < \vartheta$. Consequently, ϑ is the first measurable cardinal of the model $\mathbb{M}$.

The proof of 6.3 is complete.

To the author's knowledge, the following two problems concerning cardinalities of power sets in set theories with measurable cardinals remain open:

1) Whether the generalized continuum-hypothesis is consistent with A to E and MC;
2) Whether $2^\vartheta \neq \aleph_{\vartheta+1}$ (ϑ is the first measurable cardinal) is consistent with A to E and MC.

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