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Universal categories

Commentationes Mathematicae Universitatis Carolinae, Vol. 7 (1966), No. 2, 143--206

Persistent URL: http://dml.cz/dmlcz/105052

Commentationes Mathematicae Universitatis Carolinae
7, 2 (1966)

UNIVERSAL CATEGORIES

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In [9], A. Pultr defined universal categories as follows: a category K is called universal if every small category is isomorphic with a full subcategory of K . It is easy to see that such universal categories do exist. The problems solved in [9],[10],[12], concern further properties of universal categories, namely where from usual categories are universal.

The notion of a universal category given above requires the existence of a full embedding for every small category. Thus a universal category in this sense may be called universal for all small categories. But it is natural to consider also other "systems" of categories, for example to consider a category such that every (not necessarily small) category may be fully embedded in it. x)

In the present paper some metatheorems are given, from which there follow these results:
There exists a category in which every category may be fully embedded.
There exists a category with a singleton in which every category with a singleton may be fully embedded.

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x) The question whether there exists such a category also formulated A. Pultr in a conversation.

There exists an additive category in which every additive category may be fully additively embedded.
There exists a concrete category in which every concrete category may be fully embedded.
There exists a good bicategory x) in which every good bicategory may be fully embedded by an *isofunctor which preserves injections and projections.

The present paper is written in the set-theory with the Bernays-Gödel axioms, [4]. Although the paper is not written formally (in some details even not quite precisely) these axioms are consistently respected.

I. Preliminaries.

All needed definitions (category, functor and so on) are given in [7].

1. Notation: If K is a category, denote by K^{σ} the class of all its objects, by K^{m} the class of all its morphisms. If $a, b \in K^{\sigma}$, denote by $H_K(a, b)$ the set of all morphisms of K from a to b. If $\alpha \in H_K(a, b)$, put $\overleftarrow{\alpha} = a$, $\overrightarrow{\alpha} = b$. If $a, b, c \in K^{\sigma}$, $\alpha \in H_K(a, b)$, $\beta \in H_K(b, c)$, then the composition of α and β will be denoted by $\alpha \cdot \beta$. If K is a category such that K^{σ} is a set, then K will be called small. We shall use the symbol $K' \subset K$ to denote that K' is a subcategory of K; and the symbol $K' \subset_f K$ to denote that K is a full subcategory xx) of K.

x) For the definition of a good bicategory see section V.8 of the present paper.

xx) We recall that a subcategory K' of K is called full if $H_{K'}(a, b) = H_K(a, b)$ for all $a, b \in K'^{\sigma}$.

We shall say that a category K is one-point (or one-object) if the classes K^{σ}, K^{m} have exactly one element (or if K^{σ} has exactly one element).

Let K be a category. We recall that $a \in K^{\sigma}$ is called a singleton (or cosingleton) of K if, for every $b \in K^{\sigma}$, $H_K(b, a)$ (or $H_K(a, b)$) contains exactly one element and $H_K(a, b) \neq \emptyset$ (or $H_K(b, a) \neq \emptyset$ respectively). If Φ is a functor from a category K to H, we shall write $\Phi : K \to H$; if K, H are small then Φ is called small. If $\Phi : K \to H$, $\Psi : H \to M$ are functors, then the composition will be denoted by $\Phi \cdot \Psi$ or $\Phi \Psi$. If $\Phi : K \to H$ is a functor, $\alpha \in K^{\sigma} \cup K^{m}$, then we shall write $(\alpha)\Phi$ instead of the more usual $\Phi(\alpha)$.
All considered functors are covariant, unless otherwise expressly stated.
A one-to-one functor of a category into a category will be called an isofunctor into or an embedding. An embedding onto a full subcategory will be called a full embedding. If K' is a subcategory of K, then the inclusion functor $\iota : K' \to K$ is defined by $(\alpha)\iota = \alpha$ for every $\alpha \in K'^{\sigma} \cup K'^{m}$.

2. <u>Convention</u>: If K is a category, $\alpha \in K^{m}$, then α is always a triple, the first member of which is $\overleftarrow{\alpha}$, and the third member is $\overrightarrow{\alpha}$. Thus if K_1, K_2 are categories such that $K_1^{\sigma} \cap K_2^{\sigma} = \emptyset$, then also $K_1^{m} \cap K_2^{m} = \emptyset$.

3. <u>Convention and notation</u>: As noted before, the present paper is written in the Bernays-Gödel set-theory, [4]. Thus we distinguish between sets and classes and all axioms given in [4] are assumed. A class which is not a set is usually called a proper class. The axiom of choice for classes may be for-

mulated as follows: let X be a class, R an equivalence on X; then there exists a choice-class Y (i.e. $Y \subset X$; if $y, y' \in Y$, $y R y'$, then $y = y'$; and for every $x \in X$ there exists a $y \in Y$ such that $x R y$). It is used often in this form; for example the existence of a skeleton of a category requires it. But, as shown in [4], this form is equivalent with the following one: every class X may be well ordered (by $<$) such that for every $a \in X$ the class $\{ b \in X ; b < a \}$ is a set (the proof requires the axiom D of [4]). The last form will also be used often in the present paper x) and such a well order will be called an O_n-order for X (also when X is a set).
The properties V and W considered in the present paper are always supposed to be given by a normal formula, [4].
Let k, h be sets; then $\langle k, h \rangle$ denotes the corresponding ordered couple. If k, h are classes, we shall use the symbol $[k, h]$ for the ordered couple and it may be interpreted for example as $[k, h] = k \times \{0\} \cup h \times \{1\}$.
If f is a mapping, we shall write $(x)f$ instead of more usual $f(x)$. Every reflexive and transitive (or also antisymmetric) relation will be called a quasi-order (or partial order, respectively).

4. Definitions: Let T be an O_n-ordered class (by $<$). A collection $\{ k_\alpha ; \alpha \in T \}$ of small categories will be

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x) Metatheorems of the present paper are proved without using the axiom of choice: The axiom of choice is needed for applications only.

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called a monotone system of small categories if $\mathcal{k}_\alpha$ is a full subcategory of $\mathcal{k}_{\alpha'}$ whenever $\alpha < \alpha'$. A category $\mathcal{k}$ will be called the union of a monotone system $\{\mathcal{k}_\alpha ; \alpha \in T\}$ of small categories and denoted by $\bigcup_{\alpha \in T} \mathcal{k}_\alpha$ if $\mathcal{k}^\sigma = \bigcup_{\alpha \in T} \mathcal{k}_\alpha^\sigma$ and every $\mathcal{k}_\alpha$ is a full subcategory of $\mathcal{k}$. Evidently $\mathcal{k}$ is small if T is a set.

Let T be an O_n-ordered class (by $<$). Let $\{\mathcal{k}_\alpha ; \alpha \in T\}, \{\mathcal{h}_\alpha ; \alpha \in T\}$ be monotone systems of small categories. Let $\mathcal{G}_\alpha : \mathcal{h}_\alpha \longrightarrow \mathcal{k}_\alpha$ be a functor such that for every $\alpha < \alpha'$ there is $\mathcal{G}_\alpha \cdot {}^{k}\iota_\alpha^{\alpha'} = {}^{h}\iota_\alpha^{\alpha'} \cdot \mathcal{G}_{\alpha'}$, where ${}^{h}\iota_\alpha^{\alpha'} : \mathcal{h}_\alpha \longrightarrow \mathcal{h}_{\alpha'}$, ${}^{k}\iota_\alpha^{\alpha'} : \mathcal{k}_\alpha \longrightarrow \mathcal{k}_{\alpha'}$ are inclusion functors. Then $\{\mathcal{G}_\alpha ; \alpha \in T\}$ will be called a monotone system of small functors. A functor $\varphi : \mathcal{h} \rightarrow \mathcal{k}$, where $\mathcal{h} = \bigcup_{\alpha \in T} \mathcal{h}_\alpha$, $\mathcal{k} = \bigcup_{\alpha \in T} \mathcal{k}_\alpha$, will be called the union of $\{\mathcal{G}_\alpha ; \alpha \in T\}$ and denoted by $\varphi = \bigcup_{\alpha \in T} \mathcal{G}_\alpha$, if for every $\alpha \in T$ there is ${}^{h}\iota_\alpha \cdot \varphi = \mathcal{G}_\alpha \cdot {}^{k}\iota_\alpha$, where ${}^{h}\iota_\alpha : \mathcal{h}_\alpha \rightarrow \mathcal{h}$, ${}^{k}\iota_\alpha : \mathcal{k}_\alpha \rightarrow \mathcal{k}$ are inclusion functors.

5. Definitions: A couple $\langle \mathcal{l}, \mathcal{K} \rangle$ will be called a semiamalgam (of small categories) if $\mathcal{K}$ is a non-empty set of small categories and $\mathcal{l}$ is a full subcategory of each $\mathcal{k} \in \mathcal{K}$.

A semiamalgam $\langle \mathcal{l}, \mathcal{K} \rangle$ will be called an amalgam if $\mathcal{k}_1^\sigma \cap \mathcal{k}_2^\sigma = \mathcal{l}^\sigma$ whenever $\mathcal{k}_1, \mathcal{k}_2 \in \mathcal{K}$, $\mathcal{k}_1 \neq \mathcal{k}_2$.

An amalgam $\langle \mathcal{l}, \mathcal{K} \rangle$ will be called an unglueing of a semiamalgam $\langle \mathcal{l}, \mathcal{K}' \rangle$ if there exists a one-to-one mapping f of the set $\mathcal{K}'$ onto $\mathcal{K}$ such that to each $\mathcal{k} \in \mathcal{K}'$ there exists an isofunctor of $\mathcal{k}$ onto $(\mathcal{k})f$, which is identical on $\mathcal{l}$.

Let $\langle \mathcal{L}, \mathcal{K} \rangle$ be an amalgam. Every **small** category K such that every $k \in \mathcal{K}$ is a full subcategory of K, will be called a **filling** of the amalgam $\langle \mathcal{L}, \mathcal{K} \rangle$.

II. Categorial metatheorem

1. **Metadefinition**: Let V be a property of categories. We shall say that a semiamalgam $\langle \mathcal{L}, \mathcal{K} \rangle$ has V if $\mathcal{L}$ has V and all $k \in \mathcal{K}$ have V. We shall say that V **is amalgamic** if every amalgam with V has a filling with V.

Examples:

In [11] it is proved that every amalgam $\langle \mathcal{L}, \mathcal{K} \rangle$ has a filling K such that $K^\sigma = \bigcup_{k \in \mathcal{K}} k^\sigma$.

a) The property V_0 of being a category is amalgamic.

b) Let $\bar{k}$ be a one-point category, $a \in \bar{k}^\sigma$. Clearly, the following property V_1 (or V_1' or V_1'') is amalgamic: to contain $\bar{k}$ as a full subcategory such that a is a singleton (or a cosingleton or a null object, respectively).

c) It is easy to see that the following property V_2 is amalgamic: a category k has V_2 if and only if $\operatorname{card} H_k(a, b) \leq 1$ for every $a, b \in k^\sigma$. (If $\langle \mathcal{L}, \mathcal{K} \rangle$ is an amalgam with V_2, K its filling, identify all morphisms μ, ν such that $\overleftarrow{\mu} = \overleftarrow{\nu}$, $\overrightarrow{\mu} = \overrightarrow{\nu}$.)

d) Let $\bar{k}$ be a small category, let $\overrightarrow{S}$, $\overleftarrow{S}$ be two classes of cardinal numbers. Evidently the following property V_3 is amalgamic: a category k has V_3 if and only if it contains $\bar{k}$ as a full subcategory, and if $a \in \bar{k}^\sigma$, $b \in k^\sigma - \bar{k}^\sigma$, then $\operatorname{card} H_k(a, b) \in \overrightarrow{S}$, $\operatorname{card} H_k(b, a) \in \overleftarrow{S}$.

e) Let $\bar{k}$ be a one-point category. Clearly, the following property V_4 is amalgamic: a category k has V_4 if and only if it contains $\bar{k}$ as a full subcategory and is connected x).

2. **Metadefinition**: Let V be a property of categories. We shall say that V has a small character if every category K has V if and only if K is a union of a monotone system of small categories with V.

Examples:

a) The property V_0 of being a category is of small character. For, if K is a category, take some O_n-order $<$ for the class K^σ and let k_a be the full subcategory of K such that $k_a^\sigma = \{b \in K^\sigma ; b < a\}$. Then evidently $K = \bigcup_{a \in K^\sigma} k_a$ and $\{k_a ; a \in K^\sigma\}$ is a monotone system of small categories.

b) It is easy to see that the properties V_1 to V_3 from examples 1b) to d) are of small character.

c) Let $\bar{k}$ be a one-point category. It will now be proved that the property V_4, of containing $\bar{k}$ as a full subcategory and being connected, also is of small character. Evidently the union of a monotone system of small categories with V_4 has V_4. Now let K be a category with V_4; we attempt to express it as the union of a system with the required properties. For every small full subcategory h of K choose a small full connected subcategory $\tilde{h}$ of K,

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x) A category k is called connected if for every $a, b \in K^\sigma$ there exist $c_1, \dots, c_n \in k^\sigma$ such that $c_1 = a$, $c_n = b$, $H_k(c_i, c_{i+1}) \cup H_k(c_{i+1}, c_i) \neq \emptyset$ for $i = 1, \dots, n-1$.

which contains $\mathcal{h}$ (this is possible: for any $a, b \in \mathcal{h}^\sigma$ choose $c_1^{a,b}, \dots, c_{n_{a,b}}^{a,b} \in K^\sigma$ such that $c_1^{a,b} = a$, $c_{n_{a,b}}^{a,b} = b$ and $H_K(c_i^{a,b}, c_{i+1}^{a,b}) \cup H_K(c_{i+1}^{a,b}, c_i^{a,b}) \neq \emptyset$, and put $\tilde{\mathcal{h}}^\sigma = \bigcup_{a,b \in \mathcal{h}^\sigma} \{c_1^{a,b}, \dots, c_{n_{a,b}}^{a,b}\}$; it is easy to see that $\tilde{\mathcal{h}}$ is connected). Now let $<$ be an O_n-order for the class $K^\sigma - \bar{\mathcal{k}}^\sigma$, let a_0 be the first element. Put $\mathcal{k}_{a_0} = \bar{\mathcal{k}}$; if $a \in K^\sigma$, $a > a_0$, denote by $\mathcal{h}$ the full subcategory of K such that $\mathcal{h}^\sigma = \{b \in K^\sigma;\ b < a\} \cup \bigcup_{b<a} \mathcal{k}_b^\sigma$, and put $\mathcal{k}_a = \tilde{\mathcal{h}}$. Then evidently $K = \bigcup_{a \in K^\sigma} \mathcal{k}_a$, and $\{\mathcal{k}_a;\ a \in K^\sigma\}$ has the required properties.

3. <u>Metadefinition</u>: Let V be a property of categories. Let $\bar{\mathcal{k}}$ be a small category with V. We shall say that V <u>is</u> <u>$\bar{\mathcal{k}}$-invariant</u> if it satisfies the following conditions:

a) every category with V contains $\bar{\mathcal{k}}$ as a full subcategory;

b) if a small category $\mathcal{k}$ has V and there exists an isofunctor of $\mathcal{k}$ onto a category $\mathcal{h}$, which is identical on $\bar{\mathcal{k}}$, then $\mathcal{h}$ has V.

<u>Metadefinition</u>: Let V be a property of categories. We shall denote by $\bar{V}$ the following property of categories: a category has $\bar{V}$ if and only if it may be fully embedded into a category with V.

<u>Examples</u>: Let V_0 to V_4 be properties described in 1.

a) The property V_0 is evidently $\bar{\mathcal{k}}$-invariant, where $\bar{\mathcal{k}}$ is an empty category. Evidently $V_0 = \bar{V}_0$.

b) Every category with a singleton (or cosingleton or a system of null morphisms) has $\overline{V_1}$ (or $\overline{V_1'}$ or $\overline{V_1''}$ respectively).

c) Now prove that every connected category has $\overline{V_4}$. Let K be a connected category; one may suppose that $\bar{\mathfrak{k}}^\sigma \cap \cap K^\sigma = \emptyset$. Let H be the following category: $H^\sigma = \bar{\mathfrak{k}}^\sigma \cup K^\sigma$, $\bar{\mathfrak{k}}$, K are full subcategories of H, and for $a \in \bar{\mathfrak{k}}^\sigma$, $b \in K^\sigma$ there is $H_H(a, b) = \{\langle a, \emptyset, b \rangle\}$, $H_H(b, a) = \emptyset$. Evidently H has V_4.

4. <u>Categorial Metatheorem</u>: Let $\bar{\mathfrak{k}}$ be a small category. Let V be an amalgamic $\bar{\mathfrak{k}}$-invariant property of small character. Then there exists a category U with property V such that every category K with $\overline{V}$ may be fully embedded in U. Moreover, if K has V, then this embedding is identical on $\bar{\mathfrak{k}}$.

<u>Corollaries</u>: Using the properties V_0 to V_4 described in the examples, it is easy to see that

a) there exists a category in which every category may be fully embedded.

b) There exists a category with a singleton (or cosingleton or null object) in which every category with a singleton (or cosingleton or a system of null morphisms respectively) may be fully embedded.

c) There exists a connected category in which every connected category may be fully embedded.

d) There exists a quasi-ordered class in which every quasi-ordered class may be fully embedded.

There exists a partially ordered class in which every partially ordered class may be fully embedded.

e) Assume given a semigroup Σ with a unit. Then there exists a category U and $a \in U^{\sigma}$ such that a is a generator x) (or cogenerator) of U, $H_U(a, a)$ is isomorphic to Σ, and that every category K containing a generator (or cogenerator, respectively) $b \in K^{\sigma}$ with $H_K(b, b)$ is isomorphic to Σ, may be fully embedded in U. (Cf Appendix II a) of the present paper.)

III. Proof of the Metatheorem

In this section, $\bar{k}$ is a small category, V is an amalgamic $\bar{k}$-invariant property of small character.

1. __Lemma__: Let h, h', l be small categories with V, let h be a full subcategory of h', let $\varphi : h \xrightarrow{onto} l$ be an isofunctor identical on $\bar{k}$. Then there exists a category l' with V and an isofunctor $\varphi' : h' \xrightarrow{onto} l'$, which extends φ; furthermore l is a full subcategory of l'.

__Proof__: Evidently there exists a category l' and an isofunctor $\varphi' : h' \xrightarrow{onto} l'$, which extends φ. Also l' has V, since φ' is identical on $\bar{k}$.

2. __Lemma__: Let $\langle l, \mathcal{K}' \rangle$ be a semiamalgam with V, $k \in \mathcal{K}'$. Then there exists its unglueing $\langle l, \mathcal{K} \rangle$ with property V and such that $k \in \mathcal{K}$.

__Proof__: This is evident.

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x) We recall that a_0 is a generator of a category h if is such that, whenever $\mu, \nu \in H_h(b, c)$, $\mu \neq \nu$, then there exists $\alpha \in H_h(a_0, b)$ with $\alpha\mu \neq \alpha\nu$.

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3. <u>Notation</u>: Let α be a cardinal number, $\mathscr{k}$ and $\mathscr{h}$ small categories. The symbol $card\ \mathscr{k} \setminus \mathscr{h} \leqq \alpha$ is to mean that $\mathscr{h}$ is a full subcategory of $\mathscr{k}$, $card\ \mathscr{k}^{\sigma} - \mathscr{h}^{\sigma} \leqq \alpha$ and for $a \in \mathscr{k}^{\sigma} - \mathscr{h}^{\sigma}$, $b \in \mathscr{k}^{\sigma}$ there is $card\ H_{\mathscr{k}}(a, b) \leqq \alpha$, $card\ H_{\mathscr{k}}(b, a) \leqq \alpha$.

4. <u>Lemma</u>: Let ℓ be a small category with property V, let α be a positive cardinal. Then there exists a semi-amalgam $\langle \ell, \mathscr{K} \rangle$ with V and such that:

1) $card\ \mathscr{k} \setminus \ell \leqq \alpha$ for $\mathscr{k} \in \mathscr{K}$;

2) if $\mathscr{h}$ is a small category with V and $card\ \mathscr{h} \setminus \ell \leqq \alpha$, then there exist $\mathscr{k} \in \mathscr{K}$ and an isofunctor $\varphi : \mathscr{h} \xrightarrow{onto} \mathscr{k}$ identical on ℓ.

<u>Proof</u>: Let $\mathbb{K}$ be the class of all small categories $\mathscr{k}$ with property V and such that $card\ \mathscr{k} \setminus \ell \leqq \alpha$. Let $\wp$ be the following relation on $\mathbb{K}$: $\mathscr{k}_1 \wp \mathscr{k}_2$ if and only if there exists an isofunctor of $\mathscr{k}_1$ onto $\mathscr{k}_2$ identical on ℓ. Evidently $\wp$ is an equivalence on $\mathbb{K}$; denote by $\mathscr{K}$ some choice-class. Now it is sufficient to show that $\mathscr{K}$ is a set. Let M be a set, $M \cap \ell^{\sigma} = \emptyset$, $card\ M = \alpha$; set $S^{\sigma} = M \cup \ell^{\sigma}$. For every $\langle a, b \rangle \in S^{\sigma} \times S^{\sigma}$ let $H(a, b)$ be a set of some triples $\langle a, \alpha, b \rangle$ such that $card\ H(a, b) = \alpha$ and that for $\langle a, b \rangle \in \ell^{\sigma} \times \ell^{\sigma}$ there is $H_{\ell}(a, b) \subset H(a, b)$; set $S^m = \bigcup_{\langle a, b \rangle \in S^{\sigma} \times S^{\sigma}} H(a, b)$. For every $\mathscr{h} \in \mathscr{K}$ choose some one-to-one mapping $\varphi_{\mathscr{h}}$ of the set $\mathscr{h}^{\sigma} \cup \mathscr{h}^m$ into the set $S^{\sigma} \cup S^m$ with the following properties: if $\alpha \in \ell^{\sigma} \cup \ell^m$, then $(\alpha)\varphi_{\mathscr{h}} = \alpha$; if $a \in \mathscr{h}^{\sigma} - \ell^{\sigma}$, then $(a)\varphi_{\mathscr{h}} \in M$; if $\alpha \in \mathscr{h}^m$, then $(\alpha)\varphi_{\mathscr{h}} \in H((\overleftarrow{\alpha})\varphi_{\mathscr{h}}, (\overrightarrow{\alpha})\varphi_{\mathscr{h}})$.

Evidently one may define a composition on the set $(h^{\sigma})\varphi_h \cup (h^{m})\varphi_h$ so as to form a category (denote it by $\tilde{h}$), and $\varphi_h : h \to \tilde{h}$ will then be an isofunctor of h onto $\tilde{h}$. If for $h, k \in \mathcal{K}$ there is $\tilde{h} = \tilde{k}$, then $\varphi_h \cdot \varphi_k^{-1}$ is an isofunctor of h onto k which is identical on ℓ, and therefore $h = k$. Now it is easy to see that the $\tilde{h}$'s, where h varies over $\mathcal{K}$, form a set.

5. __Lemma__: Let α be a positive cardinal number, let $\langle \ell, \mathcal{K} \rangle$ be an amalgam with property V satisfying 1) and 2) from Lemma 4. Let h' be a small category with property V, h its full subcategory with property V, $\varphi : h \xrightarrow{onto} \ell$ an isofunctor identical on $\bar{k}$ and $card\ h' \setminus h \leq \alpha$. Then there exists an isofunctor of h' onto some $k \in \mathcal{K}$ which extends φ.

__Proof__: This follows easily from Lemma 1.

6. __Lemma__: Let $\{h_s ; s \in (S, \prec)\}$ be a monotone system of small categories. Then there exists an order-preserving mapping $f : S \to T$ into the class T of all cardinal numbers such that:

1) $(s_0)f = 0$, where s_0 is the first element of S;

2) for every $s \in S$ with $s \succ s_0$ there is $card\ (h_s \setminus \bigcup_{t \prec s} h_t) \leq (s)f$.

__Proof__: Put $(s_0)f = 0$. If $s \in S$, $s \succ s_0$, put $(s)f = 2^{m} + n$, where $m = \sum_{t \prec s} (t)f$, $n =$

$$= \sup_{a \in h_s^{\sigma},\, b \in h_s^{\sigma} - \bigcup_{t \prec s} h_t^{\sigma}} \{card\ (H_{h_s}(a, b) \cup H_{h_s}(b, a))\} .$$

Then evidently f has the required properties.

7. <u>Construction of</u> U : Let T be the class of all cardinal numbers. For $\alpha \in T$ denote by T_α the set of all $\beta \in T$ less than α . Let $p \in T$ and let $\{ k_m ;$ $m \in T_p \}$ be a monotone system of small categories with property V such that:

A) $k_0 = \bar{k}$;

B) if $m > 0$, then:

if h' is a small category with property V, h its full subcategory with property V, $\varphi : h \to \bigcup_{n<m} k_n$ an isofunctor of h onto a full subcategory of $\bigcup_{n<m} k_n$, which is identical on $\bar{k}$ and if $card\ h' \setminus h \leq m$, then there exists an isofunctor of h' onto a full subcategory of k_m , which extends φ .

Let p' follow to p . We will construct k_p so that $\{ k_m ; m \in T_{p'} \}$ is a monotone system of small categories with V satisfying A) and B). Put $k = \bigcup_{m<p} k_m$.

For every full subcategory ℓ of k with property V choose some amalgam $\langle \ell , \mathcal{H}_\ell \rangle$ satisfying 1) and 2) from Lemma 4, where one puts $\alpha = p$. Let $\langle \ell , \mathcal{H}_\ell \rangle$ be an unglueing of the semiamalgam $\langle \ell , \mathcal{K}_\ell \cup \{ k \} \rangle$ such that $k \in \mathcal{H}_\ell$, let K_ℓ be its filling with property V. Denote by $\mathcal{L}$ the set of all K_ℓ (ℓ varies over all full subcategories of k with property V). Let $\langle k, \mathcal{K} \rangle$ be an unglueing of the semiamalgam $\langle k, \mathcal{L} \rangle$. Let k_p be a filling with V of $\langle k, \mathcal{K} \rangle$.

Now it is easy to see that $\{k_m ; m \in T_{p'}\}$ is a monotone system of small categories with V satisfying A). Now prove B). It is sufficient to prove that if h', h are small categories with property V, $\operatorname{card} h' \setminus h \leq p$, $\varphi : h \to k = \bigcup_{m<p} k_m$ an isofunctor onto a full subcategory of k identical on $\bar{k}$, then there exists an isofunctor ψ of h' onto a full subcategory of k_p which extends φ. To prove this assertion, first put $\ell = (h)\varphi$. Then, using Lemma 5, there exist $\ell' \in \mathcal{K}_\ell$ and an isofunctor $\varphi' : h' \xrightarrow{onto} \ell'$ such that $\varphi'/_h = \varphi$. Then there exist $\ell'' \in \mathcal{H}_\ell$ and an isofunctor $\varphi'' : \ell' \xrightarrow{onto} \ell''$ which is identical on ℓ, consequently $(\varphi' . \varphi'')/_h = \varphi$.

Denote by $\iota_{\ell''} : \ell'' \longrightarrow K_\ell$ the inclusion functor onto a full subcategory of K_ℓ; let $K \in \mathcal{K}$, let $\chi : K_\ell \xrightarrow{onto} K$ be an isofunctor which is identical on k (and consequently also on ℓ), and denote by $\iota : K \longrightarrow k_p$ the inclusion functor onto a full subcategory of k_p. Put $\psi = \varphi' . \varphi'' . \iota_{\ell''} . \chi . \iota$. Evidently ψ is an isofunctor onto a full subcategory of k_p and $(\mu)\varphi'\varphi''\iota_{\ell''}\chi\iota = (\mu)\varphi$ for $\mu \in h$. This concludes the proof of B).

By transfinite induction one obtains a monotone system $\{k_m ; m \in T\}$ of small categories with V satisfying A) and B). Put $U = \bigcup_{m \in T} k_m$. Then evidently U has V.

8. __Proposition__: Let H be a category with property V. Then there exists an isofunctor of H onto a full subcategory of U which is identical on $\bar{k}$.

Proof: Using Lemma 6, one may suppose that $H = \bigcup_{s \in T'} h_s$, where $\{h_s ; s \in T'\}$ is a monotone system of small categories with property V, T' is a subclass of the class T of all cardinal numbers, $0 \in T'$, $h_0 = \bar{k}$ and $card(h_s \setminus \bigcup_{t<s} h_t) \leq s$ for $0 < s \in T'$. Now it is easy to construct an isofunctor Φ of H onto a full subcategory of U. Put $\Phi = \bigcup_{s \in T'} \varphi_s$, where φ_s is the following isofunctor of h_s onto a full subcategory of k_s: $\varphi_0 : h_0 = \bar{k} \longrightarrow \bar{k} = k_0$ is identical; for $s \in T'$, $s > 0$, put $\varphi = \bigcup_{t \in T', t<s} \varphi_t$, $h' = h_s$, $h = \bigcup_{t \in T', t<s} h_t$ and use B) from the construction of U.

The proof of Metatheorem is complete.

IV. Metatheorem for additive categories

1. We recall the well-known concepts of additive categories and related notions:

Definition: Let K be a category, $+$ a partial addition on K^m such that: if $\alpha + \beta$ is defined, then $\overleftarrow{\alpha} = \overleftarrow{\beta}$, $\overrightarrow{\alpha} = \overrightarrow{\beta}$, every $[H_K(a,b), +/H_K(a,b)]$ is an abelian group, and if $\mu \in H_K(c,a)$, $\alpha, \beta \in H_K(a,b)$, $\nu \in H_K(b,d)$, then $\mu \cdot (\alpha + \beta) \cdot \nu = \mu \cdot \alpha \cdot \nu + \mu \cdot \beta \cdot \nu$. We shall say that then $[K, +]$ is an a-category x). Moreover, if

x) In [8], a-categories are called preadditive categories. In the present paper the term a-category was chosen for the sake of analogies with the following parts of the paper.

every pair of objects of K has a biproduct x) in K, then $[K, +]$ will be called an additive category. For an a-category $A = [K, +]$ set $|A| = K$; K will be also called the underlying category of A. We shall say that A is small whenever K is small.

2. <u>Definition</u>: Let A, A' be a-categories. Φ will be called an a-functor of A into A' if it is a functor of $|A|$ into $|A'|$ such that $(\alpha + \beta)\Phi = (\alpha)\Phi + (\beta)\Phi$ whenever $\alpha + \beta$ is defined. Moreover if Φ is an isofunctor, then it will be called an a-isofunctor or a-embedding. If Φ is an isofunctor of $|A|$ onto a full subcategory of $|A'|$, then it will be called a full a-isofunctor or a full a-embedding. Let A, A' be a-categories. We shall say that A is an a-subcategory of A' if $|A| \subset |A'|$ and the inclusion-functor is a-embedding. Moreover if it is a full a-embedding, then A will be called a full a-subcategory of A'.

3. <u>Definition</u>: A couple $\langle \mathcal{L}, \mathcal{K} \rangle$ will be called an a-semiamalgam if $\mathcal{K}$ is a non-empty set of small a-categories and $\mathcal{L}$ is a full a-subcategory of all $k \in \mathcal{K}$.

An a-semiamalgam $\langle \mathcal{L}, \mathcal{K} \rangle$ will be called an a-amalgam if $|k_1|^{\sigma} \cap |k_2|^{\sigma} = |\mathcal{L}|^{\sigma}$ whenever k_1, $k_2 \in \mathcal{K}$, $k_1 \neq k_2$.

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x) $\langle b, \{\iota_1, \pi_1, \iota_2, \pi_2\} \rangle$ is called a biproduct of objects a_1, a_2 in a-category $[K, +]$ if $\iota_i \in H_K(a_i, b)$, $\pi_i \in H_K(b, a_i)$, $i = 1, 2$ and $\iota_i \pi_i = e_{a_i}$, $(i = 1, 2)$, $\pi_1 \iota_1 + \pi_2 \iota_2 = e_b$.

An $\mathcal{a}$-amalgam $\langle \mathcal{L}, \mathcal{K} \rangle$ will be called an $\mathcal{a}$-unglueing of an $\mathcal{a}$-semiamalgam $\langle \mathcal{L}, \mathcal{K}' \rangle$ if there exists a one-to-one mapping f of the set $\mathcal{K}'$ onto $\mathcal{K}$ such that for every $\mathcal{k} \in \mathcal{K}'$ there exists an $\mathcal{a}$-isofunctor of $\mathcal{k}$ onto $(\mathcal{k})f$, which is identical on $\mathcal{L}$.

Let $\langle \mathcal{L}, \mathcal{K} \rangle$ be an $\mathcal{a}$-amalgam. A small $\mathcal{a}$-category K such that every $\mathcal{k} \in \mathcal{K}$ is a full $\mathcal{a}$-subcategory of K, will be called an $\mathcal{a}$-filling of the $\mathcal{a}$-amalgam $\langle \mathcal{L}, \mathcal{K} \rangle$.

4. In analogy with the notions of a monotone system of small categories and its union, one may define the corresponding $\mathcal{a}$-notions of a monotone system of small $\mathcal{a}$-categories and its union.

In analogy with metanotions of amalgamic property, $\bar{\mathcal{k}}$-invariant property, property with small character, one may define the corresponding $\mathcal{a}$-metanotions, of $\mathcal{a}$-amalgamic property, $\bar{\mathcal{k}}$ - $\mathcal{a}$-invariant property (where $\bar{\mathcal{k}}$ is a small $\mathcal{a}$-category) and property of $\mathcal{a}$-small character. If V is a property of $\mathcal{a}$-categories, the definition of $\bar{V}$ is also evident.

5. <u>Metatheorem</u>: Let $\bar{\mathcal{k}}$ be a small $\mathcal{a}$-category, V an $\mathcal{a}$-amalgamic and $\bar{\mathcal{k}}$ - $\mathcal{a}$-invariant property of an $\mathcal{a}$-small character. Then there exists an $\mathcal{a}$-category U with V such that every $\mathcal{a}$-category with $\bar{V}$ may be fully $\mathcal{a}$-embedded in U. Moreover, for $\mathcal{a}$-categories with V this $\mathcal{a}$-embedding is identical on $\bar{\mathcal{k}}$.

6. In Appendix II b) of the present paper a proof of the assertion is sketched that the property of being an $\mathcal{a}$-category is $\mathcal{a}$-amalgamic. Evidently it is of $\mathcal{a}$-small cha-

racter. Thus, using the fact that every $\mathcal{a}$-category may be fully $\mathcal{a}$-embedded in an additive category, [1], we have the following results:

a) There exists an additive category in which every $\mathcal{a}$-category may be fully $\mathcal{a}$-embedded.

b) There exists an additive category U such that for every $a, b \in |U|^{\sigma}$, $H_U(a, b)$ is a torsion group (or a finite group), and with the property that every $\mathcal{a}$-category A with $H_A(a, b)$ is a torsion group (or a finite group, respectigely) for every $a, b \in |A|^{\sigma}$ may be fully $\mathcal{a}$-embedded in U. (The proof is sketched in Appendix II c),e).)

c) There exists an additive category U with a generator (or a cogenerator) $c \in |U|^{\sigma}$ such that $H_U(c, c)$ is isomorphic with a given ring with unit, and if A is any $\mathcal{a}$-category with a generator (or a cogenerator, respectively) $a \in |A|^{\sigma}$ such that the rings $H_U(c, c)$ and $H_A(a, a)$ are isomorphic, then A may be fully $\mathcal{a}$-embedded in U. The $\mathcal{a}$-embedding extends the ring-isomorphism of $H_A(a, a)$ onto $H_U(c, c)$. (The proof is sketched in Appendix II d),e).)

7. Note: It can be shown that the situation is quite analoguous if the sets of morphisms from an object to an object are not necessarily abelian groups but universal algebras of a given type and satisfy a given set of equalities (of course, the operations must be distributive with respect to the composition of morphisms).

8. <u>Proof of the Metatheorem for</u> $\mathcal{a}$<u>-categories</u>: This will only be sketched. Let $\bar{\mathcal{k}}$ be a small $\mathcal{a}$-category, let V be an $\mathcal{a}$-amalgamic $\bar{\mathcal{k}}$ - $\mathcal{a}$-invariant property of $\mathcal{a}$-small character. The lemmas analogous to Lemmas III.1 and III.2 for $\mathcal{a}$-categories and $\mathcal{a}$-functors are easily formulated and proved. We shall now formulate and prove the analogue to III.3 and III.4:

<u>Notation</u>: Let α be a cardinal number, $\mathcal{k}, \mathcal{h}$ small $\mathcal{a}$-categories. Then $card\ \mathcal{k} \setminus \mathcal{h} \leqq \alpha$ denotes that $\mathcal{h}$ is a full $\mathcal{a}$-subcategory of $\mathcal{k}$ and $card\ |\mathcal{k}| \setminus |\mathcal{h}| \leqq \alpha$.

<u>Lemma</u>: Let $\mathcal{l}$ be a small $\mathcal{a}$-category with V, let α be a positive cardinal. Then there exists an $\mathcal{a}$-semiamalgam $\langle \mathcal{l}, \mathcal{K} \rangle$ with V such that:

1) if $\mathcal{k} \in \mathcal{K}$, then $card\ \mathcal{k} \setminus \mathcal{l} \leqq \alpha$;

2) if $\mathcal{h}$ is a small $\mathcal{a}$-category with V and $card\ \mathcal{h} \setminus \mathcal{l} \leqq \alpha$, then there exist a $\mathcal{k} \in \mathcal{K}$ and an $\mathcal{a}$-isofunctor $\varphi : \mathcal{h} \xrightarrow{onto} \mathcal{k}$ which is identical on $\mathcal{l}$.

<u>Proof</u>: Let $\mathbb{K}'$ be the class of all small $\mathcal{a}$-categories $\mathcal{k}$ with V such that $card\ \mathcal{k} \setminus \mathcal{l} \leqq \alpha$, let $\mathbb{K} = \{|\mathcal{k}|;\ \mathcal{k} \in \mathbb{K}'\}$. Let ρ be the following relation on $\mathbb{K}$: $|\mathcal{k}_1|\ \rho\ |\mathcal{k}_2|$ if and only if there exists an isofunctor of $|\mathcal{k}_1|$ onto $|\mathcal{k}_2|$ which is identical on $|\mathcal{l}|$. Denote by $\mathcal{H}$ some choice-class. In the proof of Lemma III.4 it is proved that $\mathcal{H}$ is a set. For every $\mathcal{h} \in \mathcal{H}$ denote by $\mathbb{K}_{\mathcal{h}}$ the set of all $\mathcal{a}$-categories $\mathcal{k}$ such that $\mathcal{l}$ is an $\mathcal{a}$-subcategory of $\mathcal{k}$ and $|\mathcal{k}| = \mathcal{h}$; put $\mathcal{K}' = \bigcup_{\mathcal{h} \in \mathcal{K}} \mathbb{K}_{\mathcal{h}}$. The $\mathcal{a}$-semiamalgam $\langle \mathcal{l}, \mathcal{K}' \rangle$ has the required proper-

ties, concluding the proof of the lemma.
Now it is easy to complete the proof of the Metatheorem for $\mathcal{a}$-categories using the analogues to III.6, III.7, III.8; this is left to the reader.

V. Bicategorial metatheorem

1. We recall the well-known notion of bicategory, [5], and of related notions:

__Definition__: Let K be a category, I, P its subcategories such that

1) $I^m \cap P^m$ is the class of all isomorphisms of K;
2) each $\iota \in I^m$ is a monomorphism of K;
   each $\pi \in P^m$ is an epimorphism of K;
3) to every $\alpha \in K^m$ there exist $\iota \in I^m$, $\pi \in P^m$ such that $\alpha = \pi \cdot \iota$;
4) if $\iota, \iota' \in I^m$, $\pi, \pi' \in P^m$ have $\pi \cdot \iota = \pi' \cdot \iota'$, then there exists an isomorphism ρ of K such that $\pi = \pi' \cdot \rho$, $\iota' = \rho \cdot \iota$.

Then $[K, I, P]$ is termed a $\mathcal{b}$-category x); it is termed small if K is small. Let $\mathcal{B} = [K, I, P]$ be a $\mathcal{b}$-category, set $|\mathcal{B}| = K$, $I_{\mathcal{B}} = I^m$, $P_{\mathcal{B}} = P^m$. Then K will be called also an underlying category of $\mathcal{B}$, $I_{\mathcal{B}}$ the class of all injections of $\mathcal{B}$, $P_{\mathcal{B}}$ the class of all projections of $\mathcal{B}$.

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x) The term $\mathcal{b}$-category instead of bicategory, was chosen for the sake of analogies with other parts of the present paper.

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2. <u>Definition</u>: Let $\mathcal{B}$, $\mathcal{B}'$ be $\mathcal{b}$-categories. A functor Φ of $|\mathcal{B}|$ into $|\mathcal{B}'|$ is called a $\mathcal{b}$-functor if $(I_{\mathcal{B}})\Phi \subset I_{\mathcal{B}'}$, $(P_{\mathcal{B}})\Phi \subset P_{\mathcal{B}'}$. Φ will be called a $\mathcal{b}$-isofunctor of $\mathcal{B}$ into $\mathcal{B}'$ if it is an isofunctor of $|\mathcal{B}|$ into $|\mathcal{B}'|$ and $I_{\mathcal{B}'} \cap (|\mathcal{B}|^{m})\Phi = (I_{\mathcal{B}})\Phi$, $P_{\mathcal{B}'} \cap (|\mathcal{B}|^{m})\Phi = (P_{\mathcal{B}})\Phi$. If, moreover, Φ is an isofunctor of $|\mathcal{B}|$ onto a full subcategory of $|\mathcal{B}'|$, then it will be called a full $\mathcal{b}$-isofunctor or a full $\mathcal{b}$-embedding. Let $\mathcal{B}$, $\mathcal{B}'$ be $\mathcal{b}$-categories. We shall say that $\mathcal{B}$ is a (full) $\mathcal{b}$-subcategory of $\mathcal{B}'$ if $|\mathcal{B}| \subset$ $\subset |\mathcal{B}'|$ and the inclusion functor is a (full) $\mathcal{b}$-embedding.

3. The definitions of a $\mathcal{b}$-semiamalgam and its $\mathcal{b}$-unglueing, and of a $\mathcal{b}$-amalgam and its $\mathcal{b}$-filling are evident. The definition of a monotone system of $\mathcal{b}$-categories, a monotone system of $\mathcal{b}$-embeddings and their union is evident. If $\bar{k}$ is a small $\mathcal{b}$-category, then the metadefinitions of $\mathcal{b}$-amalgamic, $\bar{k}$ - $\mathcal{b}$-invariant property of $\mathcal{b}$-small character are evident. It is also evident that the following metatheorem holds:

<u>Metatheorem</u>: Let $\bar{k}$ be a small $\mathcal{b}$-category. Let V be a $\mathcal{b}$-amalgamic, $\bar{k}$ - $\mathcal{b}$-invariant property of $\mathcal{b}$-small character. Then there exists a $\mathcal{b}$-category U with property V such that every $\mathcal{b}$-category with property V may be fully $\mathcal{b}$-embedded in it; this $\mathcal{b}$-embedding is identical on $\bar{k}$.

4. However, as shown in the Appendix, II f), this metatheorem is not useful, because the property of being a $\mathcal{b}$-category is not $\mathcal{b}$-amalgamic. (The question as to whether there

exists a $\mathcal{b}$-category in which every $\mathcal{b}$-category may be fully $\mathcal{b}$-embedded remains open.) We shall give a more general metatheorem, which has more satisfactory applications.

5. <u>Metadefinition</u>: Let W be a property of $\mathcal{b}$-embeddings. It will be said that W <u>is monotonically additive</u> if the union of every monotone system of $\mathcal{b}$-embeddings with W has W. It will be said that W <u>is categorial</u> if

a) every $\mathcal{b}$-isofunctor onto has W and

b) the composition of two $\mathcal{b}$-embeddings with W has W.

6. <u>Metadefinition</u>: Let V be a property of $\mathcal{b}$-categories, W a property of $\mathcal{b}$-embeddings. It will be said that V <u>has a $\mathcal{b}$-small W-character</u> if a $\mathcal{b}$-category K has V if and only if K is the union of a monotone system $\{k_\alpha ; \alpha \in T\}$ of small $\mathcal{b}$-categories with V such that for any $\alpha < \alpha'$ the inclusion $\mathcal{b}$-functor $\iota_\alpha^{\alpha'} : k_\alpha \to k_{\alpha'}$ has W.

It will be said that V <u>is $\mathcal{b}$-amalgamic with respect to W</u> if it has the following property: if $\langle \ell , \mathcal{K} \rangle$ is a $\mathcal{b}$-amalgam with V such that the inclusion $\mathcal{b}$-functor $\iota_k : \ell \to k$ has W for every $k \in \mathcal{K}$, then there exists its $\mathcal{b}$-filling K with V such that for every $k \in \mathcal{K}$ the inclusion $\mathcal{b}$-functor $\bar{\iota}_k : k \to K$ has W.

7. <u>Metatheorem for $\mathcal{b}$-categories</u>: Let W be a categorial property of $\mathcal{b}$-embeddings. Let $\bar{k}$ be a small $\mathcal{b}$-category. Let V be a property of $\mathcal{b}$-categories, which is $\bar{k}$-$\mathcal{b}$-invariant, $\mathcal{b}$-amalgamic with respect to W and is of $\mathcal{b}$-small W-character. Then there exists a $\mathcal{b}$-category with V in which every $\mathcal{b}$-category with V may

be fully $\mathcal{b}$-embedded. The $\mathcal{b}$-embedding is identical on $\bar{k}$, and has W whenever W is monotonically additive.
Proof is analoguous to that of Metatheorem V.3 and therefore it is ommitted.

8. Definition: Let ℓ be a full $\mathcal{b}$-subcategory of k. It will be said that ℓ is a good $\mathcal{b}$-subcategory of k if it has the following property: If $\mu \in |k|^m$ and either $\overleftarrow{\mu} \in |\ell|^\sigma$ or $\overrightarrow{\mu} \in |\ell|^\sigma$, then there exist $\pi \in P_k$, $\iota \in I_k$ such that $\mu = \pi \cdot \iota$ and $\overrightarrow{\pi} \in |\ell|^\sigma$.
A $\mathcal{b}$-category K will be termed a good $\mathcal{b}$-category if $K = \bigcup_{\alpha \in T} k_\alpha$, where $\{k_\alpha ; \alpha \in T\}$ is a monotone system of small $\mathcal{b}$-categories such that for any $\alpha < \alpha'$ k_α is a good $\mathcal{b}$-subcategory of $k_{\alpha'}$.

9. Let W be the following property of $\mathcal{b}$-embeddings: $\iota : \ell \to k$ has W if and only if it is a $\mathcal{b}$-embedding onto a good $\mathcal{b}$-subcategory of k. In Appendix, II g) it is shown that W is categorial and monotonically additive. Let V be the property of being a good $\mathcal{b}$-category. Then V is of $\mathcal{b}$-small W-character; this follows immediately from the definition. V is $\bar{k}$-invariant, where $\bar{k}$ is an empty $\mathcal{b}$-category. In Appendix, II h) it is shown that V is $\mathcal{b}$-amalgamic with respect to W. Thus we have the following result:

Corollary to the Metatheorem for $\mathcal{b}$-categories:
There exists a good $\mathcal{b}$-category in which every good $\mathcal{b}$-category may be fully $\mathcal{b}$-embedded. The $\mathcal{b}$-embedding is onto a good $\mathcal{b}$-subcategory.

10. Now we give some conditions for a $\mathcal{b}$-category to be a good $\mathcal{b}$-category.

Lemma 1: A $\mathcal{b}$-category dual to a good $\mathcal{b}$-category is itself a good $\mathcal{b}$-category.

Proof: This follows immediately from the definition of a good $\mathcal{b}$-category.

Lemma 2: A $\mathcal{b}$-category is a good $\mathcal{b}$-category if and only if its skeleton is a good $\mathcal{b}$-category.

Proof: Let B be a $\mathcal{b}$-category, S be its skeleton. For every $a \in |B|^{\sigma}$ choose an isomorphism σ_a of B such that $\overleftarrow{\sigma_a} = a$, $\overrightarrow{\sigma_a} \in |S|^{\sigma}$. Let $\Gamma : B \to S$ be a $\mathcal{b}$-functor such that $(a)\Gamma = \overrightarrow{\sigma_a}$, $(\mu)\Gamma = \sigma_{\overleftarrow{\mu}}^{-1} \cdot \mu \cdot \sigma_{\overrightarrow{\mu}}$. If B is a good $\mathcal{b}$-category, then $B = \bigcup_{\alpha \in T} b_{\alpha}$, where $\{b_{\alpha} ; \alpha \in T\}$ is a monotone system of small $\mathcal{b}$-categories such that b_{α} is a good $\mathcal{b}$-subcategory of $b_{\alpha'}$, whenever $\alpha < \alpha'$. Put $s_{\alpha} = (b_{\alpha})\Gamma$. Then evidently $S = \bigcup_{\alpha \in T} s_{\alpha}$ and $\{s_{\alpha} ; \alpha \in T\}$ is a monotone system of small $\mathcal{b}$-categories, which has the required property. Consequently S is a good $\mathcal{b}$-category. Conversely, if S be a good $\mathcal{b}$-category, we shall prove that B is good. Then $S = \bigcup_{\alpha \in T} s_{\alpha}$, where $\{s_{\alpha} ; \alpha \in T\}$ is a monotone system of small $\mathcal{b}$-categories such that s_{α} is a good $\mathcal{b}$-subcategory of $s_{\alpha'}$ whenever $\alpha < \alpha'$. The property of being a $\mathcal{b}$-category is of $\mathcal{b}$-small character, as shown in Appendix, II 1). Consequently $B = \bigcup_{\beta \in Z} k_{\beta}$, where $\{k_{\beta} ; \beta \in Z\}$ is a monotone system of small $\mathcal{b}$-categories. For every $\beta \in Z$ denote by α_{β} the smallest

$\alpha \in T$ such that $(k_\beta)\Gamma$ is a $\mathcal{b}$-subcategory of $\mathcal{s}_\alpha$. Let now $\mathcal{b}_\beta$ be the full $\mathcal{b}$-subcategory of B such that $|\mathcal{b}_\beta|^\sigma = |\mathcal{s}_{\alpha_\beta}|^\sigma \cup |k_\beta|^\sigma$. Then evidently $B = \bigcup_{\beta \in Z} \mathcal{b}_\beta$ and the system $\{\mathcal{b}_\beta ; \beta \in Z\}$ has all the required properties.

We recall the well-known definition:

Definition. A $\mathcal{b}$-category $\mathcal{B}$ is termed well-powered (or co-well-powered) if for every $a \in |\mathcal{B}|^\sigma$ there exists a set $\mathcal{I}_a \subset I_\mathcal{B}$ (or $\mathcal{P}_a \subset P_\mathcal{B}$) such that, for every $\iota \in I_\mathcal{B}$, $\vec{\iota} = a$, (or $\pi \in P_\mathcal{B}$, $\overleftarrow{\pi} = a$) there exists an $\iota' \in \mathcal{I}_a$ (or $\pi' \in \mathcal{P}_a$) and an isomorphism σ such that $\iota = \sigma \cdot \iota'$ (or $\pi = \pi' \cdot \sigma$ respectively).

Lemma 3: $\mathcal{B}$ is a good $\mathcal{b}$-category if and only if it is well-powered and co-well-powered.

Proof: Let $\mathcal{B}$ be a good $\mathcal{b}$-category; let $\mathcal{B} = \bigcup_{\alpha \in T} \mathcal{b}_\alpha$, where $\{\mathcal{b}_\alpha ; \alpha \in T\}$ is a monotone system of small $\mathcal{b}$-categories such that, for $\alpha < \alpha'$, $\mathcal{b}_\alpha$ is a good $\mathcal{b}$-subcategory of $\mathcal{b}_{\alpha'}$. Let $a \in |\mathcal{B}|^\sigma$. Choose $\alpha \in T$ such that $a \in |\mathcal{b}_\alpha|^\sigma$; let $\mathcal{I}$ be the class of all $\mu \in I_\mathcal{B}$, $\vec{\mu} = a$. Since $\mathcal{b}_\alpha$ is a good $\mathcal{b}$-subcategory of $\mathcal{B}$, each $\mu \in \mathcal{I}$ may be expressed as $\mu = \sigma \cdot \nu$, where $\sigma \in P_\mathcal{B}$, $\nu \in I_\mathcal{B}$, $\overleftarrow{\nu} \in |\mathcal{b}_\alpha|^\sigma$; but then σ must be an isomorphism. Consequently $\mathcal{B}$ is well-powered. Analogously it may be proved that $\mathcal{B}$ is co-well-powered.

Conversely, let $\mathcal{B}$ be a well-powered co-well-powered $\mathcal{b}$-category. Let $\mathcal{S}$ be its skeleton. We shall prove that $\mathcal{S}$ is good. If $\mathcal{s}$ is a small full subcategory of $|\mathcal{S}|$, denote by $\bar{\mathcal{s}}$ the smallest good $\mathcal{b}$-subcategory of $\mathcal{S}$ such that $|\bar{\mathfrak{s}}| \supset \mathfrak{s}$, $\bar{\mathfrak{s}}$ is a small $\mathcal{b}$-category. Indeed, put $A_0 = \mathfrak{s}^{\sigma}$; for n odd denote by A_n the set of all $a \in |\mathcal{S}|^{\sigma}$ such that there exists $\iota \in I_{\mathcal{S}}$ with $\overleftarrow{\iota} = a$, $\overrightarrow{\iota} \in A_{n-1}$; for n even denote by A_n the set of all $a \in |\mathcal{S}|^{\sigma}$ such that there exists a $\pi \in P_{\mathcal{S}}$ with $\overrightarrow{\pi} = a$, $\overleftarrow{\pi} \in A_{n-1}$; let $\bar{\mathfrak{s}}$ be a full $\mathcal{b}$-subcategory of $\mathcal{S}$ such that $|\bar{\mathfrak{s}}|^{\sigma} = \bigcup_{n=0}^{\infty} A_n$.

If $\{\mathfrak{s}_{\alpha}; \alpha \in T\}$ is a monotone system of small categories such that $|\mathcal{S}| = \bigcup_{\alpha \in T} \mathfrak{s}_{\alpha}$, then $\{\bar{\mathfrak{s}}_{\alpha}; \alpha \in T\}$ has the required properties.

11. Now we show, using Lemmas 1 to 3, that most of the usual bicategories are good $\mathcal{b}$-categories.

Let E_{ns} be the category of all sets and all their mappings. Let $\Phi : E_{ns} \longrightarrow E_{ns}$ be a functor, covariant (or contravariant) such that

(*) for every $a \in E^{\sigma}_{ns}$ the class $\{b \in E^{\sigma}_{ns};$
$(b)\Phi = a\}$ is a set.

If $a \in E^{\sigma}_{ns}$, $\alpha \in E^{m}_{ns}$ denote $(a)\Phi$ by a^{Φ} and $(\alpha)\Phi$ by α^{Φ}.

One may then define the following category E^{Φ} x):

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x) The definition of the category E^{Φ} was given by A. Pultr and Z. Hedrlín.

$|E^{\Phi}|^{\sigma}$ is the class of all couples $\langle a, x \rangle$, where $a \in |E_{ns}|^{\sigma}, x \subset a^{\Phi}$; $H_{E^{\Phi}}(\langle a, x\rangle, \langle a', x'\rangle)$ is the set of all $\alpha^* = \langle\langle a, x\rangle, \alpha, \langle a', x'\rangle\rangle$, where $\alpha : a \to a'$ is a mapping such that $(x)\alpha^{\Phi} \subset x'$ (or $x \supset (x')\alpha^{\Phi}$ respectively).

It is easy to see that E^{Φ} may be bicategorized naturally in two ways (the contravariant case is indicated in parentheses):

P_1 is the class of all $\alpha^* : \langle a, x\rangle \to \langle a', x'\rangle$ such that $(a)\alpha = a'$, $(x)\alpha^{\Phi} = x'$ (or $(x')\alpha^{\Phi} = x \cap \cap (a'^{\Phi})\alpha^{\Phi}$).

I_1 is the class of all $\alpha^* : \langle a, x\rangle \to \langle a', x'\rangle$ such that $\alpha : a \to a'$ is one-to-one into and $(x)\alpha^{\Phi} \subset x'$ (or $x \supset (x')\alpha^{\Phi}$).

P_2 is the class of all $\alpha^* : \langle a, x\rangle \to \langle a', x'\rangle$ such that $\alpha : a \xrightarrow{onto} a'$ and $(x)\alpha^{\Phi} \subset x'$ (or $x \supset (x')\alpha^{\Phi}$).

I_2 is the class of all $\alpha^* : \langle a, x\rangle \to \langle a', x'\rangle$ such that $\alpha : a \to a'$ is one-to-one into and $(x)\alpha^{\Phi} = x' \cap \cap (a^{\Phi})\alpha^{\Phi}$ (or $x = (x')\alpha^{\Phi}$ respectively).

Then, using Lemma 2 and 3, it is easy to see that both $[E^{\Phi}, P_1, I_1]$ and $[E^{\Phi}, P_2, I_2]$ are good b-categories for every functor (covariant or contravariant).

$\Phi : E_{ns} \to E_{ns}$, satisfying (*). Also all full b-subcategories are good b-categories. Thus for every covariant functor $\Phi : E_{ns} \to E_{ns}$ satisfying (*) the category of all Φ-spaces and Φ-morphisms [6] bicategorized as before is a good b-category.

VI. Relative Metatheorem

1. Definition: Let M be a category. Let $\mathfrak{k}, \mathfrak{h}$ be subcategories of M, $\iota_{\mathfrak{k}} : \mathfrak{k} \to M$, $\iota_{\mathfrak{h}} : \mathfrak{h} \to M$ the inclusion functors, $\varphi : \mathfrak{k} \to \mathfrak{h}$ a functor. We shall say that φ is an M-functor if there exists a natural transformation of $\iota_{\mathfrak{k}}$ into $\varphi \iota_{\mathfrak{h}}$ (i.e. if for every $a \in \mathfrak{k}^{\sigma}$ there exists a morphism $\mu_a \in H_M(a, (a)\varphi)$ such that for every $\alpha \in H_{\mathfrak{k}}(a, b)$ there is $\alpha \cdot \mu_b = \mu_a \cdot (\alpha)\varphi$). If $\varphi : \mathfrak{k} \to \mathfrak{h}$ is an isofunctor into and $\iota_{\mathfrak{k}}$ and $\varphi \iota_{\mathfrak{h}}$ are naturally equivalent (i.e. all $\mu_a \in H_M(a, (a)\varphi)$ are isomorphisms of M), we shall say that φ is an M-isofunctor into or M-embedding. If φ is a full (or small) embedding and also an M-embedding, we shall say that it is a full (or small respectively) M-embedding.

2. Definition: Let M be a category. A semiamalgam $\langle \mathcal{L}, \mathcal{K} \rangle$ will be called an M-semiamalgam if all $\mathfrak{k} \in \mathcal{K}$ are subcategories of M.

The definition of M-unglueing of an M-semiamalgam is evident. The definition of an M-amalgam and its M-filling is also evident.

3. Metadefinitions: Let M be a category, W a property of M-embeddings. We shall say that W is categorial if

a) every M-isofunctor onto has W and

b) the composition of any two M-embeddings with W has W.

We shall say that W is monotonically additive if the union of every monotone system of small M -embeddings with W has W .

4. <u>Metadefinitions</u>: Let M be a category, $\bar{k}$ its small subcategory. Let V be a property of subcategories of M , W a property of M -embeddings. The metadefinitions of the following metanotions are analogous to those given before (cf V.5 and 6):

V is of M -small W -character; V is M -amalgamic with respect to W ; V is $\bar{k}$ - M -invariant.

5. We recall that a category is called replete (cf [3]) if with each object a it also contains a proper class of objects isomorphic to a .

<u>Relative Metatheorem</u>: Let M be a replete category, $\bar{k}$ its small subcategory. Let W be a categorial property of M -embeddings, which is monotonically additive. Let V be a $\bar{k}$ - M -invariant property of subcategories of M, which is of M -small W -character and is M - amalgamic with respect to W .

Then there exists a subcategory U of M with property V such that every subcategory of M with V can be fully M -embedded in U . This M -embedding is identical on $\bar{k}$ and has W .

<u>Proof</u>: This is given in the next section.

6. <u>Corollaries</u>: a) Let M be a replete category, $\bar{k}$ the empty category. It is easy to see that the property V of being a subcategory of M and also W of being an M - embedding satisfy the requirements of the Metatheorem. Thus

we have the following result: Let M be a replete category; then there exists a subcategory U in which every subcategory of M may be fully M-embedded.

b) There exists a concrete category in which every concrete category may be fully embedded.

c) There exists a concrete category with a singleton (or cosingleton or null object) in which every concrete category with a singleton (or cosingleton or null morphisms) may be fully embedded.

d) There exists a connected concrete category in which every connected concrete category may be fully embedded.

e) If M is an $\mathcal{a}$-category, then every $|M|$-isofunctor is an $\mathcal{a}$-isofunctor. Consequently we have the following result:

Let M be a replete $\mathcal{a}$-category. Then there exists an $\mathcal{a}$-subcategory in which every $\mathcal{a}$-subcategory of M may be fully $\mathcal{a}$-embedded.

f) There exists a category of (abelian) groups in which every category of (abelian) groups may be fully additively embedded.

g) If M is a $\mathcal{b}$-category, then every $|M|$-isofunctor is a $\mathcal{b}$-isofunctor. Consequently we have the following result:

Let M be a replete $\mathcal{b}$-category. Then there exists a $\mathcal{b}$-subcategory U, which is a good $\mathcal{b}$-category, and is such that every $\mathcal{b}$-subcategory of M, which is a good $\mathcal{b}$-category, may be fully $\mathcal{b}$-embedded in U.

VII. Proof of the Relative Metatheorem.

The proof of the Relative Metatheorem, which is not entirely analogous to that of the bicategorial or additive metatheorem, will be given explicitely.

1. In the following M is a replete category, $\bar{k}$ its small subcategory, W a categorial property of M-embeddings, which is monotonically additive; V is a property of subcategories of M, which is $\bar{k}$ - M-invariant, M-amalgamic with respect to W and is of M-small W-character.

__Notation__: The fact that $k, h \subset M$, $h \subset k$ and the inclusion functor $\iota : h \longrightarrow k$ has W, will be denoted by $h \overset{W}{\subset} k$. The conjunction of $h \underset{f}{\subseteq} k$ and $h \overset{W}{\subset} k$, will be denoted by $h \underset{f}{\overset{W}{\subseteq}} k$.

If $\langle \ell , \mathcal{K} \rangle$ is an M- (semi)amalgam with V and such that $\ell \underset{f}{\overset{W}{\subseteq}} k$ for every $k \in \mathcal{K}$, then it will be termed a W - M -(semi)amalgam with V.

If $\langle \ell , \mathcal{K} \rangle$ is a W - M -amalgam with V, and K is its M-filling with V such that $k \underset{f}{\overset{W}{\subseteq}} K$ for every $k \in K$, then K will be termed its W - M -filling with V.

If $\{ k_\alpha ; \alpha \in T \}$ is a monotone system of small subcategories of M with V such that $k_\alpha \underset{f}{\overset{W}{\subseteq}} k_{\alpha'}$ whenever $\alpha < \alpha'$, then we shall say that it is a W - M -monotone system with V.

If $\{ k_\alpha ; \alpha \in T \}$ is a W - M -monotone system with V and $K = \bigcup_{\alpha \in T} k_\alpha$, then evidently $k_\alpha \underset{f}{\overset{W}{\subseteq}} K$ for every $\alpha \in T$.

2. **Lemma**: Let h', h, ℓ be small subcategories of M with V, $h \overset{W}{\underset{f}{\subset}} h'$, $\varphi : h \xrightarrow{onto} \ell$ an M-isofunctor identical on $\bar{k}$. Then there exists $\ell' \subset M$ with V such that $\ell \overset{W}{\underset{f}{\subset}} \ell'$ and that there exists an M-isofunctor $\varphi' : h' \xrightarrow{onto} \ell'$ which extends φ. Moreover, if $\ell^{\sigma} \cap (h'^{\sigma} - h^{\sigma}) = \emptyset$, then $\ell'^{\sigma} - \ell^{\sigma} = h'^{\sigma} - h^{\sigma}$.

Proof: First suppose $\ell^{\sigma} \cap (h'^{\sigma} - h^{\sigma}) = \emptyset$. Let $\{\mu_a ; a \in h^{\sigma}\}$, $\mu_a \in H_M(a, (a)\varphi)$, be a natural equivalence of functors ι_h and $\varphi \iota_{\ell}$, where $\iota_h : h \to M$, $\iota_{\ell} : \ell \to M$ are the inclusion functors and $\varphi : h \to \ell$. For $a \in h'^{\sigma} - h^{\sigma}$ denote by μ_a the identity, $\mu_a \in H_M(a, a)$. ℓ' and φ' may be defined as follows: $(\ell')^{\sigma} = \ell^{\sigma} \cup (h'^{\sigma} - h^{\sigma})$, $\varphi'/_{\ell^{\sigma}} = \varphi/_{\ell^{\sigma}}$, $\varphi'/_{(h'^{\sigma} - h^{\sigma})}$ is identical, for $\alpha \in H_{h'}(a, b)$ put $(\alpha)\varphi' = \mu_a^{-1} \cdot \alpha \cdot \mu_b$ and put $\ell' = (h')\varphi'$. If $\ell^{\sigma} \cap (h'^{\sigma} - h^{\sigma}) \neq \emptyset$, choose some $\bar{h}' \subset M$ such that $\bar{h}'^{\sigma} \cap \ell^{\sigma} = \bar{k}^{\sigma}$ and that there exists an M-isofunctor $\psi : h' \xrightarrow{onto} \bar{h}'$ which is identical on $\bar{k}$. Set $\bar{h} = (h)\psi$, $\bar{\varphi} = \psi^{-1}/_{\bar{h}} \cdot \varphi$ and the first case apply to $\bar{h}, \bar{h}', \ell$ and $\bar{\varphi}$.

3. **Lemma**: Let $\langle \ell, \mathcal{K}' \rangle$ be a W-M-semiamalgam with V, let $k \in \mathcal{K}'$. Then there exists its M-unglueing $\langle \ell, \mathcal{K} \rangle$ such that $k \in \mathcal{K}$; $\langle \ell, \mathcal{K} \rangle$ is a W-M-amalgam with V.

Proof: This is evident.

4. **Definition**: A category H will be called a repletion of a category P if:

1) P is a full subcategory of H and contains some

skeleton of H ;

2) for every $a \in P^{\sigma}$, all $b \in H^{\sigma} - P^{\sigma}$ equivalent to a form a proper class.

5. <u>Lemma</u>: Let R be an equivalence on a class X such that, for every $a \in X$, $\{b \in X ; b \mathrel{R} a\}$ is a proper class. Then there exists $Y \subset X$ and a one-to-one mapping γ of X onto Y such that for every $a \in X$ there is $a \mathrel{R} (a)\gamma$ and $\{b \in X - Y ; b \mathrel{R} a\}$ is a proper class.

Proof: Let $\prec$ be an O_n-order for X ; set $X_a = \{b \in X ; b \prec a\}$, $S_a = \{b \in X ; b \succeq a, b \mathrel{R} a\}$. Put $Y^* = \{b \in X ; b$ is an isolated point of the set $S_b\}$. Then evidently for every $a \in X$ the class $\{b \in X - Y^* ; b \mathrel{R} a\}$ is proper. Now let $a \in X$ and let $\{\gamma_b ; b \in X_a\}$ be a system of one-to-one mappings $\gamma_b : X_b \to Y^*$ such that:

1) if $b \prec b'$, then $\gamma_{b'}/X_b = \gamma_b$; if b' is a non-isolated, then $\gamma_{b'} = \bigcup_{b \prec b'} \gamma_b$;

2) if $b \prec b'$, then $(b)\gamma_{b'} \mathrel{R} b$.

We shall construct $\gamma_a : X_a \to Y^*$. If a is non-isolated, put $\gamma_a = \bigcup_{b \prec a} \gamma_b$. If a succeeds a', it is sufficient to define $(a')\gamma_a$ only. Choose $(a')\gamma_a \in \{c \in Y^* ; c \mathrel{R} a', c \notin (X_{a'})\gamma_{a'}\}$. Put $\gamma^* = \bigcup_{a \in X} \gamma_a$, $Y = (X)\gamma^*$, $\gamma : X \to Y$ such that $(a)\gamma = (a)\gamma^*$ for every $a \in X$.

6. The notation from item 5 will be used. Moreover, denote by M' the full subcategory of M such that $M'^{\sigma} = M^{\sigma} - \mathfrak{K}^{\sigma}$.

<u>Lemma</u>: There exists a full subcategory P of M' and an M'-isofunctor of M' onto P such that M' is a reple-

tion of P.

Proof: Set $X = M'^{\sigma}$; let R be the equivalence on X such that $a R b$ if and only if a and b are equivalent in M'. Apply lemma 5. Let P be a full subcategory of M' such that $P^{\sigma} = Y$, Γ be an M'-isofunctor of M' onto P such that $\Gamma/_X = \gamma$. Then evidently P and Γ have the required properties.

7. **Notation**: The notation from item 6 will be used.

a) If $Z \subset M^{\sigma}$, denote by (Z) the full subcategory of M such that $[Z]^{\sigma} = Z$. Set $\tilde{k} = (\bar{k}^{\sigma})$, $\tilde{P} = (P^{\sigma} \cup \bar{k}^{\sigma})$. Let $\tilde{\Gamma}$ be an M-isofunctor of M onto $\tilde{P}$, identical on $\tilde{k}$ and such that $\tilde{\Gamma}/_{M'} = \Gamma$;

b) Choose some O_n-order $\prec$ for the class P^{σ}, which will be fixed in the following. Denote by c its first element. If $s \in P^{\sigma}$, put $p_s = (\{t \in P^{\sigma};\ t \prec s\})$, $\tilde{p}_s = (\bar{k}^{\sigma} \cup p_s^{\sigma})$.

c) For $h \subset_f h'$ put $h' \dot{-} h = (h'^{\sigma} - h^{\sigma})$.

8. **Lemma**: Let $L \subset M$, L have V. Then there exists a $H \subset M$ and an M-isofunctor of L onto H, identical on $\bar{k}$ such that $H = \bigcup_{s \in S'} h_s$; here $\{h_s;\ s \in S'\}$ is a W-M-monotone system with V such that S' is a subclass of P^{σ}, $c \in S'$, $h_c = \bar{k}$, and for every $s \in S'$, $s \succ c$ there is $h_s \dot{-} \bigcup_{t \prec s} h_t \subset p_s$

Proof: L has property V; consequently $L = \bigcup_{\alpha \in A} \ell_{\alpha}$, where $\{\ell_{\alpha};\ \alpha \in A\}$ is a W-M-monotone system with V and $\ell_{\alpha_0} = \bar{k}$ with α_0 the first element of

A . Put $H = (L)\widetilde{\Gamma}$, $\ell'_\alpha = (\ell_\alpha)\widetilde{\Gamma}$. Now it is easy to find an order-preserving mapping f of A into P^σ such that $(\alpha_0)f = c$ and that for every $\alpha \in A$ the category ℓ'_α is a subcategory of $p_{(\alpha)f}$. It is sufficient to choose $(\alpha)f \succcurlyeq max(\sup_{\beta<\alpha}(\beta)f, \sup \ell'^{\sigma}_\alpha)$ where $<$ is the order on A. Of course put $S' = (A)f$, and for $s \in S'$ put $h_s = \ell'_{(s)f^{-1}}$.

9. Construction of U : Let $s \in P^\sigma$, $s \succcurlyeq c$ and let $\{k_t; t \in P^\sigma, t \prec s\}$ be a W - M -monotone system with V such that:

A) $k_c = \bar{k}$;

B) if $t \succcurlyeq c$ then a) $k_t^\sigma \cap P^\sigma = \emptyset$;

b) if $h', h \subset M$ are small and have V, $h \overset{W}{\underset{f}{\subseteq}} h'$, $h' \dot{-} h \subset p_t$ and $\varphi: h \to \bigcup_{u \prec t} k_u$ is a full M -embedding with W identical on $\bar{k}$, then there exists a full M -embedding with W of h' into k_t, which extends φ .

We construct k_s such that $\{k_t; t \in P^\sigma, t \preceq s\}$ is a W - M -monotone system with V satisfying A) and B). Put $k = \bigcup_{t \prec s} k_t$. For every $\ell \overset{W}{\underset{f}{\subseteq}} k$ with V denote by $\mathcal{H}_\ell$ the set of all $h \subset M$ with V such that $\ell \overset{W}{\underset{f}{\subset}} h$ and $h \dot{-} \ell \subset p_s$. Let $\langle \ell, \mathcal{K}_\ell \rangle$ be an M -unglueing of the W - M -semiamalgam $\langle \ell, \mathcal{H}_\ell \cup \{k\} \rangle$ such that $k \in \mathcal{K}_\ell$, let K_ℓ be its W - M -filling with V. Let $\mathcal{L}$ be the set of all K_ℓ, where $\ell \overset{W}{\underset{f}{\subset}} k$, ℓ has V; then $\langle k, \mathcal{L} \rangle$ is a W - M -semiamalgam with V ; let $\langle k, \mathcal{K} \rangle$ be its M -unglueing. Denote by k' its W - M -filling with V. Let $\mathcal{k}_s$ be a subcategory of M such that $\mathcal{k}_s^\sigma \cap P^\sigma = \emptyset$ and that there exists an M-isofunctor of $\mathcal{k}'$ onto $\mathcal{k}_s$ identical on $\mathcal{k}$. (Such a category $\mathcal{k}_s$ exists because, for every $a \in M^\sigma$, $\{b \in M^\sigma - P^\sigma;$ b is equivalent in M with $a\}$ is a proper class.) It is easy to see that $\{\mathcal{k}_t ; t \leq s\}$ satisfies A) and B a). To prove B b) it is sufficient to show that, if h', h are small subcategories of M, $h \overset{W}{\underset{f}{\subset}} h'$, h, h' have V, $\varphi : h \rightarrow k = \bigcup_{t < s} \mathcal{k}_t$ is a full M-embedding with W identical on $\overline{\mathcal{k}}$ and $h' \dot{-} h \subset p_s$, then there exists a full M-embedding ψ with W of h' into $\mathcal{k}_s$ which extends φ. We shall prove this auxiliary assertion. Put $\ell = (h)\varphi$; then evidently $\ell \overset{W}{\underset{f}{\subset}} \mathcal{k}$ and there exists $\ell' \subset M$ such that $\ell \overset{W}{\underset{f}{\subset}} \ell'$ and there exists an M-isofunctor $\varphi' : h' \xrightarrow{onto} \ell'$ with $\varphi'/h = \varphi$ and $\ell' \dot{-} \ell = h' \dot{-} h$ (because $(h'^\sigma - h^\sigma) \cap \ell^\sigma =$ $= \emptyset$). Consequently $\ell' \in \mathcal{H}_\ell$. Now it is easy to see that there exists a full M-embedding χ with W of ℓ' into $\mathcal{k}_s$ identical on ℓ. Of course put $\psi = \varphi' \cdot \chi$. By transfinite induction one obtains a W-M-monotone system $\{\mathcal{k}_s ; s \in P^\sigma\}$ with V satisfying statements A) and B).

Put $U = \bigcup_{s \in P^\sigma} \mathcal{k}_s$. Then evidently U has V.

10. __Proposition__: Let H be a subcategory of M with property V. Then there exists a full M-embedding with W of H into U identical on $\overline{\mathcal{k}}$.

Proof: Using Lemma 8 one may suppose that $H = \bigcup_{s \in S'} h_s$, where $\{h_s ; s \in S'\}$ is a $W - M$ -monotone system with V, $S' \subset P^{\sigma}$, $c \in S'$, $h_c = \bar{k}$ and for every $s \succ c$ there is $h_s \dot{-} \bigcup_{t \prec s} h_t \subset p_s$. Now it is easy to construct a full M -embedding Φ with W of H into U. Put $\Phi = \bigcup_{s \in S'} \varphi_s$ where φ_s is the following full M -embedding with W of h_s into k_s: $\varphi_c : h_c = \bar{k} \to \bar{k} = k_c$ is identical; if $s \in S'$, $s \succ c$, set $\varphi = \bigcup_{t \prec s} \varphi_t$; then φ is a full M -embedding with W of $h = \bigcup_{\substack{t \in S' \\ t \prec s}} h_t$ into $k = \bigcup_{t \prec s} k_t$ and define φ_s by B b).

Appendix

I. Minimal universal categories

a) The following metadefinitions may be given:

Let $\mathcal{T}$ be a "system" of categories. A category U will be called universal for $\mathcal{T}$ if every category from $\mathcal{T}$ can be fully embedded into U. A category U will be called couniversal for $\mathcal{T}$ if it can be fully embedded into every category from $\mathcal{T}$. A category U will be called a minimal universal category for $\mathcal{T}$ if it is universal for $\mathcal{T}$ and couniversal for the system $\mathcal{T}'$ of all categories universal for $\mathcal{T}$.

Evidently if a category from $\mathcal{T}$ is universal for $\mathcal{T}$, then it is a minimal universal category for $\mathcal{T}$.

b) Now show that a minimal universal category for the class of all small categories does not exist.

__Definition:__ Let K' be a full subcategory of a category K. We say that K' is separated in K if for every $a \in K'^{\sigma}$, $b \in K^{\sigma} - K'^{\sigma}$ there is $H_K(a, b) \cup H_K(b, a) = \emptyset$. A category K is connected if K is the only full subcategory of K separated in K.

Let $\mathcal{K}$ be a class of small categories, let ρ be a partial order for $\mathcal{K}$. We define a category $K = \sum_{\rho} \mathcal{K}$ as follows:

The class K^{σ} is the class of all couples $m = \langle a, k \rangle$, where $k \in \mathcal{K}$, $a \in k^{\sigma}$. For any $m_1, m_2 \in K^{\sigma}$, $m_i = \langle a_i, k_i \rangle$, $i = 1, 2$, put $H_K(m_1, m_2) = \{\langle m_1, \alpha, m_2 \rangle ; \alpha \in H_{k_1}(a_1, a_2)\}$ whenever $k_1 = k_2$; if $k_1 \neq k_2$, put $H_K(m_1, m_2) = \{\langle m_1, \emptyset, m_2 \rangle\}$ whenever $k_1 \, \rho \, k_2$; and put $H_K(m_1, m_2) = \emptyset$ in the other cases. The definition of composition of morphisms in K is evident. (It is defined so that for every $k \in \mathcal{K}$ the mapping $\varphi_k : k \to K$ with $(a)\varphi = \langle a, k \rangle$ for $a \in k^{\sigma}$, $(\alpha)\varphi = \langle \langle \overleftarrow{\alpha}, k \rangle, \alpha, \langle \overrightarrow{\alpha}, k \rangle \rangle$ for $\alpha \in k^{m}$ is a full embedding of k into K.) If $\rho = \emptyset$, we shall write $\sum \mathcal{K}$ instead of $\sum_{\emptyset} \mathcal{K}$.

__Theorem__: There exists no minimal universal category for the class of all small categories.

__Proof__: Denote by $\mathcal{V}$ the class of all small categories. No category universal for $\mathcal{V}$ is small. Put $V = \sum \mathcal{V}$. Let $\prec$ be a total order for the class $\mathcal{V}$, $W = \sum_{\prec} \mathcal{V}$. Evidently V and W both are universal for $\mathcal{V}$. Every full subcategory of V which is not small is not connected. Every full subcategory of W which is not small is connected.

c) <u>Note</u>: We shall say that a category $\mathcal{k}$ may be fully separately embedded into a category K if there exists an isofunctor of $\mathcal{k}$ onto a full subcategory of K separated in K.

The following properties of the category $V = \Sigma \mathcal{V}$ may be verified:

1) Every $\mathcal{k} \in \mathcal{V}$ can be separately fully embedded into V.

2) If K is a category such that every $\mathcal{k} \in \mathcal{V}$ can be embedded in K, then V can be embedded in K.

3) If K is a category such that every $\mathcal{k} \in \mathcal{V}$ can be fully separately embedded in K, then V can be fully separately embedded in K.

II. <u>Properties of properties</u>.

Now we prove some propositions about some natural properties V.

a) <u>Let</u> $\bar{\mathcal{k}}$ <u>be a small category</u>, $a_0 \in \bar{\mathcal{k}}^{\sigma}$ <u>its generator</u>[x)]. <u>Let</u> V <u>be the following property: a category</u> K <u>has</u> V <u>if and only if it contains</u> $\bar{\mathcal{k}}$ <u>as a full subcategory</u> <u>and</u> a_0 <u>is a generator of</u> K. <u>We shall prove that</u> V

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x) We recall that a_0 is a generator of a category $\mathcal{h}$ if it is true that if $\mu, \nu \in H_{\mathcal{h}}(b, c)$, $\mu \neq \nu$, then there exists an $\alpha \in H_{\mathcal{h}}(a_0, b)$ such that $\alpha\mu \neq \alpha\nu$.

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is amalgamic. The following proposition is true: If $\langle \mathcal{L}, \mathcal{K} \rangle$ is an amalgam, $a_0 \in \mathcal{L}^\sigma$ a generator of every $k \in \mathcal{K}$, then there exists its filling K such that a_0 is a generator of K.

We prove the proposition only for the case that $\mathcal{K} = \{k_1, k_2\}, k_1^\sigma - \mathcal{L}^\sigma = \{a_1\}, k_2^\sigma - \mathcal{L}^\sigma = \{a_2\}, a_1 \neq a_2$. Let $\tilde{k}$ be the sum of categories k_1 and k_2 with the amalgamated subcategory $\mathcal{L}$, [11]. Let $\{i, j\} = \{1, 2\}$. Let Z_i be the following equivalence on $H_{\tilde{k}}(a_i, a_j)$:

$\mu Z_i \mu'$ if and only if $\alpha \cdot \mu = \alpha \cdot \mu'$ for every $\alpha \in H_{k_i}(a_0, a_i)$.

Now it is easy to see that if $\mu Z_i \mu'$, then

$\mu \cdot \sigma Z_i \mu' \cdot \sigma$ for every $\sigma \in H_{\tilde{k}}(a_j, a_j)$;

$\mu \cdot \sigma = \mu' \cdot \sigma$ for every $\sigma \in H_{\tilde{k}}(a_j, b), b \neq a_j$;

$\sigma \cdot \mu Z_i \sigma \cdot \mu'$ for every $\sigma \in H_{\tilde{k}}(a_i, a_i)$;

$\sigma \cdot \mu = \sigma \cdot \mu'$ for every $\sigma \in H_{\tilde{k}}(b, a_i), b \neq a_i$.

Let now K be a category such that $K^\sigma = k_1^\sigma \cup k_2^\sigma$, k_1 and k_2 are full subcategories of K and $H_K(a_i, a_j) = \{a_i\} \times (H_{\tilde{k}}(a_i, a_j)/Z_i) \times \{a_j\}$.

The definition of the composition in K is evident. It is easy to see that k has the required properties.

b) Now prove that the property of being an a-category is a-amalgamic. The following proposition holds:

Let $\langle \mathcal{L}, \mathcal{K} \rangle$ be an a-amalgam. Then there exists its a-filling K such that if H is an a-category, $\mathcal{G}_k : k \to H$ an a-functor such that $\mathcal{G}_k/_{\mathcal{L}} = \mathcal{G}_{k'}/_{\mathcal{L}}$ for every $k, k' \in \mathcal{K}$, then there exists exactly one

a-functor $\varphi : K \longrightarrow H$ such that $\varphi_{k} = \iota_{k} \cdot \varphi$ where $\iota_{k} : k \longrightarrow K$ is the inclusion a-functor.

(K will then be called an a-sum of a-categories with amalgamated a-subcategory ℓ .)

We prove the proposition only for the case that $\mathcal{K} = \{k_1, k_2\}$, $|k_1|^{\sigma} - |\ell|^{\sigma} = \{a_1\}$, $|k_2|^{\sigma} = \{a_2\}$, $a_1 \neq a_2$. If $|\ell|^{\sigma} = \emptyset$ then put $|K|^{\sigma} = \{a_1, a_2\}$, k_1, k_2 are full a-subcategories of K and $H_K(a_1, a_2) = \{\omega_{a_1, a_2}\}$, $H_K(a_2, a_1) = \{\omega_{a_2, a_1}\}$. Then evidently K has the required properties. Consequently we may suppose that $|\ell|^{\sigma} \neq \emptyset$. Let $\tilde{k}$ be the sum of categories $|k_1|$ and $|k_2|$ with the amalgamated subcategory $|\ell|$, [11]. Let $\{i, j\} = \{1, 2\}$. We recall that every $\mu \in H_{\tilde{k}}(a_i, a_j)$ may be expressed as $\mu = \alpha \cdot \beta$, where $\alpha \in H_{k_i}(a_i, b)$, $\beta \in H_{k_j}(b, a_j)$, $b \in |\ell|^{\sigma}$; and if also $\mu = \alpha' \cdot \beta'$, where $\alpha' \in H_{k_i}(a_i, b')$, $\beta' \in H_{k_j}(b', a_j)$, $b' \in |\ell|^{\sigma}$, then $\langle \alpha, \beta \rangle \; R^* \; \langle \alpha', \beta' \rangle$, where R^* is the smallest equivalence on the set $\bigcup_{b \in |\ell|^{\sigma}} \{H_{k_i}(a_i, b) \times H_{k_j}(b, a_j)\}$ containing the following relation $R : \langle \alpha, \beta \rangle \; R \; \langle \alpha', \beta' \rangle$ if and only if there exists a $\lambda \in |\ell|^{m}$ such that $\alpha = \alpha' \cdot \lambda$ in k_i, $\beta' = \lambda \cdot \beta$ in k_j, [11]. Then, as is easy to see, $\tilde{k}$ has a system of null morphisms. Let G_i be a free abelian group with ω_{a_i, a_j} as zero and $H_{\tilde{k}}(a_i, a_j) - \{\omega_{a_i, a_j}\}$ its set of generators.

Let h be a category defined as follows: $h^{\sigma} = |k_1|^{\sigma} \cup |k_2|^{\sigma}$, $|k_1|$ and $|k_2|$ are to be full subcategories of h, put $H_h(a_i, a_j) = \{a_i\} \times G_i \times \{a_j\}$, if $\mu, \nu \in G_i$, set $\bar{\mu} = \langle a_i, \mu, a_j \rangle$, $\bar{\nu} = \langle a_i, \nu, a_j \rangle$ and put $\bar{\mu} + \bar{\nu} = \overline{\mu + \nu}$;

consequently if $m \in H_h(a_i, a_j)$, then $m = \overline{\mu_1} + \dots + \overline{\mu_s}$ where $\mu_1, \dots, \mu_s \in H_{\tilde{k}}(a_i, a_j)$; now put

$\sigma \cdot m = \overline{\sigma \cdot \mu_1} + \dots + \overline{\sigma \cdot \mu_s}$ for every $\sigma \in H_{k_i}(a_i, a_i)$;

$\sigma \cdot m = \sigma \cdot \mu_1 + \dots + \sigma \cdot \mu_s$ for every $\sigma \in H_{k_i}(b, a_i)$, $b \in |\ell|^\sigma$;

$m \cdot \sigma = \overline{\mu_1 \cdot \sigma} + \dots + \overline{\mu_s \cdot \sigma}$ for every $\sigma \in H_{k_j}(a_j, a_j)$;

$m \cdot \sigma = \mu_1 \cdot \sigma + \dots + \mu_s \cdot \sigma$ for every $\sigma \in H_{k_j}(a_j, b)$, $b \in |\ell|^\sigma$;

and if $n \in H_h(a_j, a_i)$, $n = \overline{\nu_1} + \dots + \overline{\nu_r}$, put $m \cdot n = \sum_{t=1}^{s} \sum_{u=1}^{r} \mu_t \cdot \nu_u$, $n \cdot m = \sum_{u=1}^{r} \sum_{t=1}^{s} \nu_u \cdot \mu_t$.

The composition in h is associative because the composition in $\tilde{k}$ is associative. Moreover, if $m, n \in H_h(a_i, a_j)$, $\sigma \in H_h(b, a_i)$, $\tau \in H_h(a_j, c)$, then $\sigma \cdot (m + n)\tau = \sigma \cdot m \cdot \tau + \sigma \cdot n \cdot \tau$.

Now let T_i be the following relation on $H_h(a_i, a_j)$:

$m \; T_i \; n$ if and only if either

$m = \overline{(\alpha_1 + \alpha_2) \cdot \beta}$, where $\alpha_1, \alpha_2 \; H_{k_i}(a_i, b)$, $b \in |\ell|^\sigma$, $\beta \in H_{k_j}(b, a_j)$, $n = \overline{\alpha_1 \cdot \beta} + \overline{\alpha_2 \cdot \beta}$, or

$m = \overline{\alpha \cdot (\beta_1 + \beta_2)}$, where $\alpha \in H_{k_i}(a_i, b)$, $b \in |\ell|^\sigma$, $\beta_1, \beta_2 \in H_{k_j}(b, a_j)$ and $n = \overline{\alpha \cdot \beta_1} + \overline{\alpha \cdot \beta_2}$.

$$(*)\left\{\begin{array}{ll} \text{Evidently, if } m \; T_i \; n, \text{ then} & \\ \sigma \cdot m \; T_i \; \sigma \cdot n & \text{for every } \sigma \in H_h(a_i, a_i); \\ \sigma \cdot m = \sigma \cdot n & \text{for every } \sigma \in H_h(b, a_i), b \neq a_i; \\ m \cdot \sigma \; T_i \; n \cdot \sigma & \text{for every } \sigma \in H_h(a_j, a_j); \\ m \cdot \sigma = n \cdot \sigma & \text{for every } \sigma \in H_h(a_j, b), b \neq a_j. \end{array}\right.$$

Let S_i be the following relation on $H_h(a_i, a_j)$:

$m \; S_i \; n$ if and only if $m = p + q$, $n = p + q'$, where $p \in H_h(a_i, a_j)$, $q \; T_i \; q'$.

Evidently $(*)$ remains true if we replace T_i by S_i. Let S_i^* be the smallest equivalence on $H_h(a_i, a_j)$ which contains S_i. Then S_i^* is a congruence on the group $H_h(a_i, a_j)$, and $(*)$ remains true on replacing T_i by S_i^*.

Let now $\overline{K}$ be a category such that $\overline{K}^\sigma = h^\sigma$, $|k_1|$ and $|k_2|$ are full subcategories of $\overline{K}$ and $H_K(a_i, a_j) = \{a_i\} \times (H_h(a_i, a_j)/_{S_i^*}) \times \{a_j\}$.

Using $(*)$ with T_i replaced by S_i^*, the definition of the composition in $\overline{K}$ is evident. Now it is also easy to define the a-category K such that $|K| = \overline{K}$, and k_1, k_2 are a-subcategories of K. Let now H be an a-category, $\varphi_1 : k_1 \longrightarrow H$, $\varphi_2 : k_2 \longrightarrow H$ be a-functor such that $\varphi_1/_\ell = \varphi_2/_\ell$. Then there exists exactly one functor $\widetilde{\psi} : \widetilde{k} \longrightarrow H$ such that $\varphi_1 = \widetilde{\iota}_1 \cdot \widetilde{\psi}$, $\varphi_2 = \widetilde{\iota}_2 \cdot \widetilde{\psi}$, where $\widetilde{\iota}_1 : |k_1| \longrightarrow \widetilde{k}$, $\widetilde{\iota}_2 : |k_2| \longrightarrow \widetilde{k}$ are inclusion functors. Let $\psi : h \longrightarrow H$ be a functor such that $\psi/_{|k_1|} = \varphi_1$, $\psi/_{|k_2|} = \varphi_2$ and that if $m \in H_h(a_i, a_j)$, $m = \overline{\mu}_1 + \cdots + \overline{\mu}_s$, then $(m)\psi = (\mu_1)\widetilde{\psi} + \cdots + (\mu_s)\widetilde{\psi}$. If $m \; S_i^* \; n$, then evidently $(m)\psi = (n)\psi$. Consequently there exists an a-functor $\varphi : K \longrightarrow H$ such that $\varphi_1 = \iota_1 \cdot \varphi$, $\varphi_2 = \iota_2 \cdot \varphi$, where $\iota_1 : k_1 \longrightarrow K$, $\iota_2 : k_2 \longrightarrow K$ are inclusion functors. The unicity of φ is evident.

c) <u>Let</u> V_1 (<u>or</u> V_2) <u>be the following property of</u> a <u>-categories</u>:

<u>an</u> a<u>-category</u> K <u>has</u> V_1 (<u>or</u> V_2) <u>if and only if</u> $H_K(a, b)$ <u>is a torsion group</u> (<u>or a finite group, res-</u>

pectively) for every $a, b \in |K|^{\sigma}$. Then it is a-amalgamic.

If $\langle \ell, \mathcal{K} \rangle$ is an a-amalgam with V_1 (or V_2 respectively), then the a-sum K of a-categories from $\mathcal{K}$ with amalgamated a-subcategory ℓ has V_1 (or V_2); this follows immediately from the construction of K (cf Appendix, II b).

d) Let $\bar{k}$ be an a-category, $a_0 \in |\bar{k}|^{\sigma}$ be its generator, let V be the following property of a-categories: an a-category K has V if and only if it contains $\bar{k}$ as a full a-subcategory such that a_0 is a generator of K. Then V is a-amalgamic. We prove only that if ℓ is a full a-subcategory of a-categories k_1, k_2 such that $|k_1|^{\sigma} - |\ell|^{\sigma} = \{a_1\}$, $|k_2|^{\sigma} - |\ell|^{\sigma} = \{a_2\}$, $a_1 \neq a_2$ and $a_0 \in |\ell|^{\sigma}$ is a generator of both k_1, k_2, then there exists an a-filling k of the a-amalgam $\langle \ell, \{k_1, k_2\} \rangle$ such that a_0 is a generator of k.

Let h be an a-sum of k_1 and k_2 with amalgamated a-subcategory ℓ. Let $\{i, j\} = \{1, 2\}$. Let Z_i be the following relation on $H_h(a_i, a_j)$. $\mu Z_i \mu'$ if and only if $\alpha \cdot \mu = \alpha \cdot \mu'$ for every $\alpha \in H_h(a_0, a_i)$. Then it is easy to see that Z_i is a congruence on the group $H_h(a_i, a_j)$, and that if $\mu Z_i \mu'$, then

$\mu \cdot \sigma Z_i \mu' \cdot \sigma$ for every $\sigma \in H_h(a_j, a_j)$;

$\mu \cdot \sigma = \mu' \cdot \sigma$ for every $\sigma \in H_h(a_j, b)$, $b \neq a_j$;

$\sigma \cdot \mu Z_i \sigma \cdot \mu'$ for every $\sigma \in H_h(a_i, a_i)$;

$\sigma \cdot \mu = \sigma \cdot \mu'$ for every $\sigma \in H_h(b, a_i)$, $b \neq a_i$.

Of course put $H_k(a_i, a_j) = \{a_i\} \times (H_h(a_i, a_j)/Z_i) \times \{a_j\}$; the rest is evident.

Evidently, if $H_{k_m}(c, d)$ is a torsion group (or a finite group) for every $c, d \in |k_m^\sigma|, m = 1, 2$, then $H_k(c, d)$ is too for every $c, d \in |k|^\sigma$.

e) Let k be an a-category. We recall, [1], that k can be fully a-embedded into an additive category K. We shall sketch this construction to show that the inclusion functor has several required properties.

Every $a \in |K|^\sigma$ is a finite collection of elements of $|k|^\sigma$. If $a = \{a_i ; i = 1, \dots, n\}, b = \{b_j ; j = 1, \dots m\} \in |K|^\sigma$, then $H_K(a, b)$ is the set of all $\alpha = \langle a, \{\alpha_{i,j} ; i = 1, \dots, n, j = 1, \dots, m\}, b\rangle$ where $\alpha_{i,j} \in H_k(a_i, b_j)$. The triple $\langle a, \{\alpha_{i,j} ; i, j\}, b\rangle$ will be denoted simply by $\{\alpha_{i,j} ; i, j\}^*$. If $\alpha = \{\alpha_{i,j} ; i, j\}^*$, $\beta = \{\beta_{i,j} ; i, j\}^* \in H_K(a, b)$, then $\alpha + \beta = \{\alpha_{i,j} + \beta_{i,j} ; i, j\}^*$. If $\alpha = \{\alpha_{i,j} ; i, j\}^* \in H_K(a, b)$, $\beta = \{\beta_{j,l} ; j, l\}^* \in H_K(b, c)$, then $\alpha \cdot \beta = \{\sum_j \alpha_{i,j} \cdot \beta_{j,l} ; i, l\}^*$.

It is easy to see that k can be fully a-embedded into K and K is additive; and also that if for any $c, d \in |k|^\sigma$ the group $H_k(c, d)$ is a torsion group (or a finite group respectively) then for every $a, b \in |K|^\sigma$ the group $H_K(a, b)$ is also such.

Now prove that if c is a generator of k, then $\{c\}$ is a generator of K. Let $a, b \in |K|^\sigma$, $\alpha, \beta \in H_K(a, b)$, $\alpha \neq \beta$, $\alpha = \{\alpha_{i,j} ; i, j\}^*$, $\beta = \{\beta_{i,j} ; i, j\}^*$. Then there exist i_0, j_0 such that $\alpha_{i_0, j_0} \neq \beta_{i_0, j_0}$. Hence there exists a $\mu \in H_k(c, a_{i_0})$ such that $\mu \cdot \alpha_{i_0, j_0} \neq \mu \cdot \beta_{i_0, j_0}$. Take $\varphi = \{\varphi_i ; i\}^*$, $\varphi_i \in H_k(c, a_i)$ such that $\varphi_{i_0} = \mu$,

$\rho_i = \omega_{a, a_i}$ for $i \neq i_o$, where ω denotes the null morphism. Then $\rho \cdot \alpha \neq \rho \cdot \beta$.

f) <u>Now prove that the property of being a $\mathcal{b}$-category is not $\mathcal{b}$-amalgamic.</u> Let $\mathcal{l}, \mathcal{k}_1, \mathcal{k}_2$ be $\mathcal{b}$-categories from the following diagrams (identities are not indicated):

all diagrams are commutative and

$P_{\mathcal{l}} = \{identities\} \cup \{\pi_\chi, \pi_{\chi'}\}$, $I_{\mathcal{l}} = \{identities\} \cup \{\iota_\xi, \iota_\chi, \rho, \iota_{\xi'}, \iota_{\chi'}, \rho'\}$;

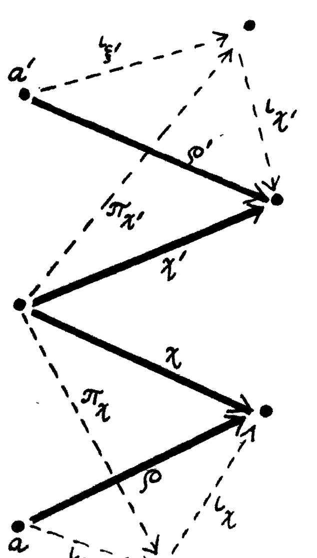
$P_{\mathcal{k}_1} = \{identities\} \cup P_{\mathcal{l}} \cup \{\alpha, \alpha', \pi_\xi, \pi_{\xi'}, \pi_\varphi\}$,

$I_{\mathcal{k}_1} = \{identities\} \cup I_{\mathcal{l}} \cup \{\iota_\varphi\}$,

$P_{\mathcal{k}_2} = \{identities\} \cup P_{\mathcal{l}} \cup \{\pi_\mu, \pi_\vartheta, \pi_{\mu'}, \pi_{\vartheta'}\}$,

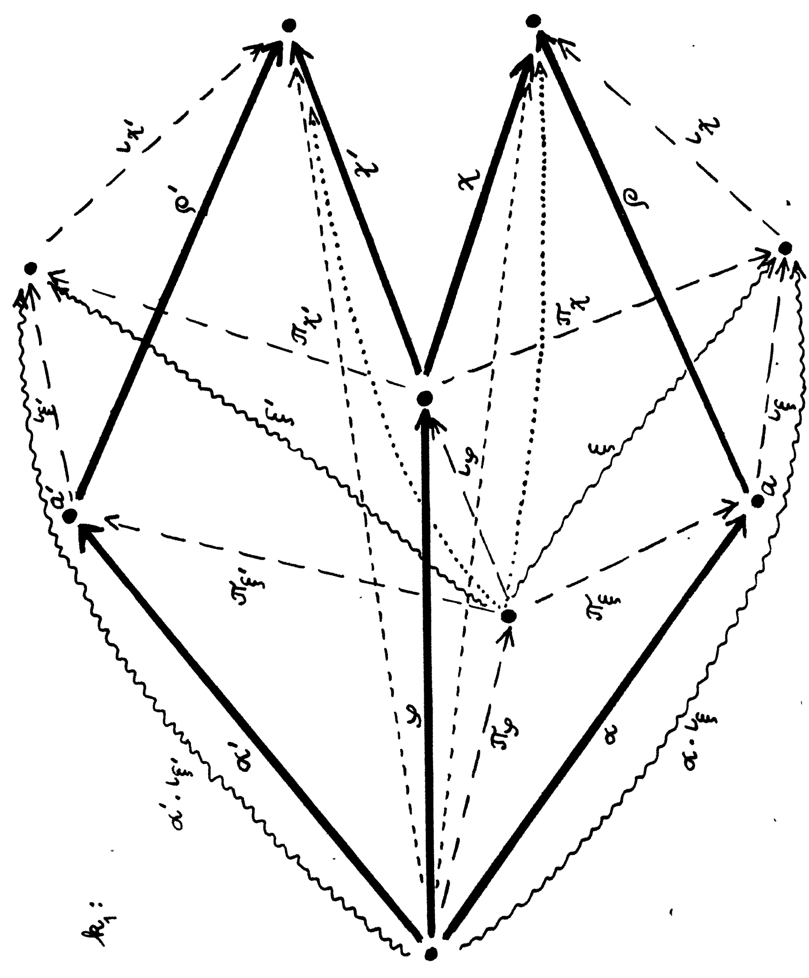
$I_{\mathcal{k}_2} = \{identities\} \cup I_{\mathcal{l}} \cup \{\iota_\mu, \iota_\vartheta, \iota_{\mu'}, \iota_{\vartheta'}, \iota_\xi \cdot \pi_\vartheta, \iota_{\xi'} \cdot \pi_{\vartheta'}, \iota_\vartheta \cdot \iota_\mu,$
$\iota_\xi \cdot \pi_\vartheta \cdot \iota_{\vartheta'}, \iota_{\xi'} \cdot \pi_{\vartheta'} \cdot \iota_\vartheta, \rho \cdot \mu, \rho' \cdot \mu', \iota_\xi \cdot \pi_\vartheta \cdot \iota_\vartheta, \iota_{\xi'} \cdot \pi_{\vartheta'} \cdot \iota_{\vartheta'}\}$.

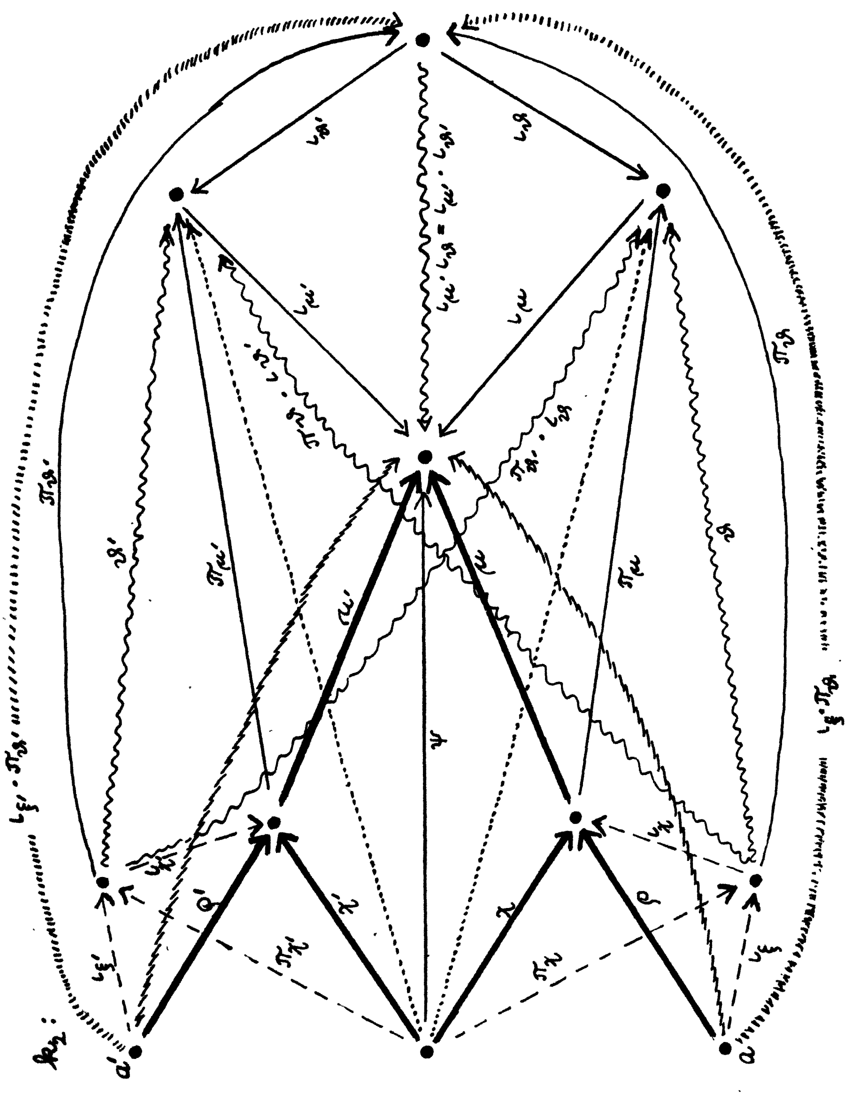
$\mathcal{l}$:



It is easy to see that $\mathcal{l}$ is a full $\mathcal{b}$-subcategory of both $\mathcal{k}_1$ and $\mathcal{k}_2$. Let $\mathcal{k}$ be a $\mathcal{b}$-category such that $\mathcal{k}_1$ and $\mathcal{k}_2$ are both full $\mathcal{b}$-subcategories of $\mathcal{k}$.

Then necessarily $\alpha, \alpha' \in P_{\mathcal{k}}$, $\rho \cdot \mu, \rho' \cdot \mu' \in I_{\mathcal{k}}$ and $\alpha \cdot \rho \cdot \mu = \alpha' \cdot \rho' \cdot \mu'$. But then necessarily there exists an isomorphism $\sigma \in H_{\mathcal{k}}(a, a')$. But $\mathcal{l}$ is a





full $\mathcal{b}$-subcategory of $\mathcal{k}$, so that $\sigma \in H_\ell(a, a')$; this is a contradiction, because $H_\ell(a, a') = \emptyset$.

g) <u>Now we prove that the property W of being a $\mathcal{b}$-embedding onto a good $\mathcal{b}$-subcategory is categorial and monotonically additive.</u>

First prove that W is categorial. It is sufficient to prove that if K is a good $\mathcal{b}$-subcategory of R, R a good $\mathcal{b}$-subcategory of S, then K is a good $\mathcal{b}$-subcategory of S. Let $\mu \in |S|^m$, $\vec{\mu} \in |K|^\sigma$. Since $|K|^\sigma \subset |R|^\sigma$, there exists $\pi \in P_S$, $\iota \in I_S$ such that $\overleftarrow{\iota} \in |R|^\sigma$ and $\mu = \pi \cdot \iota$. But then $\vec{\iota} = \vec{\mu} \in |K|^\sigma$ and consequently $\iota \in I_R$. Thus there exist $\sigma \in P_R$, $\wp \in I_R$ such that $\vec{\sigma} \in |K|^\sigma$ and $\iota = \sigma \cdot \wp$; then $\mu = (\pi \cdot \sigma) \cdot \wp$. The proof is analoguous in the case that $\overleftarrow{\mu} \in |K|^\sigma$.

Now prove that W is monotonically additive. Let $\{s_\alpha ; \alpha \in T\}$, $\{\ell_\alpha ; \alpha \in T\}$ be monotone systems of small $\mathcal{b}$-categories such that, for every $\alpha \in T$, s_α is a good $\mathcal{b}$-subcategory of ℓ_α. We shall prove that then the $\mathcal{b}$-category $S = \bigcup_{\alpha \in T} s_\alpha$ is a good $\mathcal{b}$-subcategory of $B = \bigcup_{\alpha \in T} \ell_\alpha$. Let $\mu \in |B|^m$ and let $\vec{\mu} \in |S|^\sigma$. Choose $\alpha \in T$ such that $\mu \in |\ell_\alpha|^m$, $\vec{\mu} \in |s_\alpha|^\sigma$. Evidently there then exists $\pi \in P_{\ell_\alpha} \subset P_B$, $\iota \in I_{\ell_\alpha} \subset I_B$ such that $\mu = \pi \cdot \iota$, $\vec{\pi} \in |s_\alpha|^\sigma$.

h) <u>Let V be the property of being a good $\mathcal{b}$-category. Let W be the property of being a $\mathcal{b}$-embedding onto a</u>

good $\mathcal{b}$ -subcategory.

Now prove that V is $\mathcal{b}$-amalgamic with respect to W. The following proposition holds: Let $\langle \ell, \mathcal{K} \rangle$ be a $\mathcal{b}$-amalgam such that ℓ is a good $\mathcal{b}$-subcategory of every $k \in \mathcal{K}$. Then there exists its $\mathcal{b}$-filling K such that every $k \in \mathcal{K}$ is a good $\mathcal{b}$-subcategory of K, and that if H is a $\mathcal{b}$-category and $\varphi_k : k \to H$ are $\mathcal{b}$-functors with $\varphi_k/_\ell = \varphi_{k'}/_\ell$ for every $k, k' \in \mathcal{K}$, then there exists exactly one $\mathcal{b}$-functor $\varphi : K \to H$ with $\varphi_k = \iota_k \cdot \varphi$, where $\iota_k : k \to K$ is the inclusion $\mathcal{b}$-functor.

We prove the proposition only for the case that $\mathcal{K} = \{k_1, k_2\}$.

1) Set $P_n = P_{k_n}$, $I_n = I_{k_n}$, $n = 1, 2$. Let h be a sum of the categories $|k_1|$ and $|k_2|$ with the amalgamated subcategory $|\ell|$, [11]. Put $P_n^* = \{\alpha \cdot \beta ; \alpha \in P_n, \vec{\alpha} \in |\ell|^\sigma, \beta$ is an isomorphism of $h\}$, $I_n^* = \{\beta \cdot \alpha ; \alpha \in I_n, \overleftarrow{\alpha} \in |\ell|^\sigma, \beta$ is an isomorphism of $h\}$ put $P = P_1 \cup P_1^* \cup P_2 \cup P_2^*$, $I = I_1 \cup I_1^* \cup I_2 \cup I_2^*$. Then evidently $P \subset h^m$, $I \subset h^m$.

2) Now prove that if $\mu, \nu \in P$, $\mu \cdot \nu$ is defined in h, then $\mu \cdot \nu \in P$.

Let $\{i, j\} = \{1, 2\}$. If $\mu, \nu \in P_i \cup P_i^*$, then evidently $\mu \cdot \nu \in P$.

Consequently let $\mu \in P_i \cup P_i^*$, $\nu \in P_j \cup P_j^*$; if $\mu \in P_i$, $\nu = \alpha \cdot \beta$, $\alpha \in P_j$, $\vec{\alpha} \in |\ell|^\sigma$, β is an isomorphism of h, then $\overleftarrow{\alpha} = \vec{\mu} \in |k_i|^\sigma \cap |k_j|^\sigma$, thus $\alpha \in P_\ell$ and therefore $\mu \cdot \alpha \in P_i$ and then $\mu \cdot \nu \in P$. Let $\mu \in P_i$, $\nu \in P_j$. Then $\overleftarrow{\nu} \in |\ell|^\sigma$. Since ℓ is a good ℓ-subcategory of k_j, there exists a $\tau \in P_\ell$ and an isomorphism φ of k_j such that $\nu = \tau \cdot \varphi$. But then $\mu \cdot \tau \in P_i$ and therefore $\mu \cdot \nu = (\mu \cdot \tau) \cdot \varphi \in P$. Let $\mu = P_i^* - P_i$, $\mu = \alpha \cdot \beta$, where $\alpha \in P_i$, $\overrightarrow{\alpha} \in |\ell|^\sigma$, β is an isomorphism of h; and let $\nu \in P_j \cup P_j^*$, $\nu = \gamma \cdot \sigma$, where $\gamma \in P_j$ and σ is an isomorphism of h (σ may be also identity of course). Then $\overleftarrow{\beta} \in |\ell|^\sigma$, $\overrightarrow{\beta} \in |k_j|^\sigma$, consequently β is an isomorphism of k_j and therefore $\beta \cdot \gamma \in P_j$. Since ℓ is a good ℓ-subcategory of k_j, there exists a $\tau \in P_\ell$ and an isomorphism φ of k_j such that $\beta \cdot \gamma = \tau \cdot \varphi$. But then $\alpha \cdot \tau \in P_i$ and $\varphi \cdot \sigma$ is an isomorphism of h; thus $\mu \cdot \nu = \alpha \cdot \beta \cdot \gamma \cdot \sigma = (\alpha \cdot \tau) \cdot (\varphi \cdot \sigma) \in P$.

3) Similarly one may prove that if $\mu, \nu \in I$ and $\mu \cdot \nu$ is defined, then $\mu \cdot \nu \in I$.

4) Now we prove that if $\mu \in h^m$, then there exist $\alpha \in \in P$, $\beta \in I$ such that $\mu = \alpha \cdot \beta$. If moreover $\overrightarrow{\mu} \in |k_n|^\sigma$ or $\overleftarrow{\mu} \in |k_n|^\sigma$, then $\overrightarrow{\alpha} \in |k_n|^\sigma$ $(n = 1, 2)$. This is evident whenever $\mu \in |k_n|^m$. Now let $\{i, j\} = \{1, 2\}$, $\mu \in h^m$ such that $\overleftarrow{\mu} \in \in |k_i|^\sigma$, $\overrightarrow{\mu} \in |k_j|^\sigma$. Then there exist $\varphi \in |k_i|^m$, $\psi \in |k_j|^m$ with $\mu = \varphi \cdot \psi$. Then $\varphi = \pi_\varphi \cdot \iota_\varphi$, $\psi = \pi_\psi \cdot \iota_\psi$ where $\pi_\varphi \in P_i$, $\iota_\varphi \in I_\ell$, $\pi_\psi \in P_\ell$, $\iota_\psi \in I_j$, and thus there exist $\pi \in P_\ell$, $\iota \in I_\ell$ with $\pi \iota = \iota_\varphi \cdot \cdot \pi_\psi$. Now put $\alpha = \pi_\varphi \cdot \pi$, $\beta = \iota \cdot \iota_\psi$. Evidently

$\alpha \in P$, $\beta \in I$, $\mu = \alpha \cdot \beta$ and $\vec{\alpha} \in |\ell|^{\sigma}$.

5) Let $\{i, j\} = \{1, 2\}$. We recall,[11], that if $\mu \in h^m$, $\overleftarrow{\mu} \in |k_i|^m$, $\vec{\mu} \in |k_j|^m$, then $\mu = \alpha \cdot \beta$, where $\alpha \in |k_i|^m$, $\beta \in |k_j|^m$. If also $\mu = \alpha' \cdot \beta'$ with $\alpha' \in |k_i|^m$, $\beta' \in |k_j|^m$, then $\langle \alpha, \beta \rangle R^* \langle \alpha', \beta' \rangle$, where R^* is the smallest equivalence on the set

$\bigcup_{b \in |\ell|^{\sigma}} \{H_{k_i}(\overleftarrow{\mu}, b) \times H_{k_j}(b, \vec{\mu})\}$ which contains the following relation R: $\langle \alpha, \beta \rangle R \langle \alpha', \beta' \rangle$ if and only if there exists a $\rho \in |\ell|^m$ with $\alpha \cdot \rho = \alpha'$, $\rho \cdot \beta' = \beta$.

6) Now let $\langle \alpha, \beta \rangle R \langle \alpha', \beta' \rangle$. Choose $\pi_\alpha \in P_i$, $\iota_\alpha \in I_\ell$, $\pi_\beta \in P_\ell$, $\iota_\beta \in I_j$ such that $\pi_\alpha \cdot \iota_\alpha = \alpha$, $\pi_\beta \cdot \iota_\beta = \beta$. Then choose $\pi_{\alpha,\beta} \in P_\ell$, $\iota_{\alpha,\beta} \in I_\ell$ such that $\pi_{\alpha,\beta} \cdot \iota_{\alpha,\beta} = \iota_\alpha \cdot \pi_\beta$. Analogously choose $\pi_{\alpha'}$, $\iota_{\alpha'}$, $\pi_{\beta'}$, $\iota_{\beta'}$, $\pi_{\alpha',\beta'}$, $\iota_{\alpha',\beta'}$ (i.e. $\pi_{\alpha'} \cdot \iota_{\alpha'} = \alpha'$, $\pi_{\beta'} \cdot \iota_{\beta'} = \beta'$, $\pi_{\alpha',\beta'} \cdot \iota_{\alpha',\beta'} = \iota_{\alpha'} \cdot \pi_{\beta'}$, $\pi_{\beta'}$, $\pi_{\alpha',\beta'} \in P_\ell$, $\iota_{\alpha'}$, $\iota_{\alpha',\beta'} \in I_\ell$).

Now prove that there exists an isomorphism τ of ℓ such that $\pi_\alpha \cdot \pi_{\alpha,\beta} \cdot \tau = \pi_{\alpha'} \cdot \pi_{\alpha',\beta'}$, $\tau^{-1} \cdot \iota_{\alpha,\beta} \cdot \iota_\beta = \iota_{\alpha',\beta'} \cdot \iota_\beta$. Since $\langle \alpha, \beta \rangle R \langle \alpha', \beta' \rangle$ there exists a $\rho \in |\ell|^m$ with $\alpha \cdot \rho = \alpha'$, $\beta = \rho \cdot \beta'$. Choose $\pi_\rho \in P_\ell$, $\iota_\rho \in I_\ell$ with $\pi_\rho \cdot \iota_\rho = \rho$. Choose $\pi_{\alpha,\rho} \in P_\ell$, $\iota_{\alpha,\rho} \in I_\ell$ with $\pi_{\alpha,\rho} \cdot \iota_{\alpha,\rho} = \iota_\alpha \cdot \pi_\rho$. Then $\pi_{\alpha'} \cdot \iota_{\alpha'} = \alpha' = \alpha \cdot \rho = \pi_\alpha \cdot \iota_\alpha \cdot \pi_\rho \cdot \iota_\rho = (\pi_\alpha \cdot \pi_{\alpha,\rho}) \cdot (\iota_{\alpha,\rho} \cdot \iota_\rho)$; consequently there exists an isomorphism φ of k_i such that $\pi_\alpha \cdot \pi_{\alpha,\rho} \cdot \varphi = \pi_{\alpha'}$, $\iota_{\alpha,\rho} \cdot \iota_\rho = \varphi \cdot \iota_{\alpha'}$. Since $\vec{\varphi} \in |\ell|^{\sigma}$, $\overleftarrow{\varphi} \in |\ell|^{\sigma}$, φ is an isomorphism of ℓ. Now choose $\pi_{\rho,\beta'} \in P_\ell$, $\iota_{\rho,\beta'} \in I_\ell$ such that $\pi_{\rho,\beta'} \cdot \iota_{\rho,\beta'} = \iota_\rho \cdot \pi_{\beta'}$; then $(\pi_\rho \cdot \pi_{\rho,\beta'}) \cdot$

$\cdot (\iota_{\varphi,\beta'} \cdot \iota_{\beta'}) = \pi_{\varphi} \cdot \iota_{\varphi} \cdot \pi_{\beta'} \cdot \iota_{\beta'} = \varphi \cdot \beta' = \beta = \pi_{\beta} \cdot \iota_{\beta}$, so that there exists an isomorphism ψ of k_j such that $\pi_{\beta} \cdot \psi = \pi_{\varphi} \cdot \pi_{\varphi,\beta'}$, $\psi \cdot \iota_{\varphi,\beta'} \cdot \iota_{\beta'} = \iota_{\beta}$. Now $(\pi_{\alpha,\varphi} \cdot \varphi \cdot \pi_{\alpha',\beta'}) \cdot \iota_{\alpha',\beta'} = \pi_{\alpha,\varphi} \cdot \varphi \cdot (\pi_{\alpha',\beta'} \cdot \iota_{\alpha',\beta'}) = \pi_{\alpha,\varphi} \cdot \varphi \cdot \iota_{\alpha'} \cdot \pi_{\beta'} = \pi_{\alpha,\varphi} \cdot \iota_{\alpha,\varphi} \cdot \iota_{\varphi} \cdot \pi_{\beta'} = \iota_{\alpha} \cdot \pi_{\varphi} \cdot \iota_{\varphi} \cdot \pi_{\beta'} = \iota_{\alpha} \cdot \varphi \cdot \pi_{\beta'}$ and $\iota_{\alpha} \cdot \varphi \cdot \pi_{\beta'} \cdot \iota_{\beta'} = \iota_{\alpha} \cdot \varphi \cdot \beta' = \iota_{\alpha} \cdot \beta = \iota_{\alpha} \cdot \pi_{\beta} \cdot \iota_{\beta} = \iota_{\alpha} \cdot \pi_{\beta} \cdot \psi \cdot \iota_{\varphi,\beta'} \cdot \iota_{\beta'} = \pi_{\alpha,\beta} \cdot \iota_{\alpha,\beta} \cdot \psi \cdot \iota_{\varphi,\beta'} \cdot \iota_{\beta'}$; but all the considered morphisms are morphisms of k_j and $\iota_{\beta'}$ is a monomorphism of k_j; then $\iota_{\alpha} \cdot \varphi \cdot \pi_{\beta'} = \pi_{\alpha,\beta} \cdot \iota_{\alpha,\beta} \cdot \psi \cdot \iota_{\varphi,\beta'}$. Consequently $(\pi_{\alpha,\varphi} \cdot \varphi \cdot \pi_{\alpha',\beta'}) \cdot \iota_{\alpha',\beta'} = \pi_{\alpha,\beta} \cdot (\iota_{\alpha,\beta} \cdot \psi \cdot \iota_{\varphi,\beta'})$. All the considered morphisms are elements of $|\ell|^m$ and therefore there exists an isomorphism τ of ℓ such that $\pi_{\alpha,\beta} \cdot \tau = \pi_{\alpha,\varphi} \cdot \varphi \cdot \pi_{\alpha',\beta'}$, $\tau^{-1} \cdot \iota_{\alpha,\beta} \cdot \psi \cdot \iota_{\varphi,\beta'} = \iota_{\alpha',\beta'}$. But then evidently $\pi_{\alpha} \cdot \pi_{\alpha,\beta} \cdot \tau = \pi_{\alpha} \cdot \pi_{\alpha,\varphi} \cdot \varphi \cdot \pi_{\alpha',\beta'} = \pi_{\alpha'} \cdot \pi_{\alpha',\beta'}$, $\iota_{\alpha',\beta'} \cdot \iota_{\beta'} = \tau^{-1} \cdot \iota_{\alpha,\beta} \cdot \psi \cdot \iota_{\varphi,\beta'} \cdot \iota_{\beta'} = \tau^{-1} \cdot \iota_{\alpha,\beta} \cdot \iota_{\beta}$.

7) Now it is easy to see that if $\langle \alpha, \beta \rangle R^* \langle \alpha', \beta' \rangle$, $\pi_{\alpha}, \pi_{\alpha'} \in P_i$, $\iota_{\alpha}, \iota_{\alpha'} \in I_{\ell}$, $\pi_{\beta}, \pi_{\beta'} \in P_{\ell}$, $\iota_{\beta}, \iota_{\beta'} \in I_j$, $\alpha = \pi_{\alpha} \cdot \iota_{\alpha}$, $\alpha' = \pi_{\alpha'} \cdot \iota_{\alpha'}$, $\beta = \pi_{\beta} \cdot \iota_{\beta}$, $\beta' = \pi_{\beta'} \cdot \iota_{\beta'}$, $\pi_{\alpha,\beta}, \pi_{\alpha',\beta'} \in P_{\ell}$, $\iota_{\alpha,\beta}, \iota_{\alpha',\beta'} \in I_{\ell}$, $\pi_{\alpha,\beta} \cdot \iota_{\alpha,\beta} = \iota_{\alpha} \cdot \pi_{\beta}$, $\pi_{\alpha',\beta'} \cdot \iota_{\alpha',\beta'} = \iota_{\alpha'} \cdot \pi_{\beta'}$, then there exists an isomorphism τ of ℓ such that $\pi_{\alpha} \cdot \pi_{\alpha,\beta} \cdot \tau = \pi_{\alpha'} \cdot \pi_{\alpha',\beta'}$, $\iota_{\alpha',\beta'} \cdot \iota_{\beta'} = \tau^{-1} \cdot \iota_{\alpha,\beta} \cdot \iota_{\beta}$. Consequently if $\iota, \iota' \in I$, $\pi, \pi' \in P$, $\vec{\pi} \in |\ell|^{\sigma}$, $\vec{\pi'} \in |\ell|^{\sigma}$ and $\pi \cdot \iota$, $\pi' \cdot \iota'$ are defined and equal, then there exists an isomorphism τ of ℓ such that $\pi \cdot \tau = \pi'$, $\tau \cdot \iota' = \iota$.

8) Let now $\mu \in h^m$, $\mu = \pi \cdot \iota = \pi' \cdot \iota'$, where $\pi, \pi' \in P$, $\iota, \iota' \in I$.

We shall prove that

(∗) there exists an isomorphism τ of h such that
$\pi \cdot \tau = \pi'$, $\tau \cdot \iota' = \iota$.

Let $\{i, j\} = \{1, 2\}$. Let $\mu \in |k_i|^m$.
Then there exist isomorphisms φ, φ' of h such that $\pi = \rho \cdot \varphi$, $\iota = \varphi^{-1} \sigma$, $\pi' = \rho' \cdot \varphi'$, $\iota' = \varphi'^{-1} \cdot \sigma'$, where $\rho, \rho' \in P_i$, $\sigma, \sigma' \in I_i$. Thus there exists an isomorphism τ of k_i such that $\rho \cdot \tau = \rho'$, $\tau \cdot \sigma' = \sigma$, and then $\pi \cdot (\varphi^{-1} \cdot \tau \cdot \varphi') = \pi'$, $(\varphi^{-1} \cdot \tau \cdot \varphi') \iota' = \iota$.

Now let $\mu \in h^m$, $\overleftarrow{\mu} \in |k_i|^\sigma$, $\overrightarrow{\mu} \in |k_j|^\sigma$. If $\overrightarrow{\pi} \in |\ell|^\sigma$, $\overrightarrow{\pi'} \in |\ell|^\sigma$, then (∗) is proved in 7). The following four cases are possible: ($\overrightarrow{\pi} \in |k_i|^\sigma$, $\overrightarrow{\pi'} \in |k_i|^\sigma$) or ($\overrightarrow{\pi} \in |k_i|^\sigma$, $\overrightarrow{\pi'} \in |k_j|^\sigma$) or ($\overrightarrow{\pi} \in |k_j|^\sigma$, $\overrightarrow{\pi'} \in |k_j|^\sigma$) or finally ($\overrightarrow{\pi} \in |k_j|^\sigma$, $\overrightarrow{\pi'} \in |k_i|^\sigma$) . Only the first will be considered, the remaining are analogous.
If $\overrightarrow{\pi} \in |k_i|^\sigma$, $\overrightarrow{\pi'} \in |k_i|^\sigma$, then $\pi, \pi' \in P_i$ and there exist isomorphisms φ, φ' of k_i and $\sigma, \sigma' \in I_j$ such that $\iota = \varphi \cdot \sigma$, $\iota' = \varphi' \cdot \sigma'$. But then $\overrightarrow{\pi \cdot \varphi} \in |\ell|^\sigma$, $\overrightarrow{\pi' \cdot \varphi'} \in |\ell|^\sigma$ and there exists an isomorphism τ of ℓ with $\pi \cdot \varphi \cdot \tau = \pi' \cdot \varphi'$, $\tau \cdot \sigma' = \sigma$, and then $\pi \cdot (\varphi \cdot \tau \cdot \varphi'^{-1}) = \pi'$, $(\varphi \cdot \tau \cdot \varphi'^{-1}) \cdot \iota' = \iota$.

9) Now we must prove that every $\pi \in P$ is an epimorphism of h, every $\iota \in I$ is a monomorphism of h . It is sufficient to prove this for $\pi \in P_1 \cup P_2$, $\iota \in I_1 \cup I_2$ only. Let $\{i, j\} = \{1, 2\}$, $\pi \in P_i$, $\mu, \nu \in h^m$, $\pi \cdot \mu = \pi \cdot \nu$. Then evidently $\overrightarrow{\mu} = \overrightarrow{\nu}$. If $\overrightarrow{\mu} \in |k_i|^\sigma$, then $\mu = \nu$. Let $\overrightarrow{\mu} \in |k_j|^\sigma$. Since $\overleftarrow{\mu} \in |k_i|^\sigma$, there exist $\pi_\mu, \pi_\nu \in P_i$, $\iota_\mu, \iota_\nu \in I_j$ such that $\mu = \pi_\mu \cdot \iota_\mu$, $\nu = \pi_\nu \cdot \iota_\nu$ and $\overrightarrow{\pi_\mu} \in |\ell|^\sigma$,

$\vec{\pi}_\nu \in |\ell|^\sigma$. Since $(\pi \cdot \pi_\mu) \cdot \iota_\mu = (\pi \cdot \pi_\nu) \cdot \iota_\nu$, there exists an isomorphism τ of ℓ with $\pi \cdot \pi_\mu \cdot \tau = \pi \cdot \pi_\nu$, $\tau \cdot \iota_\nu = \iota_\mu$. But π is an epimorphism of k_i, so that $\pi_\mu \cdot \tau = \pi_\nu$ and consequently $\mu = \pi_\mu \cdot \iota_\mu = \pi_\mu \cdot \tau \cdot \iota_\nu = \pi_\nu \cdot \iota_\nu = \nu$. If $\iota \in I_i$, then ι is a monomorphism of h, as may be proved analogously.

10) Now prove that $P \cap I$ is the set of all isomorphisms of h. Let σ be an isomorphism of h; we shall prove that $\sigma \in P \cap I$. This is evident whenever $\sigma \in k_1^m \cup k_2^m$. Let $\{i, j\} = \{1, 2\}$, $\overleftarrow{\sigma} \in k_i^\sigma$, $\vec{\sigma} \in k_j^\sigma$. Then evidently $\sigma = \pi \cdot \iota$, where $\pi \in P_i$, $\iota \in P_j$. It may be then shown that $\iota \cdot \sigma^{-1} = \pi^{-1}$, $\sigma^{-1}\pi = \iota^{-1}$ in h, consequently $\sigma \in P \cap I$. Conversely let $\sigma \in P \cap I$, we shall prove that σ is an isomorphism of h. This is evident if $\sigma \in (P_1 \cup P_2) \cap (I_1 \cup I_2)$ Let $i, j \in \{1, 2\}$, $\sigma \in P_i^* \cap I_j^*$. Then $\sigma = \alpha \cdot \beta = \beta' \cdot \alpha'$, where $\alpha \in P_i$, $\vec{\alpha} \in |\ell|^\sigma$, $\alpha' \in I_j$, $\overleftarrow{\alpha'} \in |\ell|^\sigma$ and β, β' are isomorphisms of h. Thus $\alpha \cdot \varphi = \alpha'$, where $\varphi = \beta \cdot (\beta')^{-1}$. It is easy to see that $\overleftarrow{\varphi} \in |\ell|^\sigma$, $\vec{\varphi} = \overleftarrow{\sigma} = \overleftarrow{\alpha} \in k_i^\sigma$. Consequently φ is an isomorphism of k_i and therefore $\alpha \cdot \varphi \in P_i \cap I_j$. Consequently σ is an isomorphism of h.

The proof of II g) is complete, and $[h, P, I]$ has the required properties.

i) <u>Now prove that the property of being a b-category is of b-small character</u>. Evidently if $\{b_\alpha ; \alpha \in T\}$ is a monotone system of small b-categories, then

$B = \bigcup_{\alpha \in T} b_\alpha$ is a b-category.

Conversely, let there be given a b-category B. Then $|B| = \bigcup_{\alpha \in T} c_\alpha$, where $\{c_\alpha ; \alpha \in T\}$ is a monotone system of small categories. Let $\alpha \in T$ and let there be already constructed a monotone system $\{b_\beta ; \beta \in T, \beta < \alpha\}$ of small b-categories such that every b_β is a full b-subcategory of B and $c_\beta \subset |b_\beta|$. We shall construct b_α. Denote by γ_0 the smallest element of the class $\{\gamma \in T ; c_\alpha^\sigma \cup \bigcup_{\beta < \alpha} |b_\beta|^\sigma \subset c_\gamma^\sigma\}$; for every $\mu \in c_{\gamma_0}^m$ choose some $a_\mu \in |B|^\sigma$ such that there exist $\pi \in P_B$, $\iota \in I_B$ such that $\mu = \pi \cdot \iota$, $\vec{\pi} = a_\mu$. Choose $\gamma_1 \in T$, $\gamma_1 \geqq \gamma_0$, such that $a_\mu \in c_{\gamma_1}^\sigma$ for every $\mu \in c_{\gamma_0}^m$. For every $\mu \in c_{\gamma_1}^m$ choose some $a_\mu \in |B|^\sigma$ such that there exist $\pi \in P_B$, $\iota \in I_B$ with $\mu = \pi \cdot \iota$, $\vec{\pi} = a_\mu$. Choose $\gamma_2 \in T$, $\gamma_2 \geqq \gamma_1$ with $a_\mu \in c_{\gamma_2}^\sigma$ for every $\mu \in c_{\gamma_1}^m$, and so on. Let b_α be a full b-subcategory of B such that $|b_\alpha| = \bigcup_{n=1}^{\infty} c_{\gamma_n}$.

III. Universal category for categories with a structure.

It is easy to see that the idea of the metatheorem and its proof is the same for a-categories and for b-categories. Now we shall apply it to obtain a corresponding metatheorem for categories with a structure.

For the definition of categories with a structure the ideas given in [2] are used.

1. In the Bernays-Gödel set-theory one may not form the

category of all categories (not necessarily small) and all their functors, nor the category of all classes and all their mappings.

Thus we shall suppose that there exists a strongly inaccessible cardinal $\aleph_\tau$, i.e. an uncountable regular cardinal such that if $\aleph_\alpha < \aleph_\tau$, then $2^{\aleph_\alpha} < \aleph_\tau$; and let $\mathcal{U}$ be a set such that

1) $card\ \mathcal{U} = \aleph_\tau$;

2) if a set A is an element of $\mathcal{U}$, then $card\ A < \aleph_\tau$;

3) if $card\ A < \aleph_\tau$ then $A \in \mathcal{U} \Longleftrightarrow A \subset \mathcal{U}$ x).

Every category K such that $K^\sigma \cup K^m \subset \mathcal{U}$ and that for every $a, b \in K^\sigma$ there is $H_K(a, b) \in \mathcal{U}$ will be called a $\mathcal{U}$-category.

A $\mathcal{U}$-category K will be called small if $K^\sigma \in \mathcal{U}$.

2. Denote by $\mathbb{M}$ the category of all sets $A \subset \mathcal{U}$ and all their mappings. Denote by $\mathbb{C}$ the category of all $\mathcal{U}$-categories and all their functors. Denote by $\mathscr{C} : \mathbb{C} \to \mathbb{M}$ the forgetful functor, i.e. the functor which to e-

x) As is well-known, the existence of a strongly inaccessible cardinal is not provable from the axioms of the Bernays-Gödel set-theory. But if we suppose it, then a set $\mathcal{U}$ with properties 1) to 3) may be easily constructed.

In [4] the ordered couple $\langle x, y \rangle$ is defined to be $\langle x, y \rangle = \{x, \{x, y\}\}$, where $\{x, y\}$ denotes the set consisting of x and y. Thus if $\mathcal{U}$ satisfies 1) to 3) then $A, B \in \mathcal{U}$ implies $A \times B \in \mathcal{U}$.

very $\mathcal{U}$-category K assigns the set K^m of all its morphisms.

3. Let $\mathbb{S}$ be a category, let $\mathcal{S}: \mathbb{S} \to \mathbb{M}$ be a functor with the following properties:

α) $\mathcal{S}$ is faithful, i.e. if $\alpha, \beta \in \mathcal{S}^m$, $\overleftarrow{\alpha} = \overleftarrow{\beta}$, $\overrightarrow{\alpha} = \overrightarrow{\beta}$, $(\alpha)\mathcal{S} = (\beta)\mathcal{S}$, then $\alpha = \beta$;

β) if $\alpha \in H_{\mathbb{S}}(s, s')$ is an isomorphism of $\mathbb{S}$, $s_0 \in \mathbb{S}^\sigma$, $(s)\mathcal{S} = (s_0)\mathcal{S}$, then in $\mathbb{S}$ there exists exactly one isomorphism β such that $\overleftarrow{\beta} = s_0$ and $(\alpha)\mathcal{S} = (\beta)\mathcal{S}$;

γ) if $m \in \mathbb{M}^\sigma \cap \mathcal{U}$, then all $s \in \mathbb{S}^\sigma$ such that $(s)\mathcal{S} = m$ form a set the power of which is less than $\aleph_\tau$.

The objects of $\mathbb{S}$ will be called structures, $\mathbb{S}$ will be called a category of structures.

4. <u>Definition</u>: Let $s, s' \in \mathbb{S}^\sigma$. We shall say that s' is a substructure of s if:

a) $(s')\mathcal{S} \subset (s)\mathcal{S}$;

b) there exists an $\iota \in H_{\mathbb{S}}(s', s)$ such that $(\iota)\mathcal{S} : (s')\mathcal{S} \to (s)\mathcal{S}$ is the inclusion mapping;

c) if $s'' \in \mathbb{S}^\sigma$, $\wp \in H_{\mathbb{S}}(s'', s)$ are such that $(s'')\mathcal{S} \subset (s')\mathcal{S}$ and $(\wp)\mathcal{S} : (s'')\mathcal{S} \longrightarrow (s)\mathcal{S}$ is the inclusion mapping, then there exists exactly one $\alpha \in H_{\mathbb{S}}(s'', s')$ such that $\wp = \alpha \cdot \iota$ and that $(\alpha)\mathcal{S} : (s'')\mathcal{S} \longrightarrow (s')\mathcal{S}$ is the inclusion mapping.

It is easy to see that ι from the definition is unique. It will be called an inclusion morphism (of s' into s in $\mathbb{S}$). It is easy to see that if s', s'' are both substructures of s and $(s')\mathcal{S} = (s'')\mathcal{S}$, then $s' = s''$; and that if s'' is a substructure of s' and s' is a substructure of s, then s'' is a substructure of s.

5. Let $\mathbb{C}_s$ be a fixed subcategory of $\mathbb{C} \times \mathbb{S}$ such that:

   a) the objects of $\mathbb{C}_s$ are some $\langle k, s \rangle$, where $k \in \mathbb{C}^{\sigma}$, $s \in \mathbb{S}^{\sigma}$, $(k)\mathcal{C} = (s)\mathcal{S}$; the morphisms of $\mathbb{C}_s$ are some $\langle \varphi, f \rangle$ where $\varphi \in \mathbb{C}^{m}$, $f \in \mathbb{S}^{m}$, $(\varphi)\mathcal{C} = (f)\mathcal{S}$;

   b) if $\langle \varphi, f \rangle \in \mathbb{C}_s^{m}$ and φ is an isomorphism of $\mathbb{C}$, f is an isomorphism of $\mathbb{S}$, then $\langle \varphi^{-1}, f^{-1} \rangle \in \mathbb{C}_s^{m}$.

The objects of $\mathbb{C}_s$ will be called s-categories, morphisms of $\mathbb{C}_s$ will be called s-functors. If $K = \langle k, s \rangle$ is an s-category, put $|K| = k$ and call it the underlying category of K. If $\Phi = \langle \varphi, f \rangle$ is an s-functor, put $|\Phi| = \varphi$ and call it the underlying functor of Φ. If $\Phi = \langle \varphi, f \rangle$ is an s-functor such that φ is an inclusion functor, f is an inclusion morphism in $\mathbb{S}$, call Φ an inclusion s-functor; moreover if φ is full, call Φ a full inclusion s-functor. If K', K are s-categories, we shall say that K' is a (full) sub-s-category of K whenever there exists an (full) inclusion s-functor from K' to K.

If K is an s-category, we shall say that K is small whenever $|K|$ is small $\mathcal{U}$-category (i.e. $|K|^{\sigma} \in \mathcal{U}$). Every isomorphism of $\mathbb{C}_s$ will be called an s-isofunctor onto. If Φ is an s-functor, $\Phi = \Phi' \cdot \iota$, where Φ'

is an $\mathcal{S}$-isofunctor onto, ι is an (full) inclusion $\mathcal{S}$-functor, then Φ will be called an (full) $\mathcal{S}$-embedding or an $\mathcal{S}$-isofunctor into (onto a full sub-$\mathcal{S}$-category).

6. Let $\{\mathcal{k}_\alpha ; \alpha \in T\}$ be a system of small $\mathcal{S}$-categories, T is an O_n-ordered set, $T \subset \mathcal{U}$ and if $\alpha < \alpha'$ then $\mathcal{k}_\alpha$ is a full sub-$\mathcal{S}$-category of $\mathcal{k}_{\alpha'}$. Then we shall say that $\{\mathcal{k}_\alpha ; \alpha \in T\}$ is a monotone system of small $\mathcal{S}$-categories. If there exists exactly one $\mathcal{k} \in \mathcal{C}_{\mathcal{S}}^{\sigma}$ such that $|\mathcal{k}| = \bigcup_{\alpha \in T} |\mathcal{k}_\alpha|$ and that every $\mathcal{k}_\alpha$ is a full sub-$\mathcal{S}$-category of $\mathcal{k}$, then we shall say that $\{\mathcal{k}_\alpha ; \alpha \in T\}$ is summable and $\mathcal{k}$ is its union, and denote by $\mathcal{k} = \bigcup_{\alpha \in T} \mathcal{k}_\alpha$.

Let $\{\mathcal{k}_\alpha ; \alpha \in T\}$, $\{\mathcal{h}_\alpha ; \alpha \in T\}$ be monotone systems of small $\mathcal{S}$-categories. Let $\Phi_\alpha : \mathcal{h}_\alpha \longrightarrow \mathcal{k}_\alpha$ be an $\mathcal{S}$-embedding for every $\alpha \in T$ such that $\Phi_\alpha \cdot {}^{\mathcal{k}}\iota_\alpha^{\alpha'} = {}^{\mathcal{h}}\iota_\alpha^{\alpha'} \cdot \Phi_{\alpha'}$ for every $\alpha < \alpha'$, where by ${}^{\mathcal{h}}\iota_\alpha^{\alpha'} : \mathcal{h}_\alpha \longrightarrow \mathcal{h}_{\alpha'}$, ${}^{\mathcal{k}}\iota_\alpha^{\alpha'} : \mathcal{k}_\alpha \longrightarrow \mathcal{k}_{\alpha'}$ are denoted the inclusion-$\mathcal{S}$-functors. Then we shall say that $\{\Phi_\alpha ; \alpha \in T\}$ is a monotone system of $\mathcal{S}$-embeddings. Let $\mathcal{k}$ or $\mathcal{h}$ be an union of $\{\mathcal{k}_\alpha ; \alpha \in T\}$ or $\{\mathcal{h}_\alpha ; \alpha \in T\}$ respectively. If there exists exactly one $\mathcal{S}$-embedding $\Phi : \mathcal{h} \rightarrow \mathcal{k}$ such that ${}^{\mathcal{h}}\iota_\alpha \, \Phi = \Phi_\alpha \cdot {}^{\mathcal{k}}\iota_\alpha$ for every $\alpha \in T$, where ${}^{\mathcal{k}}\iota_\alpha : \mathcal{k}_\alpha \longrightarrow \mathcal{k}$, ${}^{\mathcal{h}}\iota_\alpha : \mathcal{h}_\alpha \longrightarrow \mathcal{h}$ are inclusion $\mathcal{S}$-functors, then we shall say that $\{\Phi_\alpha ; \alpha \in T\}$ is summable and that Φ is its union, and denote it by $\Phi = \bigcup_{\alpha \in T} \Phi_\alpha$.

7. Let W be a property of $\mathcal{s}$-embeddings.

We shall say that W <u>is categorial</u> if:

a) every $\mathcal{s}$-isofunctor onto has W ;

b) if Φ and Φ' have W and $\Phi \cdot \Phi'$ is defined then it also has W.

We shall say that W <u>is full</u> if it has the following property:

if Φ is an $\mathcal{s}$-isofunctor onto, ι a full inclusion $\mathcal{s}$-functor with W, both from the same small $\mathcal{s}$-category, both to small $\mathcal{s}$-categories, then there exists an $\mathcal{s}$-isofunctor Φ' onto and a full inclusion $\mathcal{s}$-functor ι' with W such that $\Phi \iota' = \iota \Phi'$. We shall say that W <u>is monotonically additive</u> if every monotone system $\{\Phi_\alpha ; \alpha \in T\}$ of $\mathcal{s}$-embeddings with W such that $\{\overleftarrow{\Phi_\alpha} ; \alpha \in T\}$, $\{\overrightarrow{\Phi_\alpha} ; \alpha \in T\}$ are summable, has a union with W.

8. We shall say that $\langle \ell , \mathcal{K} \rangle$ is an $\mathcal{s}$-semiamalgam if $\mathcal{K}$ is a set of small $\mathcal{s}$-categories, $card\, \mathcal{K} < \aleph_\tau$ and ℓ is a full sub-$\mathcal{s}$-category of every $k \in \mathcal{K}$. If moreover $|k|^\sigma \cap |k'|^\sigma = |\ell|^\sigma$ whenever $k, k' \in \mathcal{K}$, $k \neq k'$, then we shall say that $\langle \ell , \mathcal{K} \rangle$ is an $\mathcal{s}$-amalgam.

The definition of an $\mathcal{s}$-unglueing of an $\mathcal{s}$-semiamalgam, and of an $\mathcal{s}$-filling of an $\mathcal{s}$-amalgam is evident.

9. Let W be a property of $\mathcal{s}$-embeddings, V a property of $\mathcal{s}$-categories.

We shall say that V is $\mathcal{s}$-<u>amalgamic with respect to</u> W if every $\mathcal{s}$-amalgam $\langle \ell , \mathcal{K} \rangle$ such that ℓ has V, that every $k \in \mathcal{K}$ has V and that, for every $k \in \mathcal{K}$ the inclusion $\mathscr{s}$ -functor $\iota_{\ell}^{k}: \ell \to k$ has W, has an $\mathscr{s}$ -filling K with V such that for every $k \in \mathcal{K}$ the inclusion $\mathscr{s}$ -functor $\iota_{k}^{K}: k \to K$ has W.

We shall say that V <u>is of</u> $\mathscr{s}$ <u>-small</u> W <u>-character</u> if it has the following property:

a) if $\{k_\alpha ; \alpha \in T\}$ is a monotone system of small $\mathscr{s}$ -categories with V such that the inclusion $\mathscr{s}$-functor $\iota_{\alpha}^{\alpha'}: k_\alpha \to k_{\alpha'}$ has W for every $\alpha < \alpha'$, then its union exists and has V;

b) if an $\mathscr{s}$ -category K has V, then $K = \bigcup_{\alpha \in T} k_\alpha$, where $\{k_\alpha ; \alpha \in T\}$ is a monotone system of small $\mathscr{s}$ -categories with V such that for every $\alpha < \alpha'$ the inclusion $\mathscr{s}$ -functor $\iota_{\alpha}^{\alpha'}: k_\alpha \to k_{\alpha'}$ has W.

Let $\bar{k}$ be a small $\mathscr{s}$ -category. We shall say that V is $\bar{k}$ - $\mathscr{s}$-invariant if the following obtains:

a) $\bar{k}$ has V;

b) every $\mathscr{s}$ -category with V contains $\bar{k}$ as a full sub- $\mathscr{s}$ -category;

c) if k is a small $\mathscr{s}$ -category with V, φ is an isofunctor of $|k|$ onto a category ℓ identical on $|\bar{k}|$, then there exists a small $\mathscr{s}$ -category h with V and an $\mathscr{s}$ -isofunctor Φ of k onto h such that $|h| = \ell$, $|\Phi| = \varphi$.

10. <u>Metatheorem for $\mathscr{s}$ -categories</u>. Let W be a property of $\mathscr{s}$ -embeddings, which is categorial, full and monotonically additive. Let $\bar{k}$ be a small $\mathscr{s}$ -category. Let V be a property of $\mathscr{s}$ -categories, which is $\bar{k}$ - $\mathscr{s}$ -invariant, $\mathscr{s}$ -amalgamic with respect to W and is of $\mathscr{s}$ -small

W-character .
Then there exists an $\mathfrak{s}$-category U with V such that every $\mathfrak{s}$-category with V can be fully $\mathfrak{s}$-embedded in U. This embedding has W and its underlying functor is identical on $|\bar{k}|$.
The proof is left to the reader.

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(Received January 31, 1966)