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AN ELEMENTARY PROOF OF NORMALITY OF THE CLASS OF ACCESSIBLE CARDINALS

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This remark is connected with the paper [2] by Keisler and Tarski. In the following we shall use the terminology and notations introduced by Keisler and Tarski without references.

In their paper [2], Keisler and Tarski have remarked on the absence of an elementary proof of theorem 1.33. In this remark we shall give such a proof.

Let J be a strongly α-complete non-principal prime ideal in $S(\alpha)$. Let f be a function such that

$$f \in {}^{\alpha}\alpha$$
$$f(\xi) \in \xi \quad \text{for } \xi \in \alpha .$$

By Lemma 1.5 and 1.16, there exists a $\xi_0 \in \alpha$ such that

$$\tau_J(f) = \xi_0 ,$$

i.e. that $\{\xi : f(\xi) = \xi_0\} \notin J$.

In the following, we shall often use this fact.

Theorem. The class AC of accessible cardinals is normal.

Proof: Let $\alpha \in C - C_1$, and J be a strongly α-complete non-principal prime ideal in $S(\alpha)$. Let d be the identity function on α. By theorem 1.18, it suffices to próve

$$AC \cap \alpha \in J .$$

Set $m_1 = \alpha - AC$

$m_2 = \{\xi : \xi \in \alpha \ \& \ (\exists\delta)(\delta \in \xi \leq 2^{\delta})\}$

$m_3 = \{\xi : \xi \in \alpha \ \& \ cf(\xi) < \xi\}$

and define

$$g(\xi) = \begin{cases} 0 & \text{for } \xi \in m_1 \cup m_3 \\ \text{the least } \delta \text{ such that } \delta < \xi \leq 2^{\delta} & \text{for } \xi \in m_2 \end{cases}.$$

We have $(\forall\xi)(\xi \in \alpha \rightarrow g(\xi) \in d(\xi))$, i.e. $\tau_J(g) < \alpha = \tau_J(d)$.

Hence there exists an $\xi_0 \in \alpha$ for which

$$\{\xi : g(\xi) = \xi_0\} \notin J .$$

If $\xi_0 > 0$, then $|\{\xi : g(\xi) = \xi_0\}| \leq 2^{\xi_0} < \alpha$, which is a contradiction ($x \notin J$ implies $|x| = \alpha$). Thus $\xi_0 = 0$, i.e.

$$m_1 \cup m_2 = \{\xi : g(\xi) = 0\} \notin J .$$

Now define $h(\xi) = cf(\xi)$ for $\xi \in \alpha$.

Let $m_3 \notin J$. Then $\tau_J(h) < \alpha$, and there exists a $\xi_1 \in \alpha$ such that $x = \{\xi : h(\xi) = \xi_1\} \notin J$.

For every $\xi \in x$ there is an ξ_1-termed sequence $\{\beta^{\xi}_{\eta}\}_{\eta \in \xi_1}$ such that $\lim_{\eta \in \xi_1} \beta^{\xi}_{\eta} = \xi$.

Define

$$h_{\eta}(\xi) = \begin{cases} \beta^{\xi}_{\eta} & \text{for } \xi \in x \\ 0 & \text{for } \xi \in \alpha - x \end{cases}.$$

It is easily shown that for every $\eta \in \xi_1$ there exists a γ_{η} such that

$$y_{\eta} = \{\xi : h_{\eta}(\xi) = \gamma_{\eta}\} \notin J .$$

Since $\xi_1 \in \alpha$, we conclude

$$y = \bigcap_{\eta \in \xi_1} y_{\eta} \notin J .$$

If $\xi \in y$, then $h_{\eta}(\xi) = \gamma_{\eta}$ and $\xi = \lim_{\eta \in \xi_1} \gamma_{\eta}$.

Thus y is a one-point set, contradicting the non-principality of $\mathcal{J}$. Hence $m_3 \in \mathcal{J}$. Since $\alpha \cap \mathbf{AC} \subseteq m_2 \cup m_3$, it follows that $\alpha \cap \mathbf{AC} \in \mathcal{J}$, and our proof is complete.

Let us remark on a connection with the theory of syntactical models of the Gödel-Bernays set theory. We use the notations introduced in [1].

It can be shown that α is representable by a function f if and only if $F_{\mathcal{J}}(f) = \alpha$ ($F_{\mathcal{J}}$ is the isomorphism between $\Gamma_{\mathcal{J}}$ and a perfect class for some α-complete prime ideal $\mathcal{J}$ in $S(\alpha)$. Thus, a class X is normal if and only if both

(i) $X \subseteq \mathbf{C}_1$, and

(ii) $F_{\mathcal{J}}(f) \neq \alpha$ for every $f \in {}^{\alpha}(X \cap \alpha)$ and every α-complete prime ideal $\mathcal{J}$ in $S(\alpha)$.

Let $\alpha \in \mathbf{C} - \mathbf{C}_1$, $F_{\mathcal{J}}(f) = \alpha$. Since α is an inaccessible cardinal, f is an inaccessible cardinal in sense of the model $\Gamma_{\mathcal{J}}$. By metatheorem 3 from [3] we have

$$\{\xi : f(\xi) \in \mathbf{AC}\} \in \mathcal{J} .$$

Thus, if α is representable by some function $f \in (\mathbf{AC} \cap \alpha)$, then $\alpha \in \mathcal{J}$; this is a contradiction.

References:

[1] L. BUKOVSKÝ and K. PŘÍKRÝ, Some metamathematical properties of measurable cardinals, to appear in Bull.Acad.Polon.Sci.

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[4] P. VOPĚNKA, Построение моделей теории множеств методом ультрапроизведения, Zeitschr.f.math. Logik und Grundlagen d.Math.8(1962),pp. 294-304.

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