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ON CONTINUITY STRUCTURES AND SPACES OF MAPPINGS

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A type of "continuity structures" [cf. 7] is considered. The spaces in question occur, e.g., under the name of "quasi-uniform spaces" in [5] (where some further references are given), and under the name of "P-Räume" in [4]. They have been also used implicitly, as tools for investigation of topological spaces, e.g., in [4]. In the present note, they are called "merotopic". The main results (section 3) concern spaces of continuous mappings (of spaces belonging to a somewhat narrower class). These results are closely related to some known theorems on spaces of mappings of topological spaces [1,3] and of "quasi-topological" [11] ones. Thus, equalities such as $(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}} \cong \mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$ are obtained, and it is proved that, roughly speaking, a merotopic space $\mathcal{X}$ has a base (in a specific sense) consisting of totally bounded sets if and only if every $\mathcal{Y}^{\mathcal{X}}$, $\mathcal{Y}$ topological, is a topological merotopic space (i.e., its structure is induced by a topology).

In addition to the main propositions, other results are included; some of these are needed in section 3, whereas others, although essentially re-formulations of known propositions, may deserve an explicit statement in a new context.

Most proofs are omitted. Some of the omitted proofs as well as some further results and examples are intended for publication in another paper.

1.

1.1. As a rule, the notation of E. Čech, Topological Spaces (rev.ed., in press) is adopted. It is close to that used by N. Bourbaki, and only some deviations need be explained.

If X is a set, then the set of all $Y \subset X$ is denoted by $\exp X$. If $\mathfrak{M}$ and $\mathfrak{N}$ are collections of sets, then $[\mathfrak{M}] \cap [\mathfrak{N}]$ denotes the collection of all $M \cap N$, $M \in \mathfrak{M}$, $N \in \mathfrak{N}$. If $\mathfrak{A}$, $\mathfrak{L}$ are collections of sets and for any $A \in \mathfrak{A}$ there is a $B \in \mathfrak{L}$ with $B \subset A$, then we shall say that $\mathfrak{L}$ minorizes $\mathfrak{A}$.

If f is a mapping (or a single-valued relation), then its value at a point (element) x will be denoted by fx; if X is a set, then $f[X]$ will denote the set of all fx, $x \in X$. If F is a set of mappings, then $F[X]$ denotes the set of all fx, $f \in F$, $x \in X$; $\mathcal{F}[\mathfrak{M}]$, where $\mathcal{F}$ is a collection of sets of mappings and $\mathfrak{M}$ is a collection of sets, denotes the collection of all $F[M]$, $F \in \mathcal{F}$, $M \in \mathfrak{M}$, etc.

There is a sharp formal distinction between families $\{x_a \mid a \in A\}$ and collections (in particular, between families of sets and collections of sets). However, this distinction will often be disregarded, and, e.g., properties defined for collections of sets will be carried over to families of sets and vice versa. We shall also often use the same symbols, e.g., for a space and for the set of its points, etc.

1.2. The following evident proposition will be useful.

Lemma. If $\mathcal{F}$ is a filter, $\mathcal{F} = \bigcup_{i=1}^{n} \mathfrak{M}_i$, then some $\mathfrak{M}_k$ minorizes $\mathcal{F}$.

1.3. The definition of closure spaces may be recalled.

Let E be a set. A (binary) relation τ on exp E which assigns exactly one set $\tau X \subset E$ to every set $X \subset E$ will be called a closure operation on E if (i) $\tau \emptyset = \emptyset$, (ii) $\tau(X \cup Y) = \tau X \cup \tau Y$, (iii) $X \subset \tau X$; $\langle E, \tau \rangle$ will then be called a closure space. If, in addition, $\tau(\tau X) = \tau X$ for every $X \subset E$, then τ is called a topological closure relation (or simply a topology) and $\langle E, \tau \rangle$ is called a topological space. Clearly, this concept is equivalent (in an obvious sense) with that of a topological space defined in the current manner by means of open sets.

1.4. Definition. Let E be a set. Let a non-void system Γ of non-void collections of subsets of E be given, such that

(1) if $\mathfrak{M} \in \Gamma$, $\mathfrak{M}_1 \subset \exp E$ and $\mathfrak{M}_1$ minorizes $\mathfrak{M}$, then $\mathfrak{M}_1 \in \Gamma$;

(2) if $\mathfrak{M}_1 \cup \mathfrak{M}_2 \in \Gamma$, then either $\mathfrak{M}_1 \in \Gamma$ or $\mathfrak{M}_2 \in \Gamma$;

(3) if $x \in E$, then $((x)) \in \Gamma$.

Then we shall say that Γ is a merotopic structure (or merotopy) on E and $\langle E, \Gamma \rangle$ is a merotopic space; every collection $\mathfrak{M} \in \Gamma$ will be called Γ-micromeric or $\mathcal{X}$-micromeric (or simply micromeric if the space is clear from the context).

Remark. In "merotopic", the first part of the word comes from the Greek "meros" - part. As shown below (3.10), important merotopic spaces are generated by topologizing certain parts of a set. Thus "merotopological" should be used, and that only for a narrower class of spaces; however, we prefer an abridged term used for a wider concept. The expression "micromeric collection" corresponds to the Russian "sistema s malymi", see e.g. [10] . - In [5], the term "quasi-uniform" is used for spaces equivalent (see below, 1.20) to merotopic spaces. If the standpoint of the present note is adopted, i.e. if merotopic spaces are the main objects of investigation (these generalize uniform, topological, etc. spaces), the name "quasi-uniform" seems less appropriate.

1.5. <u>Definition</u>. If Γ, Γ_1 are merotopies on E and $\Gamma \subset \Gamma_1$, we shall say that Γ is finer than Γ_1 and that Γ_1 is coarser than Γ, and we shall write $\Gamma \leq \Gamma_1$.

<u>Convention</u>. The set of all merotopies on a given set will always be considered with the order just described.

1.6. <u>Proposition</u>. The set of merotopies on a given set is order-complete. If $\{\Gamma_a\}$ is a family of merotopies on E, then $\sup_a \Gamma_a = \bigcup_a \Gamma_a$.

1.7. <u>Definition</u>. If $\langle E_1, \Gamma_1 \rangle$, $\langle E_2, \Gamma_2 \rangle$ are merotopic spaces, then a mapping $f : \langle E_1, \Gamma_1 \rangle \to \langle E_2, \Gamma_2 \rangle$ is called continuous (more specifically, merotopically continuous) if $f[\Gamma_1] \subset \Gamma_2$. A bijective continuous mapping f is called an isomorphism if f^{-1} is continuous.

1.8. Merotopic spaces as objects and their continuous mappings as morphisms form a category, which will be denoted by $\mathbf{M}$.

1.9. <u>Proposition and definition</u>. Let $\langle E, \tau \rangle$ be a closure space. For any $x \in E$, let $\mathcal{V}_x$ be the collection of all neighborhoods of x. Then the system Γ_τ of all $\mathfrak{M} \subset \exp E$ minorizing some $\mathcal{V}_x$ is a merotopy. We shall say that Γ_τ is induced by the closure operation τ, or that it is closure-induced. If τ is topological, then we shall call Γ_τ topological.

1.10. <u>Proposition and definition</u>. Let $\langle E, \sigma \rangle$ be a proximity space. Let Γ_σ consist of all $\mathfrak{M} \subset \exp E$ satisfying the following condition: If $E = \bigcup_{k=1}^{n} A_k$ and A_k, B_k, $k = 1, \dots, n$, are distant, then, for some k and some $M \in \mathfrak{M}$, $M \cap A_k \neq \emptyset$, $M \cap B_k = \emptyset$. Then Γ_σ is a merotopy on E. We shall say that it is induced by the proximity σ or that it is a proximally induced (or simply a proximal) merotopy.

1.11. <u>Proposition and definition</u>. Let $\langle E, \mathcal{U} \rangle$ be a uniform space. Let $\Gamma_{\mathcal{U}}$ consist of all $\mathfrak{M} \subset \exp E$ such that, for every $U \in \mathcal{U}$, there is a set $M \in \mathfrak{M}$ with $M \times M \subset U$. Then $\Gamma_{\mathcal{U}}$ is a merotopy on E. We shall say that $\Gamma_{\mathcal{U}}$ is induced by $\mathcal{U}$ or that Γ is a uniformly induced (or simply uniform) merotopy.

1.12. It is clear that if $\mathbf{U}$ is the category of all uniform spaces, the functor which assigns $\langle E, \Gamma_{\mathcal{U}} \rangle$ to $\langle E, \mathcal{U} \rangle$ and $f : \langle E, \Gamma_{\mathcal{U}} \rangle \to \langle E', \Gamma_{\mathcal{U}'} \rangle$ to $f : \langle E, \mathcal{U} \rangle \to \langle E', \mathcal{U}' \rangle$ is a one-to-one covariant functor from $\mathbf{U}$ into $\mathbf{M}$. A similar assertion holds for the categories of closure spaces and of proximity spaces.

1.13. <u>Definition</u>. Let $\mathbf{K}$ be a class of merotopic spaces; let E be a set. Let $\{f_a\}$ be a family of mappings $f : \mathcal{X}_a \to E$ where $\mathcal{X}_a$ are merotopic spaces. If there exists a merotopy Γ such that

(1) $\langle E, \Gamma \rangle \in \mathbb{K}$, and all $f_a : \mathcal{X}_a \to \langle E, \Gamma \rangle$ are continuous,

(2) if $\langle E, \Theta \rangle \in \mathbb{K}$ and if all $f_a : \mathcal{X}_a \to \langle E, \Theta \rangle$ are continuous, then $\Gamma \leqq \Theta$,

we shall say that Γ is inductively generated in $\mathbb{K}$ (for short, $\mathbb{K}$ -generated) by the family $\{f_a\}$. If $\mathbb{K}$ consists of all merotopic spaces, we shall simply say that Γ is (inductively) generated by $\{f_a\}$.

Let $\{g_b\}$ be a family of mappings $g_b : E \to \mathcal{Y}_b$, where $\mathcal{Y}_b$ are merotopic spaces. If there exists a merotopy Γ such that

(1) $\langle E, \Gamma \rangle \in \mathbb{K}$ and all $g_b : \langle E, \Gamma \rangle \to \mathcal{Y}_b$ are continuous;

(2) if $\langle E, \Theta \rangle \in \mathbb{K}$ and all $g_b : \langle E, \Theta \rangle \to \mathcal{Y}_b$ are continuous, then $\Theta \leqq \Gamma$,

we shall say that Γ is projectively generated in $\mathbb{K}$ (for short, $\mathbb{K}$ -generated) by the family $\{g_b\}$. If $\mathbb{K}$ consists of all merotopic spaces, we shall simply say that Γ is (projectively) generated by $\{g_b\}$.

1.14. <u>Proposition</u>. Let E be a set. Every non-void family $\{f_a\}$ of mappings $f_a : \mathcal{X}_a \to E$ (respectively, $f_a : E \to \mathcal{X}_a$), where $\mathcal{X}_a$ are merotopic spaces, inductively (projectively) generates a merotopy on E.

<u>Remark</u>. The system $\bigcup f_a[\Gamma_a]$ is fundamental (see 1.17) for the merotopy inductively generated by $f_a : \langle X_a, \Gamma_a \rangle \to E$.

<u>1.15. Definition</u>. If $\mathcal{X}$ is a merotopic space, then the space generated by a surjective mapping $f : \mathcal{X} \to E$ will be denoted by $\mathcal{X}/f$ and will be called the quotient space of $\mathcal{X}$ by f.

1.16. **Definition.** If $\langle Y, \Gamma \rangle$ is a merotopic space, $X \subset Y$, then the space projectively generated by the identity injection of X into Y, is called a subspace of $\langle Y, \Gamma \rangle$.

Clearly, $\langle X, \theta \rangle$ is a subspace of $\langle Y, \Gamma \rangle$ iff $X \subset Y$ and $\theta = \Gamma \cap \exp\exp X$.

1.17. **Definition.** Let $\langle E, \Gamma \rangle$ be a merotopic space. A system $\theta \subset \Gamma$ is called a Γ-fundamental system (of micromeric collections) if Γ is the least merotopy on E containing θ as a subsystem. A collection $\mathcal{L} \subset \exp E$ is called a base for $\langle E, \Gamma \rangle$ if there exists a fundamental system θ such that $\mathfrak{M} \in \theta$ implies $\mathfrak{M} \subset \mathcal{L}$.

Examples. If τ is a topology on E, then a collection of open sets is a base for $\langle E, \Gamma_\tau \rangle$ if and only if it is an open base of $\langle E, \tau \rangle$. The void system is fundamental if and only if Γ is the finest merotopy on E.

1.18. **Definition.** Let $\langle E, \Gamma \rangle$ be a merotopic space. Then a collection $\mathcal{V} \subset \exp E$ will be called a cover of $\langle E, \Gamma \rangle$ or a Γ-cover (or a merotopic cover) if, for any $\mathfrak{M} \in \Gamma$, there exist $V \in \mathcal{V}$ and $M \in \mathfrak{M}$ with $M \subset V$.

1.19. **Definition.** Let E be a set. A non-void system Ω of covers of E (i.e. of collections $\mathcal{V}$ such that $\bigcup \mathcal{V} = E$) is called a quasi-uniformity [see 5; cf. also, e.g., 6 and 9] on E if it is a filter under the refinement order, i.e. if the following hold:

(a) if $\mathcal{V} \in \Omega$, $\mathcal{W}$ is a cover of E and $\mathcal{V}$ refines $\mathcal{W}$, then $\mathcal{W} \in \Omega$ (b) if $\mathcal{V} \in \Omega$, $\mathcal{W} \in \Omega$, then $[\mathcal{V}] \cap [\mathcal{W}] \in \Omega$.

1.20. **Proposition.** Let E be a set. If Γ is a merotopy, then the system of all Γ-covers is a quasi-uniformity. For any quasi-uniformity Ω on E, there exists exactly one merotopy Γ such that Ω consists of all Γ-covers.

Remark. According to this proposition, merotopic spaces coincide, essentially, with quasi-uniform [5] spaces.

1.21. Proposition and definition. Let $\langle E, \Gamma \rangle$ be a merotopic space. For any $Y \subset E$ let τY consist of the points $x \in E$ such that the collection of all (x,y), $y \in Y$, belongs to Γ. Then τ is a closure operation on E. It will be called the closure operation (or the topology, if this is the case) induced by Γ.

1.22. Definition. Let $\langle E, \Gamma \rangle$ be a merotopic space. Let τ^* be the topology (necessarily completely regular) projectively generated, in the obvious sense, by the ring of all continuous real-valued functions on $\langle \Gamma, E \rangle$. We shall call τ^* the topology CR-induced by Γ.

Evidently, the topology τ induced by Γ is completely regular iff it is CR-induced by Γ.

2.

We are now going to consider certain special classes of merotopic spaces. The most important of these are the filter spaces (related concepts occur under various names in the literature [see, e.g., 8]).

2.1. Definition. A merotopic space $\langle E, \Gamma \rangle$ will be called a filter-merotopic space or simply a filter space if there exists a fundamental system for Γ consisting of filters. The subcategory of $\mathbf{M}$ whose objects are all filter spaces will be denoted by $\mathbf{F}$.

Examples. If $\langle E, \tau \rangle$ is a closure space, then $\langle E, \Gamma_\tau \rangle$ (see 1.9) is a filter-merotopic space. It can be shown that if $\langle E, \mathcal{U} \rangle$ is a metrisable uniform space, then $\langle E, \Gamma_{\mathcal{U}} \rangle$ is a filter-space iff $\langle E, \mathcal{U} \rangle$ is the union of a totally bounded and a uniformly discrete subspace.

2.2. Proposition. If $\langle E, \Gamma \rangle$ is a filter space, then the following condition is satisfied: if $\mathcal{U}$ is a Γ-cover and if, for every $U \in \mathcal{U}$, $\mathcal{V}_U$ is a Γ-cover, then the collection of all $U \cap V$, $U \in \mathcal{U}$, $V \in \mathcal{V}_U$, is a Γ-cover.

Proof. Let $\mathcal{M} \in \Gamma$ be a filter. There exists a set $U_0 \in \mathcal{U}$ such that $M_0 \subset U_0$ for some $M_0 \in \mathcal{M}$. Denote by $\mathcal{M}_0$ the collection of all $M \in \mathcal{M}$ contained in M_0. Clearly, $\mathcal{M}_0$ is micromeric, hence there exists a set $V_0 \in$ $\in \mathcal{V}_U$, and a set $M_0 \in \mathcal{M}$ such that $M_0 \subset V_0$, hence $M_0 \subset U_0 \cap V_0$.

Remark. The converse does not hold, however, even for uniform spaces, since a locally fine [see 5] uniform space need not be a filter space (cf. below, 2.13).

2.3. Proposition and definition. Let Γ be a merotopy on E. Then the system Γ^f of all those $\mathcal{M} \subset \exp E$ which minorize some filter $\mathcal{F} \in \Gamma$, is a filter-merotopy; it is the coarsest filter-merotopy finer than Γ. The merotopy Γ^f will be called the filter-modification of Γ.

Example. Let $\mathcal{U}$ be a uniformity on E and let $\langle \pi E, \pi \mathcal{U} \rangle$ be a completion of $\langle E, \mathcal{U} \rangle$. Then $\langle E, \Gamma^f_{\mathcal{U}} \rangle$ is a subspace of πE endowed with the merotopy induced by the topology of $\langle \pi E, \pi \mathcal{U} \rangle$.

2.4. We are now going to introduce two properties of merotopic spaces analogous to regularity (complete regularity) of topological spaces.

Definition. Let $\langle E, \Gamma \rangle$ be a merotopic space. Denote by τ (respectively, τ^*) the topology induced (CR-induced) by Γ. For any $\mathfrak{M} \subset \exp E$, denote by $\tau[\mathfrak{M}]$ (respectively, $\tau^*[\mathfrak{M}]$) the collection of all τM (respectively, $\tau^* M$) with $M \in \mathfrak{M}$. If, for every $\mathfrak{M} \in \Gamma$, $\tau[\mathfrak{M}]$ (respectively, $\tau^*[\mathfrak{M}]$) belongs to Γ, then Γ is called regular (completely regular).

Clearly, if $\langle E, \tau \rangle$ is a topological space, then $\langle E, \Gamma_\tau \rangle$ is regular (completely regular) iff $\langle E, \tau \rangle$ as a topological space is such.

2.5. Clearly, every cover of a topological space X (or of a uniform space) is refined by a cover $\mathcal{V}$ with the following property: if $\mathcal{V}^* \subset \mathcal{V}$ and $\bigcup \mathcal{V}^* = X$, then $\mathcal{V}^*$ is a cover. However, an analogous assertion does not hold for merotopic spaces (it fails even for closure spaces).

Therefore, there are two notions (at least) corresponding to that of a compact topological space.

Definition. Let $\langle E, \Gamma \rangle$ be a merotopic space. If, for any Γ-cover $\mathcal{G}$, there is a finite $\mathcal{G}^* \subset \mathcal{G}$ with $\bigcup \mathcal{G}^* = E$, then we shall call $\langle E, \Gamma \rangle$ full-bounded. (We use this term instead of the current "totally bounded" to avoid expressions such as "basically totally bounded" or even "partially totally bounded".) If every Γ-cover $\mathcal{G}$ contains a finite Γ-cover, we shall call $\langle E, \Gamma \rangle$ precompact.

Clearly, if $\langle E, \tau \rangle$ is a topological space, then the following assertions are equivalent: (i) $\langle E, \tau \rangle$ is compact, (ii) $\langle E, \Gamma_\tau \rangle$ is full-bounded, (iii) $\langle E, \Gamma_\tau \rangle$ is precompact. On the other hand, let $\langle E, \tau \rangle$ be an absolutely closed Hausdorff topological space. Define a closure as follows: $x \in \tau' X$ iff $\tau U \cap X \neq \emptyset$ whenever U is a neighborhood of x. Then $\langle E, \Gamma_{\tau'} \rangle$ is full-bounded, without, in general, being precompact.

2.6. Lemma. Let $\langle X, \Gamma \rangle$ be a merotopic space; let Y be a subspace. Then the following conditions are equivalent: (a) Y is full-bounded, (b) for any fundamental family $\{\mathfrak{M}_a\}$ and any $M_a \in \mathfrak{M}_a$ there exist $a(1), \dots, a(n)$ with $\bigcup_1^n M_{a(i)} \supset Y$, (c) there exists a fundamental family $\{\mathfrak{M}_a\}$ such that, for any choice of $M_a \in \mathfrak{M}_a$, there are $a(1), \dots, a(n)$ with $\bigcup_1^n M_{a(i)} \supset Y$.

2.7. Clearly, any subspace of a full-bounded (precompact) space is full-bounded (precompact). If $\langle E_i, \Gamma_i \rangle$, $i = 1,2$, are merotopic spaces, f is a continuous mapping of $\langle E_1, \Gamma_1 \rangle$ onto $\langle E_2, \Gamma_2 \rangle$ and $\langle E_1, \Gamma_1 \rangle$ is full-bounded, then $\langle E_2, \Gamma_2 \rangle$ is also full-bounded. However, an analogous assertion does not hold for precompactness.

2.8. Proposition. A merotopic space $\langle E, \Gamma \rangle$ is full-bounded if and only if every ultrafilter on E belongs to Γ.

Proof. I. Suppose that an ultrafilter $\mathcal{F}$ does not belong to Γ. Let $\mathcal{V}$ consist of all $E - F$, $F \in \mathcal{F}$. Let $\mathfrak{M} \in \Gamma$; as $\mathcal{F} \notin \Gamma$, there exists a set $M_0 \in \mathfrak{M}$ such that $F \in \mathcal{F}$ implies $F - M_0 \neq \emptyset$. Since $\mathcal{F}$ is an ultrafilter, we obtain $M_0 \cap F_0 = \emptyset$ for some $F_0 \in \mathcal{F}$. Thus $M_0 \subset E - F_0 \in \mathcal{V}$, and we have shown that $\mathcal{V}$ is a

Γ-cover. Since $\bigcup \mathcal{V}^* \neq E$ whenever $\mathcal{V}^* \subset \mathcal{V}$ is finite, this proves that $\langle X, \Gamma \rangle$ is not full-bounded. -

II. Suppose that $\langle E, \Gamma \rangle$ is not full-bounded; then there exists a Γ-cover $\mathcal{V}$ such that $\bigcup \mathcal{V}^* = E$ for no finite $\mathcal{V}^* \subset \mathcal{V}$. Clearly, the collection $\mathcal{Y}$ of all $E - \bigcup \mathcal{V}^*$, $\mathcal{V}^* \subset \mathcal{V}$ finite, is the base of a filter. Let $\mathcal{M} \supset \mathcal{Y}$ be an ultrafilter. Then $\mathcal{M} \notin \Gamma$, for otherwise there would exist sets $V_o \in \mathcal{V}$, $M_o \in \mathcal{M}$ such that $M_o \subset V_o$, hence $M_o \cap (E - V_o) = \emptyset$, $E - V_o \in \mathcal{Y} \subset \mathcal{M}$, which is contradiction.

2.9. <u>Example</u>. Let M consist of the numbers $\pm 1, \pm 2, \pm 3, \ldots$. Consider the merotopy Γ on M projectively generated by the bounded real-valued functions f satisfying $\lim_{n \to \infty} |f(n) - f(-n)| = 0$. Then $\langle E, \Gamma \rangle$ is completely regular (this follows at once from the fact that Γ induces the discrete topology). It is not difficult to show that $\langle E, \Gamma \rangle$ is full-bounded, but is not precompact.

2.10. <u>Proposition</u>. Every precompact merotopic space is a filter space.

<u>Proof</u>. Let $\langle E, \Gamma \rangle$ be precompact. Consider a Γ-micromeric collection $\mathcal{M}$; we may suppose that $\emptyset \notin \mathcal{M}$. Consider non-void collections $\mathcal{Y} \subset \exp E$ such that, for any $T_1, \ldots T_n \in \mathcal{Y}$, the collection of all $M \in \mathcal{M}$ contained in $T_1 \cap \ldots \cap T_n$ is Γ-micromeric; let θ be the system of all such $\mathcal{Y}$. It is clear that θ is monotonically additive. Therefore there exists a maximal $\mathcal{Y}^* \in \theta$. Clearly, $T_1 \in \mathcal{Y}^*$, $T_2 \in \mathcal{Y}^*$ implies $T_1 \cap T_2 \in \mathcal{Y}^*$, and $\emptyset \notin \mathcal{Y}^*$. We intend to show that $\mathcal{Y}^* \in \Gamma$. -

Let $\mathcal{U}$ be a Γ-cover; since $\langle E, \Gamma \rangle$ is precompact, we may suppose that $\mathcal{U}$ is finite, $\mathcal{U} = (U_1, \ldots, U_m)$. We assert that $U_k \in \mathcal{Y}^*$ for some $k = 1, \ldots, m$. Suppose not; then, for $k = 1, \ldots, m$, there exist $T_{k,1}, \ldots, T_{k,h_k} \in$ $\in \mathcal{Y}^*$ such that the collection $\mathfrak{M}_k$ of all $M \in \mathfrak{M}$ contained in $U_k \cap T_{k,1} \cap \ldots T_{k,h_k}$ does not belong to Γ. On the other hand, denoting by $\mathfrak{M}'$ the collection of all $M \in \mathfrak{M}$ contained in each $T_{k,i}$, $k = 1, \ldots, m$, $i = 1, \ldots, h_k$, we have $\mathfrak{M}' \in \Gamma$. Since $\mathfrak{M}_k \notin \Gamma$, we have $\mathfrak{M}'' = \mathfrak{M}' - \bigcup_{k=1}^{m} \mathfrak{M}_k \in \Gamma$. This is a contradiction, because $\{U_k\}$ is a Γ-cover, hence some $M \in \mathfrak{M}''$ is contained in some U_k (therefore, in $U_k \cap T_{k,1} \cap \ldots \cap T_{k,h_k}$, which is impossible).

2.11. Definition. A merotopic space $\langle E, \Gamma \rangle$ will be called filter-uniform if there is a uniformity $\mathcal{U}$ on E such that Γ is the filter-modification of the merotopy induced by $\mathcal{U}$.

2.12. Theorem. Let $\langle E, \Gamma \rangle$ be a merotopic space. If $\langle E, \Gamma \rangle$ is proximal, then it is precompact, uniform and a filter space (hence filter-uniform). If $\langle E, \Gamma \rangle$ is filter-uniform or precompact or topological, then it is a filter space.

Proof. Let $\langle E, \Gamma \rangle$ be proximal. It is well known that $\langle E, \Gamma \rangle$ is uniform and precompact; by 2.9 $\langle E, \Gamma \rangle$ is a filter space. The second assertion is clear since a precompact $\langle E, \Gamma \rangle$ is a filter space.

2.13. Theorem. A merotopic space is (1) topological and proximal if and only if its merotopy is induced by a compact topology, (2) topological and uniform if and only if its merotopy is induced by the topology (or the fine uniformity) of a paracompact space, (3) topological and filter-uniform if and only if its merotopy is induced by a complete uniformity (or the topology of a complete topological space), (4) a uniform filter space if and only if it is a dense subspace of a space whose merotopy is induced by a paracompact topology.

3.

3.1. If $\{X_a \mid a \in A\}$ is a family of sets, then the natural mapping of X_a into the sum $\sum X_a$, which assigns $\langle a, x \rangle$ to $x \in X_a$, will be denoted by inj_a. The projection of the cartesian product $\prod_a \{X_a\}$ onto X_a will be denoted by proj_a.

3.2. Definition. Let $\{\mathcal{X}_a\}$ be a family of merotopic spaces, $\mathcal{X}_a = \langle X_a, \Gamma_a \rangle$; put $X = \sum X_a$. The space $\langle X, \Gamma \rangle$ inductively generated by the mappings $\mathrm{inj}_a : \mathcal{X}_a \to X$ is called the sum of $\{\mathcal{X}_a\}$ and is denoted by $\sum \mathcal{X}_a$.

<u>Remark</u>. By 1.14, remark, the system $\bigcup \mathrm{inj}_a [\Gamma_a]$ is fundamental for $\sum \mathcal{X}_a$.

3.3. <u>Proposition</u>. If $\mathcal{X}_a$ are filter spaces, then $\sum \mathcal{X}_a$ is a filter space.

3.4. If $\mathcal{X}_a = \langle X_a, \Gamma_a \rangle$ are arbitrary merotopic spaces, then various merotopies may be introduced on the set $\prod X_a$; however, each of these seems to suffer from certain serious shortcomings. This is also true for merotopies on the set of all continuous mappings of a merotopic space into another. We shall not investigate this question here [for various kinds of products see, e.g., 2, 2a], and we confine our examination to the case of filter spaces, for which there is a natural definition.

3.5. Definition. Let $\{\mathcal{X}_a\}$ be a family of filter spaces, $\mathcal{X}_a = \langle X_a, \Gamma_a \rangle$; put $X = \prod X_a$. The space $\langle X, \Gamma \rangle$ projectively generated in $\mathbf{F}$ (see 1.13, 2.1) by the mappings $\mathrm{proj}_a : X \to \mathcal{X}_a$ will be called the cartesian (more precisely, filter-cartesian) product of the filter spaces $\mathcal{X}_a$ and will be denoted by $\prod_a \mathcal{X}_a$.

Proposition. A fundamental system for $\prod \langle X_a, \Gamma_a \rangle$ consists of all filters $\mathcal{F} \subset \exp(\prod X_a)$ such that $\mathrm{proj}_a[\mathcal{F}] \in \in \Gamma_a$ for every a.

3.6. Proposition and definition. Let $\mathcal{X} = \langle X, \Gamma \rangle$, $\mathcal{Y} = \langle Y, \Delta \rangle$ be filter spaces. Denote by C the set of all continuous mappings $f : \mathcal{X} \to \mathcal{Y}$. Then there exists exactly one filter-merotopy Θ on C such that (1) the mapping of the cartesian product $\langle C, \Theta \rangle \times \mathcal{X}$ into $\mathcal{Y}$ which assigns fx to $\langle f, x \rangle$ is continuous, (2) if a filter-merotopy Ψ on C possesses the above property, then $\Psi \leqq \Theta$. The system of all filters $\mathcal{F} \subset \exp C$ such that $\mathcal{F}[\mathcal{M}] \in \Delta$ for every $\mathcal{M} \in \Gamma$ is fundamental for Θ. The set of all continuous $f : \mathcal{X} \to \mathcal{Y}$ endowed with the merotopy Θ will be denoted by $\mathcal{Y}^{\mathcal{X}}$ and called the filter space of mappings.

3.7. To make possible a concise and exact formulation of the theorem which follows, we shall adopt the following conventions: if $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{X}_a, \mathcal{Y}_a$ are merotopic spaces, consider (1) the binary relation consisting of all pairs $\langle f, g \rangle$, $f \in \mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$, $g \in (\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$, such that

$f(x, z) = g(z)(x)$, for any $x \in \mathcal{X}$, $z \in \mathcal{Z}$; this relation will be called canonical (for $\mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$ and $(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$); (2) the binary relation consisting of all $\langle f, \{g_a\} \rangle$, $f \in \mathcal{Y}^{\Sigma \mathcal{X}_a}$, $g_a \in \mathcal{Y}^{\mathcal{X}_a}$, such that, for any a and any $x \in \mathcal{X}_a$, $f(a, x) = g_a(x)$; this relation will be called canonical (for $\mathcal{Y}^{\Sigma \mathcal{X}_a}$ and $\Pi \mathcal{Y}^{\mathcal{X}_a}$); (3) the binary relation consisting of all $\langle f, \{g_a\} \rangle$, $f \in (\Pi \mathcal{Y}_a)^{\mathcal{X}}$, $g_a \in \mathcal{Y}_a^{\mathcal{X}}$ such that for any $x \in \mathcal{X}$, $f(x) = \{g_a(x)\}$; this relation will be called canonical (for $(\Pi \mathcal{Y}_a)^{\mathcal{X}}$ and $\Pi(\mathcal{Y}_a^{\mathcal{X}})$). If it is clear from the context which of the canonical relations is meant, we shall omit an explicit mention of spaces involved.

A priori, it is not clear that the canonical relations are bijective for the spaces in questions. This assertion is contained, however, in the following proposition, in which $\mathcal{A} \cong \mathcal{B}$ means that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

3.8. <u>Theorem</u>. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{X}_a, \mathcal{Y}_a$ be filter spaces. Then

$$\mathcal{Y}^{\mathcal{X} \times \mathcal{Z}} \cong (\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}},$$
$$\mathcal{Y}^{\Sigma \mathcal{X}_a} \cong \prod_a (\mathcal{Y}^{\mathcal{X}_a}),$$
$$(\prod_a \mathcal{Y}_a)^{\mathcal{X}} \cong \prod_a (\mathcal{Y}_a^{\mathcal{X}}).$$

More specifically, the corresponding canonical relations determine isomorphisms of $\mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$ and $(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$, of $\mathcal{Y}^{\Sigma \mathcal{X}_a}$ and $\prod_a (\mathcal{Y}^{\mathcal{X}_a})$, and of $(\prod_a \mathcal{Y}_a)^{\mathcal{X}}$ and $\prod_a (\mathcal{Y}_a^{\mathcal{X}})$.

<u>Proof</u>. We shall prove only the assertion concerning $\mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$ and $(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$. Let $\mathcal{X} = \langle X, \Gamma_X \rangle$, $\mathcal{Y} = \langle Y, \Gamma_Y \rangle$, $\mathcal{Z} = \langle Z, \Gamma_Z \rangle$. Let $f \in \mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$. If $z \in Z$, let $g_z(x) = f(x, z)$ for every $x \in X$; then, for

every $\mathfrak{M} \in \Gamma_X$, $g_z[\mathfrak{M}] = f\,[(x)] \times [\mathfrak{M}]]$, hence $g_z[\mathfrak{M}] \in \Gamma_Y$; thus g_z is continuous, $g_z \in \mathcal{Y}^{\mathcal{X}}$. Let g be the mapping which assigns g_z to $z \in Z$. If $\mathfrak{N} \in \Gamma_Z$, $\mathfrak{M} \in \Gamma_X$, then $g[\mathfrak{N}][\mathfrak{M}] = f[\mathfrak{N} \times \mathfrak{M}] \in \Gamma_Y$; hence $g[\mathfrak{N}]$ is micromeric in $\mathcal{Y}^{\mathcal{X}}$. This proves that g is continuous. We have shown that the canonic relation maps $\mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$ into $(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$. Similarly, it can be shown that if $g \in (\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$ and $f(x, z) = g(z)(x)$, then $f \in \mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$. It remains to prove that the bijective mapping determined by the canonic relation is isomorphic.

Let $\mathcal{F}$ be micromeric in $\mathcal{Y}^{\mathcal{X} \times \mathcal{Z}}$. Let $\mathcal{F}^*$ denote the collection onto which $\mathcal{F}$ is mapped. Then, for any $\mathfrak{M} \in \Gamma_X$, $\mathfrak{N} \in \Gamma_Z$, the collection $\mathcal{F}^*[\mathfrak{N}][\mathfrak{M}] = \mathcal{F}[\mathfrak{M} \times \mathfrak{N}]$ belongs to Γ_Y . Therefore, $\mathcal{F}^*[\mathfrak{N}]$ is micromeric in $\mathcal{Y}^{\mathcal{X}}$, for any $\mathfrak{N} \in \Gamma_Z$; this proves that $\mathcal{F}^*$ is micromeric.

3.9. Definition. A merotopic space $\langle X, \Gamma \rangle$ will be called filter-localized (or simply "localized") if, for any micromeric filter $\mathcal{F}$, there exists a point $x \in X$ such that the collection of all $F \cup (x)$ is micromeric.

Example. A uniform merotopic space is localized iff it is complete (as a uniform space).

3.10. Proposition. A merotopic space is a localized filter space if and only if it is inductively generated by topological spaces.

Proof. Let $\langle X, \Gamma \rangle$ be a localized filter space. Then there exists a fundamental family $\{ \mathfrak{M}_a \mid a \in A \}$ such that every $\mathfrak{M}_a$ is a filter with a non-void intersection. For every $a \in A$, let a topology $\tau(a)$ on X be defined as follows: an open base for $\langle X, \tau(a) \rangle$ consists of all $M \in \mathfrak{M}_a$ and all (y) where $y \in X - \bigcap \mathfrak{M}_a$. It is easy to show that the mappings $J : \langle X, \Gamma_{\tau(a)} \rangle \to X$, where J is the identity relation, inductively generate the structure Γ. - The rest of the proof is omitted.

3.11. Lemma. Let $\mathcal{X}$, $\mathcal{Y}$ be filter spaces. If $\mathcal{Y}$ is regular, then $\mathcal{Y}^{\mathcal{X}}$ is regular.

3.12. Proposition. If $\mathcal{X}$ is a filter space, $\mathcal{Y}$ is a regular localized filter space, then $\mathcal{Y}^{\mathcal{X}}$ is also a regular localized filter space.

Proof. The space $\mathcal{Y}^{\mathcal{X}}$ is regular by 3.11; thus we have only to prove that it is localized. Let $\mathcal{F}$ be a $\mathcal{Y}^{\mathcal{X}}$-micromeric filter. Then, for any $x \in X$, the collection $\mathcal{F}[(x)]$ is a micromeric filter and therefore we can choose a point $\varphi(x) \in Y$ such that the collection of all $F[x] \cup (\varphi(x))$, $F \in \mathcal{F}$, is micromeric. Now consider the mapping $\varphi : \mathcal{X} \to \mathcal{Y}$. Clearly, for any $x \in X$ and any $F \in \mathcal{F}$, $\varphi(x)$ belongs to the closure $\tau(F[x])$ of $F[x]$. Let $\mathfrak{M}$ be $\mathcal{X}$-micromeric. If $x \in M$, $M \in \mathfrak{M}$, $F \in \mathcal{F}$, then $F[x] \subset F[M]$, hence $\varphi(x) \in \tau(F[M])$; thus, for any $M \in \mathfrak{M}$, $\varphi[M] \subset \tau(f[M])$. Since $\mathcal{Y}$ is regular, the collection of all $\tau(f[M])$ is micromeric, and therefore $\varphi[\mathfrak{M}]$ is also micromeric. We have proved that $\varphi \in \mathcal{Y}^{\mathcal{X}}$. For any $F \in \mathcal{F}$, put $F^* = F \cup (\varphi)$; let $\mathcal{F}^*$ consist of all F^*. Let $\mathfrak{M}$ be $\mathcal{X}$-micromeric. For any $M \in \mathfrak{M}$ and any $F \in \mathcal{F}$, $F^*[M] \subset \tau F[M]$; therefore, $\mathcal{Y}$ being regular, the collection

$\mathscr{J}^*[\mathfrak{M}]$ is micromeric. This proves that $\mathcal{Y}^{\mathfrak{X}}$ is localized.

3.13. Definition. Let $\mathfrak{X}$ be a merotopic space. If there exists a base $\mathscr{R}$ for $\mathfrak{X}$ consisting of full-bounded subspaces, then $\mathfrak{X}$ is called basically full-bounded.

It is easy to see that a topological space is basically full-bounded iff it is locally compact.

3.14. Remark. It can be shown that quasi-topological spaces as considered in [11], coincide essentially with basically compact merotopic spaces (i.e., spaces possessing a base consisting of subspaces whose merotopies are induced by compact topologies).

3.15. Proposition. Let $\mathfrak{X} = \langle X, \Gamma \rangle$ be a localized filter space. Then $\mathfrak{X}$ is closure-induced if and only if, for any $x \in X$, there exists an $\mathfrak{M} \in \Gamma$ such that every Γ-micromeric collection $\mathfrak{N}$ with $x \in \bigcap \mathfrak{N}$ minimizes $\mathfrak{M}$; $\mathfrak{X}$ is topological if and only if there exists a collection $\mathscr{L}$ with the following properties: for any $x \in X$, the collection $\mathscr{L}_x$ of all $B \in \mathscr{L}$ with $x \in B$ is micromeric; if $\mathfrak{M}$ is micromeric, $x \in \bigcap \mathfrak{M}$, then $\mathfrak{M}$ minimizes $\mathscr{L}_x$.

3.16. Theorem [cf. 1,3]. Let $\mathcal{Y}$ be a regular topological merotopic space, and let $\mathfrak{X}$ be a basically full-bounded filter space. Then $\mathcal{Y}^{\mathfrak{X}}$ is a regular topological merotopic space.

Proof. By 3.12, $\mathcal{Y}^{\mathfrak{X}}$ is a regular localized filter space. By 3.15, we have only to prove that there exists a collection, say $\mathscr{J}$, with the property described (for a collection $\mathscr{L}$) in 3.15. For any full-bounded $T \subset X$ and any open $U \subset Y$, let $F_{T,U}$ denote the set of all $h \in \mathcal{Y}^{\mathcal{X}}$ such that $h[T] \subset U$. Denote by $\mathcal{F}$ the collection of all $\bigcap_{i=1}^{n} F_{T_i,U_i}$ and, for any $f \in \mathcal{Y}^{\mathcal{X}}$ denote by $\mathcal{F}_f$ the collection of all $F \in \mathcal{F}$ containing f.

We are going to prove that every $\mathcal{F}_f$ is micromeric. Let $\mathfrak{M}$ be $\mathcal{X}$-micromeric; since $\mathcal{X}$ is basically full-bounded, we may assume that every $M \in \mathfrak{M}$ is full-bounded. Since $f[\mathfrak{M}]$ is micromeric and $\mathcal{Y}$ is topological, there exists a point $y \in \mathcal{Y}$ such that, for every neighborhood U of y, there is a set $M \in \mathfrak{M}$ with $f[M] \subset V = \operatorname{Int} U$. Then $F_{M,V} \in \mathcal{G}$, $f \in F_{M,V}$, $F_{M,V}[M] \subset V$. This proves that $\mathcal{F}_f[\mathfrak{M}]$ is micromeric. Thus we have proved that $\mathcal{F}_f$ is micromeric.

Now let $\mathcal{G}$ be a $\mathcal{Y}^{\mathcal{X}}$-micromeric filter such that $f_o \in G$ whenever $G \in \mathcal{G}$. Choose an arbitrary full-bounded $T \subset X$ and a neighborhood U of $f_o[T]$. It is sufficient to show that there is a set $G \in \mathcal{G}$ contained in $F_{T,U}$. Let $\mathfrak{M}$ be an $\mathcal{X}$-micromeric filter. Then $\mathcal{G}[\mathfrak{M}]$ is micromeric and every non-void $G[M \cap T]$, $G \in \mathcal{G}$, $M \in \mathfrak{M}$, intersects $f_o[T]$. It follows that there exists a set $M = M(\mathfrak{M}) \in \mathfrak{M}$ and a set $G = G(\mathfrak{M})$ belonging to $\mathcal{G}$ such that $G[M \cap T] \subset U$. Clearly, the sets $M(\mathfrak{M}) \cap T$ form a merotopic cover of T. Since T is full-bounded, there exist $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ such that $\bigcup_{k=1}^{n} M(\mathfrak{M}_k) \supset T$. Put $G^* = \bigcap_{k=1}^{n} G(\mathfrak{M}_k)$. Then, clearly, $g \in G^*$ implies $g[T] = \bigcup g[M(\mathfrak{M}_k) \cap T] \subset U$. Thus, every $g \in G^*$ belongs to $F_{T,U}$. We have proved that $G^* \subset F_{T,U}$, $G^* \in \mathcal{G}$.

3.17. <u>Proposition</u>. If a completely regular filter space

$\mathcal{X} = \langle X, \Gamma \rangle$ is not basically full-bounded, then the space $[0,1]^{\mathcal{X}}$ of its continuous mappings into the segment $[0,1]$ is not closure-induced.

Proof. Suppose that, on the contrary, $\mathcal{F} = [0,1]^{\mathcal{X}}$ is a closure-induced space. Denote by τ^* the topology CR-induced by Γ. For any $x \in X$, put $\vartheta(x) = 0$; let $\mathcal{V}$ be a complete system of neighborhoods of the function ϑ in the space $\mathcal{F}$. Since $\mathcal{X}$ is not full-bounded, there exists an $\mathcal{X}$-micromeric $\mathfrak{M}$ such that no $M \in \mathfrak{M}$ is full-bounded. Since $\mathcal{V}$ is $\mathcal{F}$-micromeric, there exist $V_0 \in \mathcal{V}$ and $M_0 \in \mathfrak{M}$ such that $V_0[M_0] \subset [0, \frac{1}{2}[$.

Let $\{\mathfrak{K}_a \mid a \in A\}$ be a fundamental family for the merotopy Γ. Since X is completely regular, the family $\{\tau^*[\mathfrak{K}_a]\}$, where $\tau^*[\mathfrak{K}_a]$ consists of all $\tau^* K$, $K \in \mathfrak{K}_a$, is also fundamental. Therefore, by 2.6, there exist sets $K_a \in \mathfrak{K}_a$ such that, for any choice of $k(1), \ldots, k(n)$, $M_0 - \bigcup_{i=1}^{n} \tau^* K_{a(i)} \neq \emptyset$.

Now, for any finite set $B \subset A$, let F_B denote the set of the functions $f \in [0,1]^{\mathcal{X}}$ such that $f(x) = 0$ for $x \in \bigcup_{a \in B} K_a$, $f(x_a) = 1$ for some $x_0 \in M_0$ (x_0 depending, of course, on B); since $M_0 - \bigcup_{a \in B} \tau^* K_a \neq \emptyset$, it is clear that no F_B is void. It is also clear that no F_B is contained in V_0 (since $f[M_0] \subset [0, \frac{1}{2}[$ for every $f \in V_0$). On the other hand, the system of all F_B is micromeric, because for any $\mathfrak{K}_{a(1)}, \ldots, \mathfrak{K}_{a(n)}$, there is a set F_B and sets $K_{a(i)} \in \mathfrak{K}_{a(i)}$ with $F_B[K_{a(i)}] = (0)$, $i = 1, \ldots, n$. This is a contradiction. Thus, $[0,1]^{\mathcal{X}}$ is not closure-induced.

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