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ON THE CHARACTERIZATION OF BANACH SPACES WITH THE STRONG KIRTZBRAUN-VALENTINE PROPERTY

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Let $X = (X, \rho)$ and $Y = (Y, \sigma)$ be metric spaces. If φ is a transformation of X into Y, then φ is said to be Lipschitzian with the constant λ, provided

$$\sigma(\varphi(x), \varphi(y)) \leq \lambda \rho(x, y)$$

for all $x, y \in X$. A Lipschitzian transformation φ with the constant $\lambda = 1$ is called a contraction.

The problem of extending of a Lipschitzian transformation A to Y (where A is a subspace of a space X) to a transformation of X to Y was studied by various authors. The existence of such an extension for $Y = E_1$ is proved by Banach in [2]. As a consequence of the result of Aronszajn and Panichpakdi [1] we get the existence of an extension for a hyperconvex space Y (i.e. spaces which have the following property: If $\mathcal{G} = \{\Omega(x_i, r_i) : i \in I\}$ is a system of σ-metric cells in Y such that for each $i \in I$, $j \in I$ there is $\sigma(x_i, x_j) \leq r_i + r_j$, then $\cap \mathcal{G} \neq \emptyset$).

Mc Shane [5], Kirtzbraun [4] and Valentine [6], [7] showed that this extension problem is associated, with the following intersection property.

A pair of metric spaces (X, ρ) and (Y, σ) is said to have a Valentine intersection property provided that:

If $\mathcal{F} = \{\Omega(x_i, r_i) : i \in I\}$

is a system of ρ-cells in X and

$$\mathcal{G} = \{ \Omega (y_i, r_i) : i \in I \}$$

a system of σ-cells in Y such that for each $i \in I$, $j \in I$ there is

$$\rho (x_i, x_j) \geq \sigma (y_i, y_j) ,$$

then

$$\cap \mathcal{F} \neq \emptyset \Rightarrow \cap \mathcal{G} \neq \emptyset .$$

In this paper we shall discuss contractions only, since the general Lipschitzian extension problem can be reduced to an adequate contraction problem (see [7]p.93) if Y is a normed linear space.

There is proved in the paper of Valentine [7], that the situation is the following one:

For any metric spaces X and Y the following two statements are equivalent:

(1) (X, Y) has the Valentine intersection property;

(2) for every $A \subset X$ and every contraction f of A into Y there exists an extension $F \supset f$ such that F is a contraction mapping X into Y.

There is also proved in [7] that for each of the following cases the Valentine intersection property is satisfied

(a) X is an arbitrary metric space, $Y = E_1$,

(b) $X = Y = E_n$;

(c) $X = Y = H$, H being a Hilbert space;

(d) $X = Y = S^n$, S^b being an n-dimensional Euclidean sphere.

In the cases (b) and (c) it may be proved that the extension F of a contraction f of $A \subset X$ into Y may be found in such a way that

$$\overline{\text{conv}}\, f(A) = \overline{\text{conv}}\, F(X);$$

where the symbol $\overline{\text{conv}}$ B denotes a closed convex hull of the set B .

In connection with the results above we formulate the following definitions:

A metric space X is said to have a Valentine intersection property if the pair (X, X) has the Valentine intersection property.

A metric linear space X is said to have a strong Valentine intersection property provided that:

if $\mathcal{F} = \{ \Omega (x_i, r_i) : i \in I \}$ and $\mathcal{G} = \{ \Omega (y_i, r_i), i \in I \}$

are systems of cells in X such that

$$\rho (x_i, x_j) \geq \rho (y_i, y_j) \quad \text{for each } i, j \in I ,$$

then

$$\cap \mathcal{F} = \emptyset \Rightarrow (\cap \mathcal{G}) \cap \overline{\text{conv}} \bigcup_{i \in I} y_i \neq \emptyset .$$

It is easy to prove the following:

Let X be a Banach space. Then the following statements are equivalent:

(1) X has a strong Valentine intersection property;

(2) for each $A \subset X$ and each contraction f of A into X there exists an extension $F \supset f$ of f such that F is a contraction of X into itself and $\overline{\text{conv}}\, f(A) = \overline{\text{conv}}\, F(X)$.

The problem of characterization of all Banach spaces with the Valentine intersection property is still unsolved. The main result of this paper is the complete characterization of all Banach spaces with the strong Valentine intersection property. The situation is described by the following theorem:

<u>Theorem</u>: Let X be a real Banach space. The following statements are equivalent:

(1) X has a strong Valentine intersection property;

(2) for each $A \subset X$ and each contraction f of A into X there exists an extension $F \supset f$ such that F is a contraction of X into X and $\overline{\text{conv}}\, f(A) = \overline{\text{conv}}\, F(X)$;

(3) for each $A \subset X$ and each contraction f of A into X there exists an extension $F \supset f$ such that F is a contraction of X into X and $\overline{\text{sp}}\, f(A) = \overline{\text{sp}}\, F(X)$, where $\overline{\text{sp}}\, B$ denotes the closed linear hull of the set $B \subset X$;

(4) either X is an inner product space (i.e. Euclidean or Hilbert space) or X is a two-dimensional space ℓ_2^∞ whose unit sphere is a parallelogram.

Proof: Obvously $(1) \Longleftrightarrow (2) \Longrightarrow (3)$. The statement $(4) \Longrightarrow (1)$ is proved in [7] for the case that X is an inner product space. As an easy consequence of [1] we get immediately the validity of (1) in the space ℓ_2^∞.

Hence it remains to prove the implication $(3) \Longrightarrow (4)$. Let X be a space with the property (3) and Z a two-dimensional subspace of X. It is clear that Z has the property (3), too. Let S be a unit cell in Z and let Σ be the unit sphere which is boundary of S. We distinguish the following two cases:

A. Let S be strictly convex (i.e. $x, y \in \Sigma$, $0 < \lambda < 1$, $\lambda x + (1 - \lambda) y \in \Sigma \Longrightarrow x = y$). In this case we shall prove that Σ is an ellipse.

Let x_1 and x_2 be different points of Z and let y_1 and y_2 be points lying in different half-planes with the stright line $\overline{x_1, x_2}$ as the common boundary and which have the property

$$\| x_1 - y_1 \| = \| x_2 - y_1 \| , \| x_1 - y_2 \| = \| x_2 - y_2 \| .$$

Let $x_0 = \frac{1}{2}(x_1 + x_2)$ and let y_0 be the common point of $\overline{x_1, x_2}$ and $\overline{y_1, y_2}$.

Let us assume $y_0 \neq x_0$. In this case exactly one of the cells

$$x_1 + \| x_1 - x_0 \| S , \quad x_2 + \| x_2 - x_0 \| S$$

contain a point y_0. Let, e.g.,

$$y_0 \in x_1 + \| x_1 - x_0 \| S .$$

Therefore

$$\| x_1 - x_0 \| = \| x_2 - x_0 \| ,$$

$$(y_1 + \| y_1 - y_0 \| S) \cap (y_2 + \| y_2 - y_0 \| S) \cap (x_1 + \| x_1 - x_0 \| S) \neq \emptyset$$

and (since S is strictly convex) we get

$$(y_1 + \| y_1 - y_0 \| S) \cap (y_2 + \| y_2 - y_0 \| S) \cap (x_2 + \| x_2 - x_0 \| S) = \emptyset,$$

which fact is a contradiction with the Valentine intersection property of Z.

Therefore there is $y_0 = x_0$ and we get that, if x_1, x_2 are arbitrary points, then the centers of all cells containing x_1, x_2 are lying in the stright line. By means of elementary geometric considerations it may be proved that ellipse is the only one possible convex cell with this property.

B. Let S be not strictly convex. At first, let us mind that the corollary of the Valentine intersection property of Z is the following property of Z :

(A) If there exists a cell with a radius r in Z such that it contains the points x_i, $i = 1, \ldots, n$ and if

$\| x_i' - x_j' \| = \| x_i - x_j \|$ for each $i, j = 1, \ldots, n$,

then there exists a cell with the radius r containing all points x_i', $i = 1, \ldots, n$.

At first we shall prove that S is a 2n-gon with sides of equal lentgh (in the sense of the norm in Z).

Let x_1, x_2 be endpoints of some maximal (straight line) segment of Σ . The cell $x_1 + \| x_2 - x_1 \| S$ has a center in the boundary of the cell $-x_1 + 2S$. Therefore the boundaries of these cells have two common points, one of these points being x_2 . Let us denote x_3 the second one.

The points $x_1, x_2, -x_1$ are contained in S . The property (A) of Z implies the existence of a cell of radius 1 , containing all points $x_1, x_3, -x_1$. Because S is the unique cell of radius 1 , which contains both x_1 and $-x_1$, there is $x_3 \in S$; since $\| x_3 - (-x_1) \| = 2$, there is $x_3 \in \Sigma$. Obviously $\| \frac{1}{2}(x_3 + x_1) \| = 1$ and therefore the whole segment $\overline{x_1, x_3}$ lies in Σ , i.e. $\overline{x_1, x_3}$ is a part of some segment of Σ with a length at least $\| x_2 - x_1 \|$.

As a consequence we get the fact that S is a 2n-gon with sides of equal (Minkowski) length.

Now, we shall prove $n = 2$, i.e., S is a parallelogram. Let $n > 2$ and let x_1, x_2, x_3 be three consecutive vertices of S . It is easy to show that

$$\| x_1 - x_2 \| < \| x_1 - x_3 \| .$$

Let us consider the points

$$y_2 = \frac{1}{2}(x_1 + x_2) + \frac{\| x_3 - x_1 \|}{2 \| x_2 - x_1 \|}(x_2 - x_1), \quad y_1 = \frac{1}{2}(x_1 + x_2) - $$

$$- \frac{\| x_3 - x_1 \|}{2 \| x_2 - x_1 \|}(x_2 - x_1) .$$

Since $\| y_2 - x_2 \| < \frac{1}{2} \| x_2 - x_1 \|$, the point

$$z = -\frac{1}{2}(x_1 + x_2) - (y_2 - x_2)$$

lies in the interior of the segment $\overline{-x_1, -x_2}$ and therefore
$\| y_2 + \frac{1}{2}(x_1 + x_2) \| = \| x_2 - z \| = 2 = \| x_3 - (-x_2) \|$.

Further we are able to prove that the segments
$\overline{y_1, -\frac{1}{2}(x_1 + x_2)}$ and $\overline{x_1, -x_2}$
have the same length. The points $x_1, -x_2, x_3$ are contained in the cell of radius 1 , but the points $y_1, y_2,$
$-\frac{1}{2}(x_1 + x_2)$ are not contained in any cell of radius 1 ; which fact is a contradiction with the property (A) of the space Z .

Meanwhile the following statement was proved:

If Z is a two-dimensional subspace of X and if X has the strong Valentine intersection property, then the unit cell in Z is either an ellipse, or a parallelogram.

Let Σ be a unit sphere in a normed linear space with the strong Valentine property X . Let us denote by Δ the set of all intersections of Σ with the two-dimensional subspaces of X . Let ρ be the metric in X ; for $S \in \Delta$, $S' \in \Delta$ put

$$h(S, S') = \max(\sup_{x \in S} \rho(x, S'), \sup_{x' \in S'} \rho(x', S)).$$ The function h is a metric on Δ (so called Hausdorff metric). The mapping φ of $\Sigma \times \Sigma$ into Δ , which to every $(i, j) \in \Sigma \times \Sigma$ consignates the intersection Σ with the plane $sp(i, j)$ is obviously continuous as the mapping into the metric space (Δ, h) . As the subsets
$\{(i, j) \in \Sigma \times \Sigma : \varphi(i, j)$ is an ellipse $\}$ and
$\{(i, j) \in \Sigma \times \Sigma : \varphi(i, j)$ is a parallelogram $\}$
are evidently open in $\Sigma \times \Sigma$, one of these sets must be empty.

Then for all spaces satisfying the condition (3) only one of the following situations can occur.

(α) The intersection Σ with every plane containing the origin is ellipse.

(β) The intersection Σ with every plane containing the origin is a parallelogram.

The situation (β) can occur only if X is a two-dimensional space $X = \ell_2^{\infty}$.

If the (α) occurs, very two-dimensional subspace of X is Euclidean and therefore in a consequence of [3] p.115 (JN_1) X is an inner product space.

R e f e r e n c e s :

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