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A THEOREM OF HAHN-BANACH TYPE FOR LOCALLY CONVEX TOPOLOGICAL SPACES

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By the well known Hahn-Banach theorem every continuous linear functional defined on a linear subspace of real or complex normed linear space can be extended onto the whole space so as to remain linear, continuous, and with preserving the norm. The extension of continuous linear transformations between two normed linear spaces has been studied by several authors. This problem is connected with the question of the existence of projections of norm 1 of a normed linear space E into different subspaces of E (see [3]).

Let E be a given normed linear space, E is said to have the <u>extension property</u> if, for any normed linear space X and every continuous linear transformation f of the linear subspace $Y \subset X$ into E there exists a continuous linear extension $\mathcal{F} \supset f$ such that $\mathcal{F}$ maps X into E and $\|\mathcal{F}\| = \|f\|$. The Hahn-Banach theorem says simply that every one-dimensional normed linear space has the extension property.

Nachbin [5] has proved: Normed linear spaces with the extension property are exactly the spaces with the following <u>binary intersection property</u>: If $\{\Omega_i, i \in I\}$ is a system of cells in E such that every two members of them intersect, then the intersection $\bigcap_{i \in I} \Omega_i$ is non-void. Another characterization of normed linear spaces with the extension property is due to Kelley ([4]). His result is that E has the extension property if and only if $E = C(Q)$, where Q is an extremally disconnected compact Hausdorff topological space.

In this paper we shall give a convenient definition of the extension property and the binary intersection property of a locally convex topological linear space, which are a generalization of those properties for normed linear spaces. Theorem 1 is a generalization of Nachbin's result [5] and theorem 2 gives a topological characterization of locally convex topological linear spaces with the extension property in our sense.

1. Definition and terminology

Let E be a real locally convex topological linear space and $\mathcal{L}$ a subbase consisting of convex, symmetric and closed neighbours of zero in E. For $\mathcal{U}$, $\mathcal{V}$ in $\mathcal{L}$ we put $\mathcal{U} \sim \mathcal{V}$ if there exist positive numbers λ and μ such that

$$\lambda \mathcal{U} \subset \mathcal{V} \quad \text{and} \quad \mu \mathcal{V} \subset \mathcal{U} .$$

Evidently, $\sim$ is an equivalence on $\mathcal{L}$ and $\mathcal{L}$ is divided by $\sim$ into classes of mutually equivalent elements. A set which contains exactly one element of each class of this equivalence will be called a skeleton of the space E. As the definition of the skeleton of E depends on the subbase $\mathcal{L}$, the skeleton of E is not determined uniquely by the topology of E. For example, if E_2 is the Euclidean plan, the class $\mathcal{R}_1$ of one set $\{(x, y): x^2 + y^2 \leqq 1\}$ and the class $\mathcal{R}_2$ of two sets $\{(x, y) : |x| \leqq 1\}$ and $\{(x, y) : |y| \leqq 1\}$ are both skeletons of E_2. If $\mathcal{R}$ is a skeleton of E, $\mathcal{U} \in \mathcal{R}$, $x \in E$ and λ is a positive number, then the set

$$x + \lambda \mathcal{U} = \{ x + \lambda y : y \in \mathcal{U} \}$$

is called a $\mathcal{R}$-solid in E . Two $\mathcal{R}$-solids

$$x_1 + \lambda_1 \mathcal{U} \quad \text{and} \quad x_2 + \lambda_2 \mathcal{U}$$

are called similar.

A skeleton $\mathcal{R}$ of E is said to have a binary intersection property if any system of $\mathcal{R}$-solids whose every two similar $\mathcal{R}$-solids intersect, has a non-void intersection.

It is clear that if E is a normed linear space, then the set containing only the unit cell $\{ x \in E : \|x\| \leq 1 \}$ is a skeleton of E ; E has the binary intersection property in the usual sense if and only if this one-element skeleton has the binary intersection property. On the other hand, E_2 has not the binary intersection property in usual sense, but there exists a skeleton $(\{(x, y) : |x| \leq 1\},$ $\{(x, y) : |y| \leq 1\})$ of E_2 which has this property.

Let Y be a locally convex topological linear space and f be a continuous linear operator on Y into E ; let $\mathcal{R}$ be a skeleton of E . From the continuity of f , for each $\mathcal{U} \in \mathcal{R}$ there exists a neighbourhood $\mathcal{V}$ of zero in Y such that $f(\mathcal{V}) \subset \mathcal{U}$. Let X be a locally convex topological linear space which contains Y as a subspace. Then there exists a neighbourhood $\mathcal{W}$ of zero in X such that $\mathcal{W} \cap Y = \mathcal{V}$. Let us denote by $\langle \mathcal{R}, f, X \rangle$ the set of all mappings τ of $\mathcal{R}$ into the set of all neighbourhoods of zero in X such that $f(\tau(\mathcal{U}) \cap Y) \subset \mathcal{U}$ for any $\mathcal{U} \in \mathcal{R}$. Let $\tau \in \langle \mathcal{R}, f, X \rangle$; a linear operator $\mathcal{F}$ with domain Z, $Y \subset Z \subset X$, into E which satisfies the conditions

1. $\mathcal{F} \supset f$ (i.e. $\mathcal{F}(x) = f(x)$ for $x \in Y$),

2. $\mathcal{F}(\tau(\mathcal{U}) \cap Z) \subset \mathcal{U}$ for any $\mathcal{U} \in \mathcal{R}$,

is said to be a τ -continuous extension of f . It is obvious that any τ -continuous extension is a continuous linear operator. A skeleton $\mathcal{R}$ of E is said to have an extension property if for each topological linear spaces $X \supset Y$ each continuous linear mapping f of Y into E and each $\tau \in \langle \mathcal{R}, f, X \rangle$ there exists a τ -continuous extension $\mathcal{F}$ of f with domain X . A locally convex topological linear space E is said to have the extension property if there exists a skeleton $\mathcal{R}$ of E with the extension property.

It is clear that if E is a normed linear space, then E has the extension property in Nachbin's sense if and only if the skeleton of E , consisting only of the unit cell in E , has the extension property in our sense. Therefore, if a normed linear space has the extension property in the usual sense, then it has the extension property in our sense. On the other hand it is obvious, that the n-dimensional Euclidean space has the extension property in our sense, but if $n \geq 2$ it has not the usual extension property.

The following main theorem of this paper is a generalization of a theorem of Nachbin ([5] p. 30, theorem 1):

Theorem 1: A skeleton of a real locally convex topological linear space has the extension property if and only if it has the binary intersection property.

Corollary: A real locally convex topological linear space E has the extension property if and only if there exists a skeleton of E with the binary intersection property.

2. Proof of sufficiency:

Let $\mathcal{R}$ be a skeleton of E which has the binary intersection property. Let X be any locally convex topological linear space, Y a subspace of X, f a continuous linear mapping of Y into E and $\tau \in \langle \mathcal{R}, f, X \rangle$. Let us denote by Φ the set of all τ-continuous extensions of f partially ordered by inclusion of their graphs. By the Zorn's theorem there exists a maximal element $\varphi \in \Phi$. Let Z' be the domain of φ; it is sufficient to prove that $Z' = X$. If $Z' \neq X$, then there exists an element $r_0 \in X \dot{-} Z'$. We shall show that there exists a τ-continuous extension of f with domain

$$Z'' = \{ r + \lambda r_0 : r \in Z', \ \lambda \text{ is real} \} .$$

Let $\mathcal{O}\!\ell(f)$ be the set of all $\mathcal{R}$-solids of the form

$$\mu \mathcal{U} - f(r) ,$$

such that $r_0 + r \in \mu \tau(\mathcal{U})$, $\mathcal{U} \in \mathcal{R}$. For every two similar $\mathcal{R}$-solids

$$\mu_1 \mathcal{U} - f(r_1) \quad \text{and} \quad \mu_2 \mathcal{U} - f(r_2)$$

in $\mathcal{O}\!\ell(f)$ we have

$$r_0 + r_1 \in \mu_1 \tau(\mathcal{U}), \quad r_0 + r_2 \in \mu_2 \tau(\mathcal{U}),$$

$$r_1 - r_2 = (r_1 + r_0) - (r_2 + r_0) \in (|\mu_1| + |\mu_2|) \tau(\mathcal{U})$$

and therefore

$$f(r_1) - f(r_2) = f(r_1 - r_2) \in (|\mu_1| + |\mu_2|) \mathcal{U} = \\ = \mu_1 \mathcal{U} - \mu_2 \mathcal{U} .$$

This implies that every two elements of $\mathcal{O}\!\ell(f)$ have at least one common point and because $\mathcal{R}$ has the binary intersection property it follows that there exists a point β in the common of all solids in $\mathcal{O}\!\ell(f)$. Let us define an operator φ' from Z'' into E by the equality

$$\varphi'(r + \lambda r_0) = \varphi(r) + \lambda \beta \quad .$$

By this definition it is evident that φ' is a τ-continuous extension of f , $\varphi' \supset \varphi$, $\varphi' \neq \varphi$ and it is a contradiction with the assumption of maximality of φ . Thus the sufficiency is proved.

<u>3. Proof of necessity</u>:

Let us assume that a skeleton $\mathcal{R}$ of a locally convex topological linear space E has the extension property. We shall show that it has the binary intersection property.

Let $\mathcal{A}$ be a system of $\mathcal{R}$-solids whose any two similar elements have a common point. For $\mathcal{U} \in \mathcal{R}$ we denote by $\mathcal{A}_{\mathcal{U}}$

1) The collection of all elements of $\mathcal{A}$ which are similar to $\mathcal{U}$ if this collection is not empty.

2) The class of one element $\mathcal{U}$ if there exists no element in $\mathcal{A}$ which is similar to $\mathcal{U}$.

Let

$$A_{\mathcal{U}} = \{ x \in E : \mu \mathcal{U} + x \in \mathcal{A}_{\mathcal{U}} \quad \text{for some } \mu \}$$

and let $\xi_{\mathcal{U}}$ be a fixed element in $A_{\mathcal{U}}$. Let $p_{\mathcal{U}}$ be the semi-norm induced by $\mathcal{U} \in \mathcal{R}$, i.e.

$$p_{\mathcal{U}}(x) = \inf \{ |\lambda| \ : x \in \lambda \mathcal{U} \} \ ;$$

and let $\lambda_{\mathcal{U}}$ be a function defined by the equations

$$\lambda_{\mathcal{U}}(x) \begin{cases} \inf \{ |\lambda| : \lambda \mathcal{U} + x \in \mathcal{A}_{\mathcal{U}} \} & \text{if } x \in A_{\mathcal{U}} \\ \lambda_{\mathcal{U}}(\xi_{\mathcal{U}}) + p_{\mathcal{U}}(x - \xi_{\mathcal{U}}) & \text{if } x \in E \dot{-} A_{\mathcal{U}} . \end{cases}$$

Now we shall prove that the function $\lambda_{\mathcal{U}}$ has the following property

(1) $\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(y) \geq p_{\mathcal{U}}(x - y)$ for each $x, y \in E$,

which is equivalent with the proposition

(1′) $x - y \in (\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(y))\,\mathcal{U}$ for each $x, y \in E$.

Let us consider the following three cases:

(a) $x \in A_{\mathcal{U}}$, $y \in A_{\mathcal{U}}$: By the definition of $\lambda_{\mathcal{U}}$ for every $\varepsilon > 0$ there exist numbers λ and μ such that

$$\lambda_{\mathcal{U}}(x) + \varepsilon > \lambda \geq \lambda_{\mathcal{U}}(x), \quad \lambda\,\mathcal{U} + x \in \mathfrak{A}_{\mathcal{U}}$$

$$\lambda_{\mathcal{U}}(y) + \varepsilon > \mu \geq \lambda_{\mathcal{U}}(y), \quad \mu\,\mathcal{U} + y \in \mathfrak{A}_{\mathcal{U}}$$

Since every two elements of $\mathfrak{A}_{\mathcal{U}}$ have a non-void intersection, we have

$$x - y \in (\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(y))\,\mathcal{U} .$$

(b) $x \in A_{\mathcal{U}}$, $y \notin A_{\mathcal{U}}$: Since $x - y = (x - \xi_{\mathcal{U}}) + (\xi_{\mathcal{U}} - y)$ and $x \in A_{\mathcal{U}}$, $\xi_{\mathcal{U}} \in A_{\mathcal{U}}$, we obtain by (a)

$$x - y \in [\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(\xi_{\mathcal{U}})]\,\mathcal{U} + \rho_{\mathcal{U}}(\xi_{\mathcal{U}} - y)\,\mathcal{U} .$$

The convexity of $\mathcal{U}$ implies that

$$x - y \in [\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(\xi_{\mathcal{U}}) + \rho_{\mathcal{U}}(\xi_{\mathcal{U}} - y)]\,\mathcal{U} =$$
$$= [\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(y)]\,\mathcal{U} .$$

(c) $x \notin A_{\mathcal{U}}$, $y \notin A_{\mathcal{U}}$: By the definition of $\lambda_{\mathcal{U}}$, if $z \in E \dot{-} A_{\mathcal{U}}$ then $\lambda_{\mathcal{U}}(z) \geq \rho_{\mathcal{U}}(z - \xi_{\mathcal{U}})$. Hence

$$\xi_{\mathcal{U}} - z \in \lambda_{\mathcal{U}}(z)\,\mathcal{U}$$

and therefore $\lambda_{\mathcal{U}}(x)\,\mathcal{U} + x$ and $\lambda_{\mathcal{U}}(y)\,\mathcal{U} + y$ have the common point $\xi_{\mathcal{U}}$. Therefore we have

$$x - y \in (\lambda_{\mathcal{U}}(x) + \lambda_{\mathcal{U}}(y))\,\mathcal{U}$$

and the proof of property (1) is closed.

Put

$$N_{\mathcal{U}} = \{x \in E : \rho_{\mathcal{U}}(x) = 0\} .$$

It is easy to show that $\lambda_{\mathcal{U}}$ can be considered as a function defined on the factor space $E/N_{\mathcal{U}}$. The property (1) of $\lambda_{\mathcal{U}}$, by the lemma 2 of [5] p. 34 applied in the Banach space $E/N_{\mathcal{U}}$, implies the existence of a real-valued function with domain E having the following properties

(1) $r_{\mathcal{U}}(x) + r_{\mathcal{U}}(y) \geqq \rho_{\mathcal{U}}(x - y)$ for $x, y \in E$,

(2) $|r_{\mathcal{U}}(x) - r_{\mathcal{U}}(y)| \leqq \rho_{\mathcal{U}}(x - y)$ for $x, y \in E$,

(3) $r_{\mathcal{U}}(\lambda x + (1 - \lambda) y| \leqq \lambda r_{\mathcal{U}}(x) + (1 - \lambda) r_{\mathcal{U}}(y)$

for and such that

$\lambda_{\mathcal{U}}(x) \geqq r_{\mathcal{U}}(x)$ for each $x \in E$.

Let us consider the system $\mathcal{O}\!\!\mathcal{L}_0$ of all $\mathcal{R}$-solids of the form $\lambda_{\mathcal{U}}(x)\,\mathcal{U} + x$, where $\mathcal{U} \in \mathcal{R}$, $x \in E$. To prove that $\mathcal{O}\!\!\mathcal{L}$ has a non-void intersection it is evidently sufficient to show that $\mathcal{O}\!\!\mathcal{L}_0$ has a non-void intersection.

Let us consider the following situations:

(a) There exists an element $x_0 \in E$ such that $r_{\mathcal{U}}(x_0) = 0$ for each $\mathcal{U} \in \mathcal{R}$.

In this case we have

$$x_0 - x \in \rho_{\mathcal{U}}(x_0 - x)\,\mathcal{U} \subset (r_{\mathcal{U}}(x_0) + r_{\mathcal{U}}(x))\,\mathcal{U} = r_{\mathcal{U}}(x)\,\mathcal{U}$$

and x_0 is therefore a common point of

$r(x)\,\mathcal{U} + x$ for all $x \in E$ and $\mathcal{U} \in \mathcal{R}$.

(b) For every $x \in E$ there is a $\mathcal{U} \in \mathcal{R}$ such that $r_{\mathcal{U}}(x) > 0$.

Let ζ be an abstract element and

$$E' = \{ x + \lambda \zeta : x \in E , \ \lambda \text{ is real} \} .$$

For $\mathcal{U} \in \mathcal{R}$ we shall put

$$\rho'_{\mathcal{U}}(x + \lambda \xi) = \begin{cases} |\lambda| \cdot r_{\mathcal{U}}(-\frac{1}{\lambda} x) & \text{if } \lambda \neq 0 , \\ \rho_{\mathcal{U}}(x) & \text{if } \lambda = 0 . \end{cases}$$

By means of the same considerations used in Nachbin's lemma 1 [5] p. 33 it can be proved that $\rho'_{\mathcal{U}}$ is a semi-norm on E'. We shall consider the space E' as a locally convex topological linear space with a topology induced by the class $\{\rho'_{\mathcal{U}} : \mathcal{U} \in \mathcal{R}\}$ of semi-norms: For $\mathcal{U} \in \mathcal{R}$ put

$$\mathcal{U}' = \{x' \in E' : \rho'_{\mathcal{U}}(x) \leq 1\} .$$

As for each $\mathcal{U} \in \mathcal{R}$ we have $E \cap \mathcal{U}' = \mathcal{U}$, it follows that E is a subspace of E'.

Let e be the identity mapping on E and τ the mapping of $\mathcal{R}$ into the neighborhood system of zero in E' defined by $\tau(\mathcal{U}) = \mathcal{U}'$. It is clear that $\tau \in \langle \mathcal{R}, e, E' \rangle$ and therefore it must exist a τ-continuous extension e' of e defined on E'. Hence if $x' \in \tau(\mathcal{U}) = \mathcal{U}'$ then $e'(x') \in \mathcal{U}$ and $e'(-x+\xi) \in \rho'_{\mathcal{U}}(-x+\xi)\,\mathcal{U}$. If $x \in \mathcal{U}$ it is clear that $\rho'_{\mathcal{U}}(-x+\xi) = r_{\mathcal{U}}(x)$ and $e'(-x+\xi) = e'(\xi) - x$. Therefore $e'(\xi)$ belongs to all $\mathcal{R}$-solids of the form $r_{\mathcal{U}}(x)\,\mathcal{U} + x$ for $x \in E$, $\mathcal{U} \in \mathcal{R}$. Since

$$\lambda_{\mathcal{U}}(x)\,\mathcal{U} + x \supset r_{\mathcal{U}}(x)\,\mathcal{U} + x ,$$

the element $e'(\xi) \in E$ is contained in all $\mathcal{R}$-solids in $\mathcal{O}_0$ and the proof is complete.

__Note:__ In the special case if E is a normed linear space and $\mathcal{R}$ consists of one unit cell in E theorem 1 of this paper is equivalent to the cited Nachbin's result ([5], theorem 1, p. 30).

4. Topological characterization of locally convex topological linear spaces with the extension property:

If $\{E_\alpha : \alpha \in A\}$ is a system of locally convex topological linear spaces, we denote by $\mathsf{X}_{\alpha \in A} E_\alpha$ the cartesian product of this system with algebraical operations and topology defined as usual.

Let $\mathcal{R}$ be a skeleton of a locally convex topological linear space and $\mathcal{U} \in \mathcal{R}$. Let $\rho_{\mathcal{U}}$ be the semi-norm induced by $\mathcal{U}$, defined by

$$\rho_{\mathcal{U}}(x) = \inf\{|\lambda| : x \in \lambda\,\mathcal{U}\}$$

and $N_{\mathcal{U}}$ the set $\{x : \rho_{\mathcal{U}}(x) = 0\}$. Since $N_{\mathcal{U}}$ is a subspace of E we can consider the factor space $E/N_{\mathcal{U}}$, the elements of which are sets of the form $\pi_{\mathcal{U}}(x) = x + N_{\mathcal{U}}$. Algebraic operations in $E/N_{\mathcal{U}}$ are defined obviously and the norm is defined by the equation $\| \pi_{\mathcal{U}}(x) \| = \rho_{\mathcal{U}}(x)$. Thus the factor space becomes normed linear. Now let us consider the topological product

$$\underset{\mathcal{U}\in\mathcal{R}}{\mathsf{X}} E/N_{\mathcal{U}} \quad .$$

Then there exists a natural linear homeomorphism φ of E into the space $\underset{\mathcal{U}\in\mathcal{R}}{\mathsf{X}} E/N_{\mathcal{U}}$, defined by

$$\varphi(x) = \{\pi_{\mathcal{U}}(x)\}_{\mathcal{U}\in\mathcal{R}} \in \underset{\mathcal{U}\in\mathcal{R}}{\mathsf{X}} E/N_{\mathcal{U}} \quad .$$

In general $\varphi(E)$ is not necessarily equal to $\underset{\mathcal{U}\in\mathcal{R}}{\mathsf{X}} E/N_{\mathcal{U}}$.

The following lemmas will be useful.

Lemma 1. If E has the extension property, then there exists a skeleton $\mathcal{R}$ of E such that for all $\mathcal{U} \in \mathcal{R}$ the skeleton $\mathcal{R}_{\mathcal{U}}$ consisting of unit cell in normed linear space $E/N_{\mathcal{U}}$ has the extension property.

Proof: Let $\mathcal{R}$ be a skeleton of E with the binary intersection property. Then, for $\mathcal{U} \in \mathcal{R}$, $\mathcal{R}_{\mathcal{U}}$ has clearly the binary intersection property and by theorem 1 it has the extension property.

Lemma 2. A locally convex topological linear space E with the extension property is complete.

Proof: Let $\hat{E}$ be the completion of E , i.e. a complete locally convex topological linear space containing E as a dense subspace. Let $\{x_\alpha\}_{\alpha\in A}$ be a Cauchy net in E , which is convergent to a point $\hat{x} \in \hat{E}$. Let $\hat{e}$ be the continuous extension of the identity mapping e of E to $\hat{E}$. Since $\hat{e}(x_\alpha) = e(x_\alpha) = x_\alpha$ for $\alpha \in A$ and $\lim \hat{e}(x_\alpha) = \hat{e}(\hat{x})$, by uniqueness of the limit we obtain $\hat{x} = \hat{e}(\hat{x}) \in E$ and the net $\{x_\alpha\}_{\alpha \in A}$ converges to a point of E and thus E is complete.

Now we are able to give a characterization of locally convex spaces with the extension property.

Theorem 2. A locally convex topological linear space has the extension property if and only if it is linear homeomorphic with a topological product of normed linear spaces with the usual extension property.

a) Proof of sufficiency: Let $E = \mathop{X}_{\alpha \in A} E_\alpha$ and E_α be spaces with the extension property. By theroem 1 for each $\alpha \in A$ there exists a skeleton $\mathcal{R}_\alpha$ of E_α with the binary intersection property. Let $\mathcal{R}$ be the collection of neighbouhhoods of zero in E, consisting of all

$$\mathcal{U} = \mathop{X}_{\alpha \in A} \mathcal{U}_\alpha ,$$

where $\mathcal{U}_\alpha \in \mathcal{R}_\alpha$ and $\mathcal{U}_\alpha = E_\alpha$ except for a finite number of $\alpha \in A$. It is easy to see that $\mathcal{R}$ is a skeleton in E with the binary intersection property and therefore, by theorem 1, E has the extension property.
(Note: It is not necessary to assume that E_α are normed linear.)

b) Proof of necessity: Let E be a locally convex topological linear space with the extension property. By lemma 1 there exists a skeleton $\mathcal{R}$ of E such that for each $\mathcal{U} \in \mathcal{R}$ the normed linear space $E/N_{\mathcal{U}}$ has the usual extension property. By lemma 2 E is complete and since φ is a linear homeomorphic mapping $\varphi(E)$ is also complete. To prove that

$$\varphi(E) = \mathop{X}_{\mathcal{U} \in \mathcal{R}} E/N_{\mathcal{U}}$$

it is sufficient to show that $\varphi(E)$ is dense in $\mathop{X}_{\mathcal{U} \in \mathcal{R}} E/N_{\mathcal{U}}$

Let $\{x_{\mathcal{U}}\}_{\mathcal{U} \in \mathcal{R}} \in \mathop{X}_{\mathcal{U} \in \mathcal{R}} E/N_{\mathcal{U}}$ be an arbitrary element and $\mathcal{W}$

be a neighborhood of zero of the form

$$\mathcal{W} = \mathop{X}_{\mathcal{U}\in\mathcal{R}} \mathcal{W}_{\mathcal{U}} \quad ,$$

where

$\mathcal{W}_{\mathcal{U}_j} = \{ z \in E/N_{\mathcal{U}_j} : \| z \|_{\mathcal{U}_j} \leq \lambda_j \}$ for $\mathcal{U}_j \in \mathcal{R}$, $j = 1, \ldots, k$ and $\mathcal{W}_{\mathcal{U}_j} = E/N_{\mathcal{U}}$ except for $\mathcal{U} = \mathcal{U}_j$, $j = 1, \ldots, n$. For $j = 1, \ldots, n$ choose $y_{\mathcal{U}_j}$ in $\pi^{-1}_{\mathcal{U}_j}(x_{\mathcal{U}_j})$; then the collection $\{ \lambda_j \mathcal{U}_j + y_{\mathcal{U}_j} : j = 1, \ldots, n \}$ is a set of $\mathcal{R}$-solids any two of which are not similar, hence all $\mathcal{R}$-solids $\lambda_j \mathcal{U}_j + y_{\mathcal{U}_j}$ have a common point $x \in E$. Thus we have $\pi_{\mathcal{U}_j}(x) \in \pi_{\mathcal{U}_j}(\lambda_j \mathcal{U}_j + y_{\mathcal{U}_j}) = \lambda_j \pi_{\mathcal{U}_j}(\mathcal{U}_j) + x_{\mathcal{U}_j} =$

$= \mathcal{W}_{\mathcal{U}_j} + x_{\mathcal{U}_j}$ for all $j = 1, \ldots, n$. This shows that $x \in \mathcal{W} + \{x_{\mathcal{U}}\}_{\mathcal{U}\in\mathcal{R}}$ and therefore $\varphi(E)$ is dense in $\mathop{X}_{\mathcal{U}\in\mathcal{R}} E/N_{\mathcal{U}}$ and

$$\varphi(E) = \mathop{X}_{\mathcal{U}\in\mathcal{R}} E/N_{\mathcal{U}} \quad .$$

E is therefore linear homeomorphic with the topological product of normed linear spaces which have the usual extension property.

<u>Corollary 1</u>: A locally convex topological linear space E has the extension property if and only if E is linear homeomorphic with a topological product of normed linear spaces E_α of continuous function over extremally disconnected compact Hausdorff topological spaces Q_α .

This is an immediate consequence of theorem 2 and the result of Kelley [4] p. 323.

<u>Corollary 2</u>: If a normed linear space E has the extension property (in our sense), then there exists an equivalent norm in E such that E with this norm has the usual extension property.
<u>Proof</u>: By theorem 2 we have $E = \mathop{X}_{\alpha\in A} E_\alpha$, where E_α are normed linear spaces with the usual extension property. As E is

a normed space, A must be finite. If $\| \|_{\alpha}$ is the norm in E_{α}, we put for $x = \{x_{\alpha}\} \in E$, $\|x\| = \max_{\alpha \in A} \|x_{\alpha}\|_{\alpha}$ and this norm has the required properties.

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