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CONCERNING CONGRUENCE RELATIONS ON COMMUTATIVE SEMIGROUPS

(Preliminary communication)

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Let S be a commutative semigroup. For any congruence relation C on S let $[C]$ denote the ideal consisting exactly of all $x \in S$ with the following property: there exists a positive integer ρ such that $x^\rho u C x^\rho v$ is true for all u and v in S . A primary congruence relation is a congruence relation C satisfying the following condition: if $xu C xv$ and u (non) $C v$ hold for some u and v in S , then $x \in [C]$. If C is primary, then $xy \in [C]$ and $y \notin [C]$ implies $x \in [C]$.

A decomposition

$$(1) \quad C = C_1 \cap C_2 \cap \dots \cap C_n$$

is said to be a standard decomposition of C , if every C_i ($i = 1, 2, \dots, n$) is a primary congruence relation, if $[C_i] \neq [C_j]$ for $i \neq j$ and if no C_i in (1) can be omitted. S is said to satisfy the maximality condition for congruence relations, if every non empty set of congruence relations on S contains a maximal one (in the sense of the well-known partial ordering of congruence relations). In this case for every C at least one standard decomposition is possible. Moreover, there is a unicity theorem: In any standard decomposition (1) the number n and the ideals $[C_i]$ ($i = 1, 2, \dots, n$) are uniquely determined by C . Further on, as in the classical ideal theory

of commutative rings, the second unicity theorem can be proved: if

$$\begin{aligned} C &= C_1 \cap C_2 \cap \dots \cap C_k \cap C_{k+1} \cap \dots \cap C_n = \\ &= C'_1 \cap C'_2 \cap \dots \cap C'_k \cap C'_{k+1} \cap \dots \cap C'_n \end{aligned}$$

are two standard decompositions of C , if $[C_i] = [C'_i]$ for all $i = 1, 2, \dots, n$ and if $[C_j] \not\subseteq [C_i]$ for all $i = 1, 2, \dots, k$ and $j = k+1, \dots, n$, then

$$C_1 \cap C_2 \cap \dots \cap C_k = C'_1 \cap C'_2 \cap \dots \cap C'_k.$$

The preceding theory can be treated as an ideal theory as well. A subset \mathcal{J} of S is called a congruence ideal of S , if there is a congruence relation \mathcal{C} on S such that $x \in \mathcal{J}$ holds if and only if $xu \mathcal{C} xv$ is true for all u and v in S . If, among all possible congruence relations \mathcal{C} corresponding to \mathcal{J} a primary can be found, then \mathcal{J} is called primary.

A congruence ideal \mathcal{J} is always an ideal. By $[\mathcal{J}]$ we denote the ideal consisting exactly of all $x \in S$ such that there is a positive integer ρ with $x^\rho \in \mathcal{J}$. The intersection of any system of congruence ideals is always a congruence ideal.

A decomposition

$$(2) \quad \mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2 \cap \dots \cap \mathcal{J}_k$$

is said to be a standard decomposition of \mathcal{J} if all

\mathcal{J}_i ($i = 1, 2, \dots, k$) are primary congruence ideals, if $[\mathcal{J}_i] \not\subseteq [\mathcal{J}_j]$ for $i \neq j$ and if no \mathcal{J}_i in (2) can be omitted. If S satisfies the maximality condition for congruence relations, then for every congruence ideal \mathcal{J} at least one standard decomposition of \mathcal{J} is possible. More-

over, both unicity theorems are true: In (2) the number κ and the ideals $[\mathcal{I}_i]$ are uniquely determined by \mathcal{I} and if $\mathcal{J} =$

$$= \mathcal{I}'_1 \cap \mathcal{I}'_2 \cap \dots \cap \mathcal{I}'_k \cap \mathcal{I}'_{k+1} \cap \dots \cap \mathcal{I}'_\kappa = \mathcal{I}'_1 \cap \mathcal{I}'_2 \cap \dots \cap \mathcal{I}'_k \cap \mathcal{I}'_{k+1} \cap \dots \cap \mathcal{I}'_\kappa$$

are two standard decompositions of \mathcal{I} with $[\mathcal{I}_i] = [\mathcal{I}'_i]$ for all $i = 1, 2, \dots, \kappa$ and if $[\mathcal{I}_j] \not\subseteq [\mathcal{I}_i]$ holds for all $i = 1, 2, \dots, k$ and $j = k+1, \dots, \kappa$, then $\mathcal{I}'_1 \cap \mathcal{I}'_2 \cap \dots \cap \mathcal{I}'_k = \mathcal{I}'_1 \cap \mathcal{I}'_2 \cap \dots \cap \mathcal{I}'_k$.