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CONCERNING CONGRUENCE RELATIONS ON COMMUTATIVE SEMIGROUPS

(Preliminary communication)

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Let S be a commutative semigroup. For any congruence relation C on S let $[C]$ denote the ideal consisting exactly of all $x \in S$ with the following property: there exists a positive integer ρ such that $x^\rho u \, C \, x^\rho v$ is true for all u and v in S. A primary congruence relation is a congruence relation C satisfying the following condition: if $xu \, C \, xv$ and $u \, (\text{non } C) \, v$ hold for some u and v in S, then $x \in [C]$. If C is primary, then $xy \in [C]$ and $y \notin [C]$ implies $x \in [C]$.

A decomposition

(1) $$C = C_1 \cap C_2 \cap \dots \cap C_r$$

is said to be a standard decomposition of C, if every C_i $(i = 1, 2, \dots, r)$ is a primary congruence relation, if $[C_i] \neq [C_j]$ for $i \neq j$ and if no C_i in (1) can be omitted. S is said to satisfy the maximality condition for congruence relations, if every non empty set of congruence relations on S contains a maximal one (in the sense of the well-known partial ordering of congruence relations). In this case for every C at least one standard decomposition is possible. Moreover, there is a unicity theorem: In any standard decomposition (1) the number r and the ideals $[C_i]$ $(i = 1, 2, \dots, r)$ are uniquely determined by C. Further on, as in the classical ideal theory of commutative rings, the second unicity theorem can be proved: if

$$C = C_1 \cap C_2 \cap \dots \cap C_k \cap C_{k+1} \cap \dots \cap C_r =$$
$$= C_1' \cap C_2' \cap \dots \cap C_k' \cap C_{k+1}' \cap \dots \cap C_r'$$

are two standard decompositions of C, if $[C_i] = [C_i']$ for all $i = 1, 2, \dots, r$ and if $[C_j] \not\subseteq [C_i]$ for all $i = 1, 2, \dots, k$ and $j = k+1, \dots, r$, then

$$C_1 \cap C_2 \cap \dots \cap C_k = C_1' \cap C_2' \cap \dots \cap C_k' .$$

The preceding theory can be treated as an ideal theory as well. A subset $\mathfrak{I}$ of S is called a congruence ideal of S, if there is a congruence relation C on S such that $x \in \mathfrak{I}$ holds if and only if $xu \, C \, xv$ is true for all u and v in S. If, among all possible congruence relations C corresponding to $\mathfrak{I}$ a primary can be found, then $\mathfrak{I}$ is called primary.

A congruence ideal $\mathfrak{I}$ is always an ideal. By $[\mathfrak{I}]$ we denote the ideal consisting exactly of all $x \in S$ such that there is a positive integer ρ with $x^{\rho} \in \mathfrak{I}$. The intersection of any system of congruence ideals is always a congruence ideal.

A decomposition

$$\mathfrak{I} = \mathfrak{I}_1 \cap \mathfrak{I}_2 \cap \dots \cap \mathfrak{I}_r \tag{2}$$

is said to be a standard decomposition of $\mathfrak{I}$ if all $\mathfrak{I}_i$ $(i = 1, 2, \dots r)$ are primary congruence ideals, if $[\mathfrak{I}_i] \neq [\mathfrak{I}_j]$ for $i \neq j$ and if no $\mathfrak{I}_i$ in (2) can be omitted. If S satisfies the maximality condition for congruence relations, then for every congruence ideal $\mathfrak{I}$ at least one standard decomposition of $\mathfrak{I}$ is possible. Moreover, both unicity theorems are true: In (2) the number r and the ideals $[\mathfrak{I}_i]$ are uniquely determined by $\mathfrak{I}$ and if $\mathfrak{I} =$

$$= \mathfrak{I}_1 \cap \mathfrak{I}_2 \cap \ldots \cap \mathfrak{I}_k \cap \mathfrak{I}_{k+1} \cap \ldots \cap \mathfrak{I}_r = \mathfrak{I}'_1 \cap \mathfrak{I}'_2 \cap \ldots \cap \mathfrak{I}'_k \cap \mathfrak{I}'_{k+1} \cap \ldots \cap \mathfrak{I}'_r$$

are two standard decompositions of $\mathfrak{I}$ with $[\mathfrak{I}_i] = [\mathfrak{I}'_i]$ for all $i = 1, 2, \ldots, r$ and if $[\mathfrak{I}_j] \not\subset [\mathfrak{I}_i]$ holds for all $i = 1, 2, \ldots, k$ and $j = k+1, \ldots, r$, then $\mathfrak{I}_1 \cap \mathfrak{I}_2 \cap \ldots \cap \mathfrak{I}_k = \mathfrak{I}'_1 \cap \mathfrak{I}'_2 \cap \ldots \cap \mathfrak{I}'_k$.