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# ON CYCLIC AND RADIAL VARIATIONS OF A PLANE PATH

(Preliminary communication)

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We shall identify the set of finite complex numbers with the Euclidean plane $E_2$ . By a path we mean any continuous complex-valued function $\psi$ on a compact inverval $\langle a, b \rangle$ . Given such a path $\psi$ , a point $z \in E_2$ and a real number $\alpha$ , we denote by $\mu^{\psi}(\alpha; z)$ the number (possibly zero or infinite) of points in $\{t; t \in \langle a, b \rangle, \psi(t) - z = |\psi(t) - z| \exp i\alpha, \psi(t) \neq z\}$.

Then $\mu^{\psi}(\alpha; z)$ is Lebesgue measurable with respect to the variable $\alpha$ on $\langle 0, 2\pi \rangle$ and we may put

$$v^{\psi}(z) = \int_0^{2\pi} \mu^{\psi}(\alpha; z)\, d\alpha \; .$$

In a similar way, let $\nu^{\psi}(\rho; z)$ stand for the number of points in $\{t; t \in \langle a, b \rangle, |\psi(t) - z| = \rho\}$ $(0 \leqq \nu^{\psi}(\rho; z) \leqq +\infty)$.
Since $\nu^{\psi}(\rho; z)$ is Lebesgue measurable with respect to $\rho$ on $(0, +\infty)$ we may form the integrals

$$u_{\kappa}^{\psi}(z) = \int_0^{\kappa} \nu^{\psi}(\rho; z)\, d\rho \; ; \; \kappa > 0 \; .$$

$v^{\psi}(z)$ will be called the cyclic variation of $\psi$ with respect to $z$ and $u^{\psi}(z) = u_{\infty}^{\psi}(z)$ will be called the radial variation of $\psi$ with respect to $z$ . The quantities $v^{\psi}(z)$, $u_{\kappa}^{\psi}(z)$ were introduced in Comment. Math. Univ. Carolinae 3,1 (1962), 3-10 in connection with investigations concerning the boundary behaviour of the logarithmic potential. In present paper we shall collect some results on $v^{\psi}(z)$, $u_{r}^{\psi}(z)$ which are useful in above mentioned topics in potential theory and may be of some interest in themselves. We shall start with several simple theorems connecting rectifiability of $\psi$ and the behaviour of $v^{\psi}(z)$, $u^{\psi}(z)$.

Theorem 1. Suppose that the points $z^1, z^2, z^3$ are not situated on a single straight-line. If $\psi$ is a path and

$$v^{\psi}(z^1) + v^{\psi}(z^2) + v^{\psi}(z^3) < +\infty$$

then $\psi$ is rectifiable.

Theorem 2. Let $z^1 \neq z^2$ and let $\psi$ be a path not meeting the straight-line through $z^1$ and $z^2$. If

$$v^{\psi}(z^1) + v^{\psi}(z^2) < +\infty$$

then $\psi$ is rectifiable.

Theorem 3. If $\psi$ is a rectifiable path on $\langle a, b \rangle$ then $v^{\psi}(z)$, considered as a function of the variable $z$, is finite and locally Lipschitzian on $E_2 - \psi(\langle a, b \rangle)$. Further $\lim_{|z| \to \infty} v^{\psi}(z) = 0$.

Remark 1. In theorem 3, $v^{\psi}$ need not be finite on $\psi(\langle a, b \rangle)$. If we do not assume that $\psi$ is rectifiable then $v^{\psi}$ may be infinite in some points of $E_2 - \psi(\langle a, b \rangle)$ and may be discontinuous there. This can be shown by simple examples.

Similar theorems hold about the radial variation $u^{\psi}$.

Theorem 4. Let $\psi$ be a rectifiable path on $\langle a, b \rangle$. Then $u^{\psi}(z)$ is a finite and

continuous function of the variable $z$ on $E_2$.
Moreover, $u^{\psi}(z)$ is locally Lipschitzian on $E_2 - \psi(\langle a, b \rangle)$.

Remark 2. It is easily shown by an example that, in theorem 4, $u^{\psi}$ need not be Lipschitzian on $E_2$. If the assumption that $\psi$ be rectifiable is dropped then $u^{\psi}$ may be infinite and discontinuous in some points of $E_2 - \psi(\langle a, b \rangle)$.

Theorem 5. Let $z^1 \neq z^2$ and suppose that $\psi$ is a path not meeting the straight-line through $z^1$ and $z^2$. If

$$u^{\psi}(z^1) + u^{\psi}(z^2) < +\infty$$

then $\psi$ is rectifiable.

Theorem 6. Suppose that $\psi$ is a path and $z^1, z^2, z^3$ are points in $E_2$ not lying on a single straight-line. If

$$u^{\psi}(z^1) + u^{\psi}(z^2) + u^{\psi}(z^3) < +\infty$$

then $\psi$ is rectifiable.

Remark 3. Simple example can be given of a path of infinite length such that $u^{\psi}(z)$ is finite for an infinity of points $z$; according to theorem 6, all $z$'s with $u^{\psi}(z) < +\infty$ must then be situated on the same straight-line.

Theorem 7. Let $\psi$ be a path and suppose that

$$u^{\psi}(z) + v^{\psi}(z) < +\infty$$

for a certain $z \in E_2$. Then $\psi$ is rectifiable.

Now we shall announce two estimates concerning $v^{\psi}(z)$ and $u^{\psi}_{r}(z)$ which are useful for the investigation of non-tangential limits of the logarithmic potential of the double distribution.

Theorem 8. Let $\psi$ be a path on $\langle a, b\rangle$, $z \in E_2$. Fix $\beta \in E_1$, $r > 0$ and put $\zeta = z + r \exp i\beta$. Suppose that

(1) $\quad z \pm \rho \exp i\alpha \notin \psi(\langle a, b\rangle)$

whenever $0 < \rho < r$, $|\alpha - \beta| < \delta \; (0 < \delta < \frac{\pi}{2})$. Then

$$r^{-1} u_r^{\psi}(z) \leqq K[v^{\psi}(z) + v^{\psi}(\zeta)]$$

with $K$ depending on $\delta$ only.

Theorem 9. Let $z \in E_2$, $r > 0$ and suppose that $\psi$ is a path not meeting $\{\eta;\ \eta \in E_2, |\eta - z| > r\}$. Fix a $\beta \in E_1$ and assume that (1) takes place for every couple $\rho$, $\alpha$ with $0 < \rho < r$, $|\alpha - \beta| < \delta \; (0 < \delta < \frac{\pi}{2})$. Let $0 < x \leqq r$ and put $\zeta = z + x \exp i\beta$. Then

$$v^{\psi}(\zeta) \leqq L\{v^{\psi}(z) + \sup_{0<t<r} t^{-1} u_t^{\psi}(z)\}$$

with $L$ depending on $\delta$ only.

Theorem 10. If $\psi$ is a path on $\langle a, b\rangle$ then $v^{\psi}(z)$ is lower semicontinuous on $E_2$. If $\psi$ is rectifiable then $v^{\psi}(z)$ is finite, continuous and subharmonic on $E_2 - \psi(\langle a, b\rangle)$.

Notation. If $\psi$ is a path on $\langle a, b\rangle$ and $\zeta \in E_2$ we shall denote by $N_{\psi}(\zeta)$ the number of points in

$\{t;\ t \in \langle a, b\rangle, \psi(t) = \zeta\} \; (0 \leqq N_{\psi}(\zeta) \leqq +\infty)$.

We say that $\psi$ is simple provided $\psi(t_1) \neq \psi(t_2)$ whenever $a \leqq t_1 < t_2 \leqq b$, $t_2 - t_1 < b - a$. $\psi$ is termed closed if $\psi(a) = \psi(b)$.

Theorem 11. Let $\psi$ be a path on $\langle a, b\rangle$, $C = \psi(\langle a, b\rangle)$ and suppose that $\sup_{\zeta \in C} N_{\psi}(\zeta) < +\infty$.

In order that $v^{\psi}(z)$ be bounded on $E_2 - C$ it is necessary and sufficient that it be bounded on $C$.

As a consequence of theorem 11 we obtain the following

Corollary. Let $\psi$ be a simple closed path on $\langle a, b \rangle$ and let $G$ be a complementary domain of $C = \psi(\langle a, b \rangle)$. Then the following conditions (a) and (b) are equivalent to each other:

(a) $$\sup_{z \in C} v^{\psi}(z) < +\infty ,$$

(b) $$\sup_{z \in G} v^{\psi}(z) < +\infty .$$

Remark 4. The above corollary makes it possible to prove a necessary and sufficient condition that, for every continuous double distribution on $C$, the corresponding logarithmic potential be uniformly continuous on $G$ (cf. theorems 3 and 4 in Comment. Math. Univ. Carolinae 3,1 (1962), p. 9). The above corollary can also be used to obtain a solution of the problem N° 4 raised by I. Babuška in Časopis pro pěstování matematiky 79 (1954), N° 2, p. 164.

The following alternative of theorem 11 is also useful in potential theory.

Theorem 12. Let $\psi$ be a path on $\langle a, b \rangle$, $C = \psi(\langle a, b \rangle)$ and let $G \subset E_2$ be a bounded open set disjoint with $C$. Write $B(G)$ for the boundary of $G$ and suppose that $N_{\psi}(\xi)$ is bounded on $C \cap B(G)$. Further suppose that a neighborhood $\mathcal{U}(\xi)$ can be associated with every $\xi \in C \cap B(G)$ that the following condition (c) be fulfilled:

(c) For every $z \in \mathcal{U}(\xi) \cap G$ the segment of end-points $z, \zeta$ meets $C$ at $\zeta$ only.

Then $v^{\psi}(z)$ is bounded on $G$ provided it is bounded on $B(G)$.

Notation. If $u \neq 0 \neq v$ are complex numbers we write $\sphericalangle\, u, v \,\not> = \arccos\left(\mathcal{Re}\,\frac{u}{v} \cdot \left|\frac{v}{u}\right|\right)$ for the non-oriented angle of corresponding vectors.

Let now $\psi$ be a simple path on $\langle a, b \rangle$. With any subdivision $D = \{a = t_0 < t_1 < \dots < t_n = b\}$ $(n > 1)$ of $\langle a, b \rangle$ we associate the number

$$\sigma(D) = \sum_{j=1}^{n-1} \sphericalangle\, \psi(t_{j+1}) - \psi(t_j),\ \psi(t_j) - \psi(t_{j-1}) \,\not>$$

and define

$$\operatorname{bnd} \psi = \sup_D \sigma(D).$$

Remark 5. The quantity $\operatorname{bnd} \psi$, to be called the bend of $\psi$, could be defined for more general paths (cf. a number of papers by Ka. Iséki in Proc. Japan Academy 1961, 1962). We shall not go into details and shall only notice that simple plane paths $\psi$ with $\operatorname{bnd} \psi < +\infty$ coincide with those introduced by J. Radon in connection with his investigations on the logarithmic potential (Über Randwertaufgaben beim logarithmischen Potential, Sitzber. Akad. Wiss. Wien 128 (1919), 1123-1167) and called "Kurven beschränkter Drehung" ("courbes à rotation finie" - cf. F. Riesz et B. Sz. Nagy: Leçons d'analyse fonctionelle, chap. IV, n$^{o}$ s 81, 91). A comparison of these paths $\psi$ with those for which $v^{\psi}$ is bounded is given by the following

Theorem 13. Let $\psi$ be a simple path on $\langle a, b \rangle$, $C = \psi(\langle a, b \rangle)$. Then

$$\sup_{z \in C} v^{\psi}(z) \leq \operatorname{bnd} \psi.$$

Remark 6 . While

$$\sup_{z \in C} v^{\psi}(z) < + \infty \tag{2}$$

whenever $bnd\ \psi < + \infty$ , the converse is not true as simple examples show. It is also easily seen that the class of smooth paths fulfilling so-called Ljapunoff conditions (compare N.M. Gjuntěr: Těorija potěnciala i jejo priměněnija k osnovnym zadačam matěmatičeskoj fiziki, Gostěchizdat 1953, where analogous surfaces in 3-space are considered) is included in the class of all $\psi$ with (2). It is worth mentioning that some investigations on the logarithmic potential which are usually connected with Radon´s "Kurven beschränkter Drehung" can be carried out for more general paths fulfilling (2).