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ON COMMON FIXED POINTS OF COMMUTATIVE MAPPINGS

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We use the following notation: if Φ is a system of mappings from the set Y into Y, then, for any $Z \subset Y$, $\Phi(Z)$ is the set of all $f(z)$, $f \in \Phi$, $z \in Z$; instead of $\Phi(\{z\})$, $\Phi(z)$ is written. If $Z \subset Y$, $\Phi(Z) \subset Z$, then Φ / Z denotes the set of all $f \in \Phi$ restricted to Z.

The operation in all semi-groups throughout this remark is the composition of mappings.

<u>Theorem</u>. Let F be a commutative semi-group of continuous mappings from a compact interval X into itself; let F contain a unity element. If $F(e)$ is connected for some $e \in X$, then all mappings from F have a common fixed point.

First we shall prove a few lemmas.

<u>Lemma 1</u>. Let G be a commutative semi-group of mappings from a given set Y into itself. Let $G(e) = Y$ for some $e \in Y$.

Then G is a group if and only if $G(x) = Y$ for every $x \in Y$. If G is a group, then every $f \in G$ is one-to-one onto, and $f_1 \in G$, $f_2 \in G$, $f_1(x) = f_2(x)$ for some $x \in Y$, implies $f_1 = f_2$.

Proof. Let G be a group. We can find for every $x \in Y$ a mapping $f \in G$ such that $f(e) = x$. The mapping f^{-1} belongs to G and $e = f^{-1}(x)$. Therefore $Y \supset G(x) \supset G(e) = Y$.

Let $G(x) = Y$ for every $x \in Y$. We can find $f \in G$ and $g \in G$ such that $f(e) = x$ and $g(x) = e$. Therefore $f[g(e)] = e$. If $z = h(e)$, $h \in G$, then

$$f[g(z)] = f\{g[h(e)]\} = h\{f[g(e)]\} = h(e) = z$$

Therefore $f[g(z)] = z$ for every $z \in Y$ and $g = f^{-1}$. Evidently G contains the identity mapping.

Let G be a group and $f \in G$, $x_1 \in Y$, $f(x_1) = f(x_2)$. Then $x_1 = g_1(e)$, $x_2 = g_2(e)$, $g_1 \in G$, $g_2 \in G$ and

$$f(x_1) = f[g_1(e)] = g_1[f(e)] = g_2[f(e)] .$$

As $G[f(e)] = Y$ we can write for every $y \in Y : y = h[f(e)]$, $h \in G$; therefore, $g_1(y) = g_1\{h[f(e)]\} = h\{g_1[f(e)]\} =$ $= h\{g_2[f(e)]\} = g_2\{h[f(e)]\} = g_2(y)$. Hence $g_1 = g_2$, $x_1 = x_2$.

If $f_1(x) = f_2(x)$, $f_1 \in G$, $f_2 \in G$, $x \in Y$, then for every $y \in Y$ we can find a mapping $g \in G$ such that $y = g(x)$. Then

$$f_1(y) = f_1[g(x)] = g[f_1(x)] = g[f_2(x)] = f_2[g(x)] =$$
$$= f_2(y) \text{ and } f_1 = f_2.$$

<u>Lemma 2</u>. Let F be a commutative semi-group of mappings from a set X into X ; suppose that F contains a unity element. If $x \in X$, $F(x) = X$, $x' \in X$, then either

(a) $F \mid F(x')$ is a group or

(b) for some $y \in X$, x' non $\in F(y)$.

Proof. If (b) does not hold, then $x' \in F(x)$ for every $x \in X$. Clearly, $F[E(x)] \subset F(x)$ for every $x \in X$. Put $X' = F(x')$, $F' = F \mid X'$. Evidently, $X' = F'(x')$ and, for any $x \in X'$, $X' = F'(x') \subset F'[F'(x)] \subset F'(x)$, hence $F'(x) = X'$.

By Lemma 1, F' is a group.

<u>Lemma 3</u>. Let G be a commutative group of continuous mappings from a given bounded connected subset Y of the real line into Y ; let Y ontain more than one point. Let $G(e) = Y$ for some $e \in Y$. Then Y is an open interval. If we put $Y = (a; b)$, then $\lim_{x \to a+} f(x) = a$ and $\lim_{x \to b-} f(x) = b$ for every $f \in G$.

Proof. According to Lemma 1, every $f \in G$ is a one-to-one mapping from Y onto Y and the values of two different mappings from G are distinct at every point. As identity mapping belongs to G, every $f \in G$ is an increasing function. As every mapping $f \in G$ is onto, $\lim_{x \to a+} f(x) = a$ and $\lim_{x \to b-} f(x) = b$. If $a \in Y$, then $f(a) = a$, as f is continuous, and therefore $G(a) = a$. As Y contains more than one point we have $G(a) \neq Y$ and $a \notin Y$. The same is valid for b.

If Z is a metric space, we shall denote by $d(Z)$ its diameter.

<u>Lemma 4</u>. Let X_0 be a compact interval of the real line, c its centre. Let F be a commutative semi-group of continuous mappings of X_0 into X_0; suppose that F possesses a unity element. Suppose that, for some $x_0 \in X_0$, $F(x_0)$ is connected, $\overline{F(x_0)} = X_0$. Then either (1) $F(c) = (c)$, or (2) the endpoints of the interval $F(c)$ are fixed points for F, or (3) there exists $x_1 \in F(x_0)$ such that $F(x_1)$ is connected, $d(F(x_1)) \leqq \frac{1}{2} d(X_0)$.

Proof. For any $x \in F(x_0)$, the set $F(x)$ is connected since, for some $f \in F$, $F(x) = F[f(x_0)] = f[F(x_0)]$. Consider the semi-group $F_0 = F \mid F(x_0)$. By Lemma 2, either $F_0 | F(x)$ is a group or there exists $x_1 \in F(x_0)$ such that c non $\in F(x_1)$. In the first case, apply Lemma 3 (the case $F(c) = (c)$ is trivial). In the second case,

$$d(F(x_1)) \leqq \frac{1}{2} d(F(x_0)) = \frac{1}{2} d(X) \quad \text{since} \quad c \text{ non} \in F(x_1).$$

Now we can prove the main theorem.

We put $X_0 = \overline{F(e)}$ and consider the semi-group

$F \mid X_0$.

By Lemma 4, either the endpoints of $F(c)$ (or c itself) are fixed for F, or there exists $x_1 \in F(c)$ such that $d(F(x_1)) \leqq \frac{1}{2} d(X_0)$, and $X_1 = \overline{F(x_1)}$ satisfies the conditions required for X_0 in the Lemma 4. Proceeding by induction, either we obtain, at some step, a fixed point for F, or a sequence of intervals $\{X_n\}$ is obtained with $X_n \supset X_{n+1}$, $d(X_{n+1}) \leqq \frac{1}{2} d(X_n)$, $F(X_n) \subset X_n$; in this last case, clearly, $\bigcap X_n$ is one point-set (z), and z is fixed for F.