Commentationes Mathematicae Universitatis Carolinae

Ivo Marek

C-convergence of iterations of linear bounded operators

Commentationes Mathematicae Universitatis Carolinae, Vol. 2 (1961), No. 4, 22–24

Persistent URL: http://dml.cz/dmlcz/104897

Commentationes Mathematicae Universitatis Carolinae

C - CONVERGENCE OF ITERATIONS OF LINEAR BOUNDED OPERATORS

Ivo MAREK, Praha

Let X be a complex Banach space, $X_1 = (X \to X)$ the Banach space of linear transformations of the space X into itself with the usual norm

$$\|T\|_{X_1} = \sup_{\|x\|_X \leq 1} \|Tx\|_X .$$

Let us suppose that the operator $T \in X_1$ has p eigenvalues $\mu_1, \dots, \mu_p$; $p \geq 1$, which are poles of multiplicities $q_1, \dots, q_p$ of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$.

Let the inequalities

(1) $$|\mu_1| = \dots = |\mu_p| > |\lambda|$$

hold for every point $\lambda \in \sigma(T), \lambda \neq \mu_j, j = 1, \dots, p$, where $\sigma(T)$ is the spectrum of the operator T .

Let

$$R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_j)^k T_{jk} + \sum_{k=1}^{q_j} (\lambda - \mu_j)^{-k} B_{jk}$$

be the Laurent expansion of the resolvent of the operator $T \in X_1$ in the neighborhood of the point μ_j . It is well known [3] that

$$B_{j1} = \frac{1}{2\pi i} \int_{C_{oj}} R(\lambda, T) d\lambda , \quad B_{j\,k+1} = (T - \mu_j I) B_{jk} ,$$

$$j = 1, \dots, p; \quad k = 1, 2, \dots ,$$

where C_{oj} is the boundary of the circle K_{oj} having the property $\overline{K_{oj}} \cap \sigma(T) = \{\mu_j\}$ ($\overline{K_{oj}}$ denotes the closure of K_{oj}).

Let us investigate the Cesaro sums

$$S_{jn} = \frac{1}{n} \sum_{m=1}^{n} m^{-q_j+1} \mu_j^{-m} T^m , \quad n = 1, 2, \dots$$

Theorem 1. If the inequalities (1) hold for the eigenvalues $\mu_1, \dots, \mu_p$ of the operator $T \in X_1$, and $q_j \geqq q_r$ for $r = 1, \dots, p$, then we have in the norm of the space X_1

$$\lim_{n \to \infty} S_{jn} = \frac{\mu_j^{-q_j+1}}{(q_j - 1)!} B_{j\,q_j} ,$$

where the rapidity of the convergence is given by the estimation of the rest

$$\left\| S_{jn} - \frac{\mu_j^{-q_j+1}}{(q_j - 1)!} B_{j\,q_j} \right\|_{X_1} \leqq O\,(n^{-1} \log n) .$$

Corollary: If the operator $T \in X_1$ has a spectrum lying in a unit circle $|\lambda| \leqq 1$, on the boundary of which lies a finit number of simple isolated poles of the resolvent $R(\lambda, T)$, and if $\mu_1 = 1$ is one of the eigenvalues, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} T^m = B_{11} ,$$

where B_{11} is the projection corresponding to the value $\mu_1 = 1$.

Let us suppose that $x_j^{(o)} \in X$ is such a vector that

$$B_{j1}\, x^{(o)} \neq o,$$

so that an index s, $1 \leqq s \leqq q_j$ exists for which

(2) $$B_{js}\, x^{(o)} \neq o, \quad B_{js+1}\, x^{(o)} = o ,$$

where o is a zero vector in X. The vector

$\tilde{x}_{oj} = B_{js}\, x_j^{(o)}$ is evidently an eigenvector of operator T corresponding to μ_j.

Supposing that we know the eigenvalues $\mu_1, \dots, \mu_p$, we can construct the eigenvectors corresponding to some of the eigenvalues with the hekp of the Cesaro iterations.

<u>Theorem 2</u>. Let (1) hold for the eigenvalues $\mu_1, \dots, \mu_p$ of the operator $T \in X_1$, let $q_j \geqq q_r$ for $r = 1, \dots, p$ and let $x_j^{(o)}$ be such a vector that (2) holds. Then we have in the norm of space X

$$\lim_{n \to \infty} S_{jn}\, x_j^{(o)} = x_{oj} ,$$

where

$$x_{oj} = \frac{\mu_j^{-s+1}}{(s-1)!} B_{js}\, x_j^{(o)}$$

is an eigenvector of the operator T, corresponding to the eigenvalue μ_j.

The mentioned eigenvalue can be considered to be known, if we know that they are the roots of a certain known algebraic equation. This is the case for instance of stochastic cyclical kernels [2], page 152 and stochastic matrices [1], chapter XIII. In these cases the eigenvalues of interest lie on a unit circle and are the roots of a binomial equation

$$\lambda^d = 1$$

where d is the index of imprimitivity [1], page 345.

References

[1] F.R.GANTMACHER, Těorija matric, Moskva 1953.

[2] T.A. SARYMSAKOV, Osnovy těorii processov Markova, Moskva 1954.

[3] A.E.TAYLOR, Spectral theory of closed distributive operators. Acta Math. 84,1951,189-223.