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FURTHER APPLICATIONS OF ITERATIONS OF LINEAR BOUNDED OPERATORS IN NOT SELF - ADJOINT EIGENVALUE PROBLEMS

(Summary of author´s results)

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1. Introduction. Definitions and designations.

In this paper the results published in [6] are extended to the case of dominant eigenvalues which are multiple poles of the resolvents of linear bounded operators. Besides this a general schedule for the construction of Kellog´s iterations is given, which generalises or contains many of the iteration schedules used in specific Banach or Hilbert spaces. It is to be remarked that for made the conclusions in [6] to be valid, it is necessary to add to the conditions given in [6], that the dominant eigenvalue μ_0 must be positive. Besides that under the conditions listed in [6], it cannot be asserted that $BX = Y$ is a closed subspace in X. To ensure the validity of theorems 5 and 7 of [6] it is sufficient to demand that the operator B should be an open transformation mapping X into BX.

Proofs of the theorems of paper [6] and this paper will be published in the Czech. Math. Journ.

Let X be a complex Banach space, X^* its adjoint space of continuous linear forms. We will denote elements of the space X by small Roman characters, elements of the space X^* by the same characters with asteresks. We denote the null-vector of both these spaces by the symbol o. Let T be a linear bounded operator mapping X into itself. The set of linear bounded operators mapping X into itself forms a Banach space, which we will denote X_1. We will distinguish norms in the spaces X, X^*, X_1, by the corresponding index to the norm sign, i.e. for $x \in X$, $x^* \in X^*$ we write $\|x\|_X$, $\|x^*\|_{X^*}$ and for $T \in X_1$:

$$\|T\|_{X_1} = \sup_{\|x\|_X \leq 1} \|Tx\|_X .$$

We will drop the indices in case when no misunderstandings can arise. We denote the null and indentity operators by the symbols Θ, I . Let Π be an open complex plane. We denote the spectrum of the operator T by the symbol $\sigma(T)$.

Let $T \in X_1$ and let $R(\lambda, T)$ be the resolvent of the operator in the point $\lambda \in \Pi$. Let G be an open bounded part of complex plane; let the boundary of the set G consist of a finite number of disjunct rectifi - cable Jordan curves and let $G \supset \sigma(T)$.

Then ([8]) we have for every polynom f :

$$f(T) = \frac{1}{2\pi i} \int_C f(\lambda) R(\lambda, T) d\lambda ,$$

where C is boundary of the set G , oriented in the e- vident way. We will say, that the operator T has the property R_q in the point $\mu_0 \in \sigma(T)$, if μ_0 is an isolated pole of the resolvent $R(\lambda, T)$.

If the operator T has the property R_q in the point μ_0 , the resolvent $R(\lambda, T)$ can be developed into Laurent series in the neighborhood of the point μ_0 ([8]) :

$$R(\lambda, T) = \sum_{k=0}^{\infty} (\lambda - \mu_0)^k T_k + \sum_{k=1}^{q} (\lambda - \mu_0)^{-k} B_k ,$$

where

(1) $B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T) d\lambda$, $B_{k+1} = (T - \mu_0 I) B_k$, $k = 0, 1, \dots, q-1$,

where C_0 is such a positively oriented circle with the centre μ_0 that no other point of the spectrum $\sigma(T)$ ex- cept μ_0 lies on or in the circle C_0 .

If f is a polynom and if the operator T has the property R_q in the point μ_0 , we put

$$H[\mu_0 T, f(\lambda)] = \frac{1}{2\pi i}\int_{C_0} f(\lambda) R(\lambda, T)\, d\lambda =$$

$$= \sum_{k=1}^{q} \frac{f^{(k-1)}(\mu_0)}{(k-1)!} B_k ,$$

where C_0 has the sense defined above.

Especially we put

$$H_m[\mu_0, T] = H[\mu_0, T, (\frac{\lambda}{\mu_0})^m], \quad m \geq 0$$

<u>Definition</u>: We will call the point $\mu_0 \in \sigma(T)$ the dominant point of the spectrum of the operator T, if

$$|\lambda| < |\mu_0|$$

for any point $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$.

2. Iterations of linear bounded operators and iteration processes.

Let the following assumptions be fulfilled in all the statements of this paragraph, if nothing else is asserted.

1) The operator T is a linear bounded operator mapping the space X into itself.

2) The value μ_0 is a pole of the order q of the re - solvent $R(\lambda, T)$ $(0 \leq q < +\infty)$.

3) The value μ_0 is the dominant point of the spectrum of the operator T.

The proof of the convergence of Kellog´s iteration process is based on the following lemmas.

<u>Lemma 1</u>. In the norm of the space X_1 we have:

$$\lim_{m\to\infty} m^{-q+1} H_m[\mu_0, T] = \frac{\mu_0}{(q-1)!} B_q .$$

<u>Lemma 2</u>. For m large enough we have

$$\| H_m[\mu_0, T] - \mu_0^{-m} T^m \|_{X_1} \leq c_1 \left(\frac{\mu}{|\mu_0|}\right)^m$$

where c_1 is independent on m and μ is the radius of a circle C_1 which contains the whole spectrum except point μ_0.

Let $x^{(0)} \in X$ be a definite fixed vector, for which

(2) $$B_q x^{(0)} \neq 0, \quad B_{q+1} x^{(0)} = 0$$

where s is a definite index, $1 \leq s \leq q$ and B_s, B_{s+1} are defined in (1), $B_{q+1} = \Theta$.

Lemma 3. Let (2) hold for the vector $x^{(0)} \in X$. In the norm of the space X we then have:

$$\lim_{m \to \infty} m^{-s+1} \mu_0^{-m} T^m x^{(0)} = \frac{\mu_0^{-s+1}}{(s-1)!} B_s x^{(0)} .$$

Theorem 1. In the norm of the space X_1, we have

$$\lim_{m \to \infty} m^{-q+1} \mu_0^{-m} T^m = \frac{\mu_0^{-q+1}}{(q-1)!} B_q .$$

Let $\{x^*_m\}$, $\{y^*_m\}$, $\{z^*_m\}$ be sequences of linear forms mapping X into Π. Let such forms $x^* \in X^*$, $y^* \in X^*$ exist, that

$$x^*(x) = \lim_{m \to \infty} x^*_m(x),$$

(3)

$$y^*(x) = \lim_{m \to \infty} y^*_m(x) = \lim_{m \to \infty} z^*_m(x)$$

for every vector $x \in X$.

If the inequalities (2) hold for the vector $x^{(0)} \in X$, and when

(4) $$x^*(B_s x^{(0)}) \neq 0, \quad y^*(B_s x^{(0)}) \neq 0,$$

then we put

(5) $$x_0 = \frac{B_s x^{(0)}}{x^*(B_s x^{(0)})} .$$

Kellog´s iterations are constructed according to the following formulas:

(6) $$x^{(m)} = T x^{(m-1)}, \quad x_{(m)} = \frac{x^{(m)}}{x^*_m(x^{(m)})},$$

(7) $$\mu_{(m)} = \frac{z^*_m(x^{(m+1)})}{y^*_m(x^{(m)})} .$$

Theorem 2. Let (3) hold for the forms x^*_m, y^*_m, z^*_m, x^*, y^*. Let $x^{(0)}$ be such a vector, that (2) and (4) hold.

Then $$\lim_{m \to \infty} x_{(m)} = x_0$$

holds for the sequence (6) in the norm of the space X and

$$\lim_{m \to \infty} \mu_{(m)} = \mu_0$$

for the numerical sequence (7) where x_0 is the eigen-vector of the operator T, corresponding to the eigen-value μ_0.

Remark. If we take sequences of continuous functionals $\tilde{y}_m$, $\tilde{z}_m$ such that for every $x \in X$, $y \in X$ hold

$$\tilde{y}_m(\lambda x) = |\lambda| \tilde{y}_m(x), \quad \tilde{z}_m(\lambda x) = |\lambda| \tilde{z}_m(x),$$

$$|\tilde{y}_m(x) - \tilde{y}_m(y)| + |\tilde{z}_m(x) - \tilde{z}_m(y)| \leqq c\,\|x - y\|,$$

where c is independent on m and for such exists the functional $\tilde{y}$ that

$$\lim_{m \to \infty} \tilde{y}_m(x) = \lim_{m \to \infty} \tilde{z}_m(x) = \tilde{y}(x)$$

for every vector $x \in X$, instead of the sequence of linear forms $\{y^*_m\}, \{z^*_m\}$ in formula (7), then under assumptions analogous to those of Theorem 2, when $\mu_0 > 0$ we obtain the following equality:

$$\lim_{m \to \infty} \frac{\tilde{z}_m(x^{(m+1)})}{\tilde{y}_m(x^{(m)})} = \mu_0 .$$

Specifically for $\tilde{x}_m(x) = \tilde{y}_m(x) = \tilde{z}_m(x) = \|x\|_X$ we obtain the classical Kellog´s iteration sequence $\{\|x^{(m+1)}\| / \|x^{(m)}\|\}$ and the formula

$$\lim_{m \to \infty} \frac{\|x^{(m+1)}\|}{\|x^{(m)}\|} = \mu_0 .$$

If we choose the sequences of forms $\{y^*_m\}$, $\{z^*_m\}$, $\{x^*_m\}$ or functionals $\{\tilde{y}_m\}$, $\{\tilde{z}_m\}$, $\{\tilde{x}_m\}$ in a specific way, we obtain some well known iteration processes ([1], [2], [3], [5], [7], [9]).

In [2] and [5] the authors give iteration formulas for the constructions of eigenvectors, which differ from Kellog´s original formula (6). Both iteration processes, [2] and [5] can be summed up in one general schedule:

$$y_{(m+1)} = \frac{1}{\mu_{(m)}} T y_{(m)}, \tag{8}$$

where $\mu_{(m)}$ is defined as

$$\mu_{(m)} = \frac{z^*_m(T y_{(m)})}{y^*_m(y_{(m)})} \tag{9}$$

i.e. by the formula (7).

Theorem 3. Let operator T have the property R_1 in the point μ_0. Let the forms x^*_m, y^*_m, z^*_m, y^*, x^*, fulfilling the conditions of Theorem 2, have the following property: for large enough m the inequality

(10) $$|x^*_m(x) - x^*(x)| + |y^*_m(x) - y^*(x)| + |z^*_m(x) - y^*(x)| \leq c(x)\, m^{-1-\delta}, \quad \delta > 0$$

holds. Let (2) and (5) hold for the vector $x^{(0)}$ and let for $\mu_{(m)}$ in (9) $\mu_{(m)} \neq 0$ for $m = 0, 1, \ldots$.

Then

(11) $$\lim_{m \to \infty} y_{(m)} = y_0$$

holds in the norm of the space X, where y_0 is the eigenvector of the operator T corresponding to the eigenvalue μ_0.

Theorem 4.

Let $z^*_m(x) = y^*_{m+1}(x)$ for $m = 0, 1, \ldots$ and let y^*_m together with y^* satisfy the conditions of Theorem 2. Let (2) and (5) hold for the vector $x^{(0)}$. Let $y^*_m(x^{(m)}) \neq 0$ for $m = 0, 1, \ldots$.

Then (11) holds for the sequence (8) with $\mu_{(m)}$ defined by (9) and the vector y_0 is the eigenvector of the operator T corresponding to the value μ_0.

For the applicability of Kellog's iterations (6) and (7) it is sufficient if the order q of the value μ_0 is finite. It is not necessary to know q explicitely. If we do know q we can use this fact in calculations (see Theorem 5).

Lemma 4.

In the norm of the space X_1 we have

$$\lim_{m \to \infty} \mu_0^{-m} T^m (T - \mu_0 I)^q = \theta.$$

Theorem 5.

If the conditions of Theorem 2 are fulfilled then

$$\lim_{m \to \infty} \mu_0^{-m} \Big\{ y^*_m(x^{(m+q)}) - \binom{q}{1} \mu_0\, y^*_m(x^{(m+q-1)}) + \cdots + (-1)^q \mu_0^q\, y^*_m(x^{(m)}) \Big\} = 0.$$

3. Modified iteration processes.

The iterations investigated in the previous paragraph can also be applied to the construction of characteristic values and eigenvectors of equations of the type

(12) $$Lx = \lambda B x,$$

where L and B are linear operators. Just as in paragraph 2 we will list the assumptions about the operators L and B separately, so as not to repeat their formulations in most of the statements.

<u>Assumptions.</u> (A) Operator B is a bounded linear operator mapping X into itself.

(B) A bounded iverse operator L^{-1} mapping X into $\mathfrak{D}(L)$ where $\mathfrak{D}(L)$ is the domain of the operator L, exists for the bounded operator L.

(C) The operator $T = L^{-1}B$ fulfils the assumptions 1. - 3. of paragraph 2. with $\mu_0 = \lambda_0^{-1}$.

The following modified Kellog's iterations are analogous to Kellog's iterations (6) and (7):

$$v^{(m)} = B u^{(m)}, \quad L u^{(m+1)} = v^{(m)}, \quad u^{(0)} = x^{(0)},$$

$$u_{(m)} = \frac{u^{(m)}}{x^*_m\left(u^{(m)}\right)},$$

$$\lambda_{(m)} = \frac{y^*_m\left(u^{(m)}\right)}{z^*_m\left(u^{(m+1)}\right)},$$

where $\{y^*_m\}$, $\{z^*_m\}$, $\{x^*_m\}$ are sequences of linear forms defined together with the forms x^*, y^* in Theorem 2.

<u>Theorem 6.</u>

Let the forms x^*_m, y^*_m, z^*_m, y^*, x^* and vector $x^{(0)}$ fulfil the conditions of Theorem 2. Then

$$\lim_{m\to\infty} \lambda_{(m)} = \lambda_0,$$
$$\lim_{m\to\infty} u_{(m)} = u_0$$

in the norm of the space X, where u_0 is the eigenvector of the equation (12), corresponding to the characteristic value λ_0.

The following iterations are analogous to the iteration process (8):

$$(13) \quad u^{(m)} = B w_{(m)}, \quad L w_{m+1} = u^{(m)}, \quad w_{(m+1)} = \lambda_{(m)} w_{m+1},$$

$$w_{(0)} = x^{(0)},$$

where $\lambda_{(m)}$ are given by

$$\lambda_{(m)} = \frac{y_m^*\,(u^{(m)})}{z_m^*\,(L^{-1} B w_{(m)})}.$$

As a special case, when X is a Hilbert space, correctly choosing the forms x_m^*, y_m^*, z_m^*, we obtain some known modified iteration processes [2], [9].

Theorem 7.

Let the operator $T = L^{-1} B$ have the property R_1 in the point λ_0^{-1}. Let the forms y_m^*, z_m^*, y^*, x_m^*, x^* fulfil the conditions of Theorem 3. Let (2) and (5) hold for the vector $x^{(0)}$.

Then the following holds in the norm of the space X:

$$\lim_{m \to \infty} w_{(m)} = w_0, \quad \lim_{m \to \infty} \lambda_{(m)} = \lambda_0,$$

where w_0 is the eigenvector of the equation (12) corresponding to the characteristic value λ_0.

Theorem 8.

Let the forms y_m^*, y^* and the vector $x^{(0)}$ fulfil the conditions of Theorem 2.

Then

$$\lim_{m \to \infty} \lambda_0^m \left\{ \lambda_0^q\, y_m^*\,(u^{(m+q)}) + \binom{q}{1} \lambda_0^{q-1}\, y_m^*\,(u^{(m+q-1)}) + \right.$$

$$\left. + \dots + (-1)^q\, y_m^*\,(u^{(m)}) \right\} = 0.$$

4. Modified iterations in a reduced part of space.

Let the conditions of paragraph 3 hold in this paragraph, only instead of (C) let us have:

(C′) Operator $T = BL^{-1}$ fulfils the conditions 1.– 3. of paragraph 2 with $\mu_0 = \lambda_0^{+1}$.

Lemma 5.

Let $y \in X$ be an eigenvector of the operator BL^{-1}

corresponding to the characteristic value λ_0. Then the vector $x = L^{-1} y$ is an eigenvector of the equation (12) corresponding to the same value λ_0.

Kellog's iterations for the operator BL^{-1} can be obtained directly for equation (12). Thus it is not necessary to construct the operator BL^{-1} and its higher powers. We thus obtain the following iteration process:

$$
(14) \quad \begin{cases} L u^{(m+1)} = v^{(m)}, \; v^{(m+1)} = B u^{(m+1)}, \; v^{(0)} = B x^{(0)}, \\[2ex] u_{(m)} = \dfrac{u^{(m)}}{x^*_{m-1}\left(v^{(m-1)}\right)}, \\[2ex] \lambda_{(m)} = \dfrac{y^*_m\left(v^{(m)}\right)}{z^*_m\left(v^{(m+1)}\right)}; \end{cases}
$$

where $\{x^*_m\}$, $\{y^*_m\}$, $\{z^*_m\}$ are sequences of linear forms defined together with the forms x^*, y^* in Theorem 2.

<u>Theorem 9.</u>

Let the forms x^*_m, y^*_m, z^*_m, y^*, x^*_m, x^* fulfil the conditions of Theorem 2 for the operator $T = BL^{-1}$.

Let

(15) $B_s y^{(0)} \neq 0, \; B_{s+1} y^{(0)} = 0, \; y^*(B_s y^{(0)}) \neq 0, \; x^*(B_s y^{(0)}) \neq 0$

hold for the vector

$$y^{(0)} = B x^{(0)}$$

where s is a certain index, $1 \leq s \leq q \; (B_{q+1} = \Theta)$.

Then
$$\lim_{m \to \infty} \| u_{(m)} - u_0 \|_X = 0, \quad \lim_{m \to \infty} \lambda_{(m)} = \lambda_0$$

holds for the sequences (14), where u_0 is the eigenvector of the equation (12) corresponding to the characteristic value λ_0.

The following iterations are analogous to iterations (13):

$$
(16)\quad \begin{cases} L z_{(m)} = z^{(m)}, \quad z_{m+1} = B z_{(m)}, \quad z^{(m+1)} = \lambda_{(m)} z_{m+1}, \\ z^{(0)} = B x^{(0)}, \\ \lambda_{(m)} = \dfrac{y_m^*\,(z^{(m)})}{z_m^*\,(z^{(m+1)})}. \end{cases}
$$

Theorem 10.

Let the forms y_m^*, z_m^*, y^*, x_m^*, x^* fulfil the conditions of Theorem 2. Let (15) hold for the vector $y^{(0)} = bx^{(0)}$.

Let the operator $T = BL^{-1}$ have the property R_1 in the point $\mu_0 = \lambda_0^{-1}$. Then

$$\lim_{m \to \infty} z_{(m)} = z_0$$

in the norm of the space X; the vector z_0 here is the eigenvector of equation (12) corresponding to the value which is the limit of the sequence (16):

$$\lim_{m \to \infty} \lambda_{(m)} = \lambda_0 .$$

Theorem 11.

Let the conditions of Theorem 9 be fulfilled for the operator $T = BL^{-1}$. Then the following holds for the sequences defined in (13):

$$\lim_{m \to \infty} \lambda_0^m \left\{ \lambda_0^q\, y_m^*\,(u^{(m+q)}) - \binom{q}{1} \lambda_0^{q-1}\, y_m^*\,(u^{(m+q-1)}) + \right.$$

$$\left. + \dots + (-1)^q\, y_m^*\,(u^{(m)}) \right\} = 0.$$

Remark.

If the operator T has a finite number of values $\mu_1, \dots, \mu_p$ such that $|\mu_s| = |\mu_1|$ for $s = 1, \dots, p$ and the operator T has the property R_{q_s} in these points, one can also use Kellog´s itera - tions. Instead of the operator T one investigates the operator $T - \nu_s I$, where ν_s is an adequate complex number.

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