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BIORTHOGONAL SYSTEMS AND BASES IN BANACH SPACE
(Preliminary Note)

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In all this paper B denotes a Banach space, $\{x_n, f_n\}$ a biorthogonal system in B, (i.e. $x_n \in B, f_n \in B^*$, $f_i(x_j) = 0$, for $i \neq j$ and $f_n(x_n) = 1$ for $n = 1, 2, \dots$) such that $\{x_n\}$ is fundamental in B (i.e. $\overline{[\{x_n\}]} = B$). The sequence $\{x_n\}$ is the base B, if $\sum_{n=1}^{\infty} f_n(x) x_n = x$.

THEOREM 1. [14] : If $\sup_{k=1}^{\infty} \| \sum_{n=1}^{k} f_n(x) x_n \| < +\infty \Rightarrow x =$

$$= \sum_{n=1}^{\infty} f_n(x) x_n ,$$

then $\{x_n\}$ is the base B .

The consequence of results of N.S.Foguel [5] is, that in the case that $\{x_n\}$ is no base of B , no two of the sets

$$E_1 = \mathop{\mathcal{E}}_{x \in B} \left(x = \sum_{n=1}^{\infty} f_n(x) x_n\right), \quad E_2 = \mathop{\mathcal{E}}_{x \in B} \left(\sup_{k=1}^{\infty} \left\| \sum_{n=1}^{k} f_n(x) x_n \right\| < +\infty\right)$$

$$E_3 = \mathop{\mathcal{E}}_{x \in B} \left(\sup_{k=1}^{\infty} \left\| \sum_{n=1}^{k} f_n(x) x_n \right\| = +\infty\right), \quad E_4 = \mathop{\mathcal{E}}_{x \in B} \left(\lim_{k \to \infty} \left\| \sum_{n=1}^{k} f_n(x) x_n \right\| = +\infty\right)$$

are coincident. N.S.Foguel has proved that E_2 and E_4 are the first category in B in the case mentioned above. The generalization of this result gives the following theorem.

THEOREM 2. [14] : If $\sum_{n=1}^{\infty} c_i y_i = 0$ only if $c_i = 0$ for $i = 1, 2, \dots$, then the set of those $x \in B$, which can be represented in the form $x = \sum_{n=1}^{\infty} c_i y_i$, is the first category in itself.

An infinite semimatrix $T = (t_{ij})_{\substack{i=1,2,\dots \\ j=1,\dots,i}}$ is called quasi - consistent, if $\lim_{i \to \infty} \sum_{j=k}^{i} t_{ij} = 1$ for all $k = 1, 2, \dots$.

This conception is more general than the so called Toeplitz' matrix usually employed for summabilitation of infinite series. If there exists $\sum_{j=1}^{\infty} t_{ij} y_j$ for all i and $\lim_{i \to \infty} \sum_{j=1}^{\infty} t_{ij} y_j = y$, it is said that the sequence $\{y_n\}$ converges to y according to the semimatrix T

and we shall use the denotation $(T)\lim\limits_{n\to\infty} y_n = y$. There follows:

THEOREM 3. [15] : If T is a quasi-consistent semimatrix and $(T)\sum\limits_{n=1}^{\infty} c_n x_n = x$ for any numerical sequ^nce $\{c_n\}$, then $c_n = f_n(x)$ for $n = 1, 2, \dots$.

If $\{t_n\}$ is total in B ($f_n(x) = 0$ for $n=1,2,\dots \Rightarrow x=0$) and if for the quasi-consistent semimatrix T is $(T)\sum\limits_{n=1}^{\infty} f_n(x) x_n = y$, then $y = x$.

THEOREM 4. [15] : If for $x \in B$ only a finite number of elements of the sequence $\{f_n(x)\}$ is equal to zero, then exists a quasi-consistent semimatrix T such that $(T)\sum\limits_{n=1}^{\infty} f_n(x) x_n = x$.

An example may easily be found where it can be shown that the assumption of the finite number of zero terms in the expansion of the element x from theorem 4 is substantial. It is by no means clear whether this assumption may be omitted, when the totality of the sequence $\{f_n\}$ is required.

Let us denote H_r the Banach space, the elements of which are functions, holomorphic for $|z| < r$ and continuous for $|z| \leqq 1$, with the usual algebraic operations and the norm $\|f\| = \max\limits_{|z| \leqq r} |f(z)|$.

Let $\{r_n\}$ be a decreasing sequence of real number, converging to 1. Let η be a strict inductive limit [4] of the spaces H_{r_n} . It may be shown that the powers $1, z, z^2, \dots$, form a basis of the local convex topological space η (which is of course nonmetrizable), but no basis of the Banach space H_1 . The question of the existence of a base in H_1 remains unreplied at present. The method of finding this base by means of base in the space of harmonic functions is ineffectual [13] .

Further there may be proved:

THEOREM 5 [17] : Let B be a separable Banach space. Then there exists a subspace of the space H_1 isomorfic (algebraical and topological) with B .

Therefore and from the generalized James' result 3 there follows

THEOREM 6 [17] : There exists no unconditional x) basis in the space H_1 .

Let us denote $s_n(x) = \sum_{j=1}^{n} f_j(x) x_j$. It is useful to investigate such sequences where either $s_n(x)$ is the best approximation of the element x by means of the elements from $[\{x_i\}_{i=1}^{n}]$, or the residue $x - s_n(x)$ is the best approximation of the element x from $\overline{[\{x_i\}_{i=n+1}^{\infty}]}$.

DEFINITION [16] : Let $\{x_n\}$ be a fundamental sequence in B , composed exclusively of nonzero elements. We then say that

1) $\{x_n\}$ is approximating from the right, if for any numbers $t_1, \dots, t_{p+1}$ there is $\|\sum_{i=1}^{p} t_i x_i\| \leq \|\sum_{i=1}^{p+1} t_i x_i\|$,

2) $\{x_n\}$ is well approximating from the right, if for any numbers $t_1, \dots, t_p$ and $t_{p+1} \neq 0$ there is $\|\sum_{i=1}^{p} t_i x_i\| < \|\sum_{i=1}^{p+1} t_i x_i\|$

3) $\{x_n\}$ is approximating from the left, if for any numbers $t_n, \dots, t_s$ there is $\|\sum_{i=n+1}^{s} t_i x_i\| \leq \|\sum_{i=n}^{s} t_i x_i\|$,

4) $\{x_n\}$ is well approximating from the left, if for any numbers $t_{n+1}, \dots, t_s$, $n < s$ and $t_n \neq 0$ there is $\|\sum_{i=n+1}^{s} t_i x_i\| < \|\sum_{i=n}^{s} t_i x_i\|$

5) $\{x_n\}$ is approximating, if it is approximating simultaneously from the left and right,

6) $\{x_n\}$ is well approximating, if it is well approximating simultaneously from the left and right.

THEOREM 6 [16] : A sequence $\{x_n\}$ non-zero elements of B is approximating from the right, if and only if there exists a sequence $\{f_n\}$ such that $\{x_n, f_n\}$ is a biorthogonal system and $\|s_n\| = 1$ for $n = 1, 2, \dots$.

x) It is said that $\{x_n\}$ is an unconditional base B , if for each $x \in B$ and each order $\{n_i\}$ of natural numbers there is $\sum_{i=1}^{\infty} f_{n_i}(x) x_{n_i} = x$

A sequence $\{x_n\}$ non-zero elements of B is approximating from the left, if and only if there exists a sequence $\{f_n\}$ such that $\{x_n, f_n\}$ is a biorthogonal system and $\|I - s_n\| = 1$, where I is an identical mapping of B

THEOREM 7 [16] : If $\{x_n\}$ is a sequence of non-zero elements of B, approximating from the left or right respectively, then $\{x_n\}$ is a base of B.

THEOREM 8 [16] : If $\{x_n\}$ is a base of B, then exists an equivalent norm $\|\!|\ \|\!|$, by means of which $\{x_n\}$ is the well approximating sequence of $(B, \|\!|\ \|\!|)$.

THEOREM 9 [16] : Let $\{x_n\}$ be a sequence of non-zero elements of B. Then

1) $\{x_n\}$ is approximating from the right, if and only if $\|s_n(x)\| = \rho(x, [\{x_i\}_{i=n+1}^{\infty}])$.

2) $\{x_n\}$ is well approximating from the right, if and only if, furthermore, $\|x - y\| < \rho(x, [\{x_i\}_{i=n+1}^{\infty}])$ is valid, as soon as $y \in [\{x_i\}_{i=n+1}^{\infty}]$, $y \neq x - s_n(x)$.

3) $\{x_n\}$ is approximating from the left, if and only if $\|x - s_n(x)\| = \rho(x, [\{x_i\}_{i=1}^{n}])$.

4) $\{x_n\}$ is well approximating from the left, if and only if, furthermore, $\|x - y\| < \rho(x, [\{x_i\}_{i=1}^{n}])$ is valid, as soon as $y \in [\{x_i\}_{i=1}^{n}]$, $y \neq s_n(x)$

From the following general theorem there follows that there need not exist a base approximating from the left in each separable Banach space.

THEOREM 10 [16] : Let $\{x_n\}$ be a base of the space $\mathcal{C}\langle 0,1\rangle$ and n a natural number. Then there exist two elements $h \in \mathcal{C}\langle 0,1\rangle$, $h' \in [\{x_i\}_{i=1}^{n}]$ such that $\|h - s_n(h)\| > \|h - h'\|$.

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