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REMARKS ON CHARACTERS AND PSEUDOCHARACTERS

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The character $\chi(M,S)$ of a set M in a topological space S is defined as the least cardinal of a base around M ; for the definition of the pseudocharacter $\psi(M,S)$, a pseudobase replaces a base. In a topological space S, a base (pseudobase) around a set $M \subset S$ is a collection $\mathcal{U}$ of neighborhoods of M such that any neighborhood of M contains some $U \in \mathcal{U}$ (the intersection of all $U \in \mathcal{U}$ is equal to M). These notions have been introduced, essentially by P. Alexandrov and P. Urysohn, Mémoire sur les espaces compacts, 1929. Varions important results concerning the existence of spaces with prescribed characters and pseudocharacters of points are due to B. Pospíšil (e.g. Čas. pěst. mat. fys. 67 (1938), 249-255). Little seems to be known concerning characters of sets ; from the few known results, recall the following one (J. Novák, Čas. pěst. mat. fys. 66 (1937), 206-209): if S is metrizable, $M \subset S$, then $\chi(M,S) \leqq \aleph_0$ if and only if $M - \operatorname{Int} M$ is compact.

The present remarks, arisen in connection with the problem (considered by J. Novák) of the equality $\chi = \psi$ for the set of rational numbers (in the real line), contain several simple results concerning characters and pseudocharacters of sets as well as some related notions. It is to be noted that the equality $\aleph_1 = 2^{\aleph_0}$ is not assumed. We consider completely regular topological spaces (called simply " spaces ") only. The terminology of J. Kelley, General Topology, 1955, is used (with slight differences). The prover of a set M is denoted card M ; the letter S always denotes a space.

1.

1.1. Definition. A k-base (a k-pseudobase) of is a collection $\mathcal{O}\!\iota$ of compact subsets such that every compact

$K \subset S$ (respectively, every $x \in S$) is contained in some $A \in \mathfrak{A}$. The k-character of S, denoted $k\chi(S)$ (respecti-vely, k-pseudocharacter, denoted $k\psi(S)$) is the least cardinal of a k-base (k-pseudobase) of S. Clearly, every k-base of S contains a k-base $\mathfrak{A}$ with card $\mathfrak{A} = k\chi(S)$ and a k-pseudobase $\mathfrak{B}$ with card $\mathfrak{B} = k\psi(S)$.

1.2. If S is compact, $A \cup B = S$, $A \cap B = \emptyset$, then $\chi(A, S) = k\chi(B)$, $\psi(A, S) = k(B)$.

1.3. If S_1, S_2 are spaces, and g is a continuous mapping of S_1 onto S_2 such that $g^{-1}(K)$ is compact whenever $K \subset S_2$ is so, then $k\chi(S_1) = k\chi(S_2)$, $k\psi(S_1) = k\psi(S_2)$.

1.4. Theorem. Let S_1, S_2 be locally compact, $M_1 \subset S_1$, $\overline{M_1} = S_1$, $M_2 \subset S_2$, $\overline{M_2} = S_2$; let M_1, M_2 be homeomorphic. Then $\chi(M_1, S_1) = \chi(M_2, S_2)$, $\psi(M_1, S_1) = \psi(M_2, S_2)$.

Proof. Consider only χ, the proof for ψ being quite analogous. Suppose first that S_1 is compact. Let f be a continuous mapping of the Čech - Stone compactification βM_1 onto S_1, $f(x) = x$ for $x \in M_1$. Then $f(\beta M_1 - M_1) = S_1 - M_1$, and the restriction g of f to $\beta M_1 - M_1$ satisfies the conditions from 1.3. Hence $k\chi(\beta M_1 - M_1) = k\chi(S_1 - M_1)$ and therefore, by 1.2, $\chi(M_1, \beta M_1) = \chi(M_1, S_1)$. This implies the validity of the theorem for compact S_1, S_2. If S_i are locally compact, choose compact $T_i \supset S_i$ with $\overline{S_i} = T$. Then S_i are open in T_i and therefore $\chi(M_i, S_i) = \chi(M_i, T_i)$ from which the theorem follows.

1.5. By 1.4, for a given S, the cardinals $\chi(S, K)$, $\psi(S, K)$ where $S \subset K$, $\overline{S} = K$, K is compact, do not depend on K; they will be denoted $e\chi(S)$, $e\psi(S)$ and called external character (pseudocharacter) of S. Two spaces S_1, S_2 will be called associated if the these is a compact space K and subspaces $S_i' \subset K$ homeomorphic with S_i such that $S_1' \cup S_2' = K$, $S_1' \cap S_2' = \emptyset$,

$\overline{S_1'} = \overline{S_2'} = K$. Clearly, if S_1, S_2 are associated, then $e\chi(S_1) = k\chi(S_2)$, $e\psi(S_1) = k\psi(S_2)$.

Clearly, $\chi(S, R) \leqq e\chi(S)$ if S is dense in the space R; if not, it may happen e. g. that $\chi(S, R) > \aleph_0$, $e\chi(S) = 1$, $\chi(\bar{S}, R) = \aleph_0$.

1.6. If S is locally compact σ-compact, then $k\chi(S) \leqq \aleph_0$.

1.7. If $k\chi(S) \leqq \aleph_0$, and $\chi(x, S) \leqq \aleph_0$ for every $x \in S$, then S is locally compact σ-compact.

Proof. Suppose that S is not locally compact at $a \in S$. Let A_n, $n = 1, 2, \ldots$, form a k-base of S; let G_n form a base around a and let $G_1 \supset G_2 \supset \ldots$. Since $G_n - A_n \neq \emptyset$, choose $x_n \in G_n - A_n$, $x_n \neq a$; denote K the set consisting of a and all x_n. Then K is compact, $K - A_n \neq \emptyset$, $n = 1, 2, \ldots$, which is a contradiction.

Remark. It is easy to see that the assumption $\chi(x, S) \leqq \aleph_0$ cannot be omitted.

2.

2.1. If R is an ordered set, let the least cardinal of a cofinal set in R be called cofinality character of R. Let N^N denote the set of all sequences of natural numbers ordered as follows: $\{\xi_n\} \leqq \{\eta_n\}$ if (and only if) $\xi_n \leqq \eta_n$ for every n. The cofinality character of N^N will be denoted b.

It is clear that $\aleph_1 \leqq b \leqq 2^{\aleph_0}$; by the author's knowledge neither of the equalities $\aleph_1 = b$, $b = 2^{\aleph_0}$ has been proved as yet (nor disproved, of course).

Order the set F of all sequences of positive numbers as follows: $\{\xi_n\}$ precedes $\{\eta_n\}$ if (and only if)

$\xi_n \geq \beta_n$ for every n. Evidently, $\mathfrak{b}$ is the cofinality character of F.

2.2. If S is metrizable, $M \subset S$ is σ-compact, then $\chi(M, S) \leq \mathfrak{b}$.

Proof. Let $M = \bigcup_{n=1}^{\infty} K_n$, K_n compact. Let A be cofinal in N^N. Choosing a metric ρ for S, put $G_{n,k} = \{x \in S : \rho(x, K_n) < \frac{1}{k}\}$, and, for any $x = \{\xi_n\} \in N^N$, $U_x = \bigcup_{n=1}^{\infty} G_{n,\xi_n}$.
If H is a neighborhood of M, choose k_n with $G_{n,k_n} \subset H$ and $x \in A$ with $\{k_n\} \leq x$; then $M \subset U_x \subset H$. Hence U_x, $x \in A$, form a base around M.

2.3. Let S be metrizable, $M \subset S$. If $M - \mathrm{Int}\, M$ is not compact, then $\chi(M, S) \geq \mathfrak{b}$.

Proof. There exist (distinct) points $b_n \in M - \mathrm{Int}\, M$ such that $\{b_n\}$ has no cluster point in M. Choose a metric ρ for S and put, for any neighborhood G of M, $\varphi(G) = \{\rho(b_n, S - G)\} \in F$ (see 2.1). Let $\mathcal{U}$ be a base around M. If $\{\xi_n\} \in F$, choose $x_n \in S - M$ with $\rho(x_n, b_n) < \min(\frac{1}{n}, \xi_n)$. Since $H = S - \bigcup(x_n)$ is a neighborhood of M, there is $U \in \mathcal{U}$ with $U \subset H$. Since $\rho(b_n, S - U) \leq$ $\leq \rho(b_n, x_n)$, $\{\xi_n\}$ precedes $\varphi(U)$ in F. Thus $\varphi(U)$; $U \in \mathcal{U}$, form a cofinal set in F.

2.4. Theorem. Let S be metrizable; let $M \subset S$ be σ-compact. Then $\chi(M, S) = \mathfrak{b}$ if and only if $M - \mathrm{Int}\, M$ is not compact.

Remark. For instance, in E_n the character of every non-compact closed set (different from E_n) is $\mathfrak{b}$.

3.

3.1. Definition. A space S will be called a λ-space if there is a transitive relation σ on S and a set A such that the sets $\{x \in S : x \sigma a\}$, $a \in A$, form a

k-base of S.

Clearly, any well ordered space is a λ-space.

Remark. It is easy to prove that S is a λ-space if and only if it satisfies one of the following equivalent conditions : (a) there is a k-base $\mathcal{A}$ such that, for any $A \in \mathcal{A}$, $A - \bigcup_{\substack{X \in \mathcal{A} \\ A - X \neq \emptyset}} X \neq \emptyset$, (b) there is a k-pseudobase $\mathcal{A}$ and a mapping φ of the system $\mathcal{K}$ of all compact $K \subset S$ into S such that $K \in \mathcal{K}$, $A \in \mathcal{A}$, $\varphi(K) \in A$ implies $K \subset A$.

3.2. Theorem. If S is a λ-space, then $k\chi(S) = k\psi(S)$.

Proof. Let σ, A be as in 3.1. Clearly, there is $B \subset A$ with card $B = k\psi(S)$ such that the system $\mathcal{B}$ of all $\{x \in S : x \sigma b\}$, $b \in B$ is a k-pseudobase. It is easy to prove that $\mathcal{B}$ is also a k-base.

3.3. Let $\mathcal{A}$ be a system of compact sets $A \subset S$ such that (1) for any compact $K \subset S$, $K \subset \bigcup_{i=1}^{n} A_i$ for some $A_i \in \mathcal{A}$, (2) if $\mathcal{A}' \subset \mathcal{A}$, $\bigcup_{A \in \mathcal{A}'} A = S$ then $\mathcal{A}' = \mathcal{A}$. Then S is a λ-space.

Proof. By (2), we can choose, for any $A \in \mathcal{A}$, a point $z(A) \in A$ contained in no $X \in \mathcal{A}$, $X \neq A$. Let $\mathcal{A}$ be directed by a relation $\leqq$ in such a way that all $\{X \in \mathcal{A} : X \leqq A\}$, $A \in \mathcal{A}$, are finite. For $x \in S$, $y \in S$ put $x \sigma y$ if (and only if) there are $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{A}$ with $x \in A_1$, $A_1 \leqq A_2$, $y = z(A_2)$. It is easy to see that σ is transitive. If $y = z(A)$, then $\{x \in S : x \sigma y\}$ is equal to $\bigcup_{\substack{X \in \mathcal{A} \\ X \leqq A}} X$, hence compact. Condition (1) implies (since $\mathcal{A}$ is directed) that $\{x \in S : x \sigma y\}$,

$y = x(A)$, $A \in \mathcal{A}$, form a k-base.

3.4. The cartesian product of λ-spaces is a λ-space.

Proof. Let S_ξ , $\xi \in Z$, be λ-spaces, $S = \mathrm{P} S_\xi$. Let σ_ξ , A_ξ be (for S_ξ) as in 3.1. Put $\{x_\xi\} \sigma \{y_\xi\}$ if (and only if) $x_\xi \sigma y_\xi$ for every ξ ; put $A = \mathrm{P} A_\xi$. Then A , ξ possess (for S) properties required in 3.1.

3.5. Theorem. The cartesian product of locally compact paracompact spaces is a λ-space.

Proof. Let S be locally compact paracompact. Then there is a locally finite open cover $\{U_\alpha\}$ such that $\overline{U_\alpha}$ are compact. Clearly, there exists a subcover $\{U_\beta\}$ and points $x_\beta \in U_\beta$ such that no x_β lies in $U_{\beta'}$, $\beta \neq \beta'$. By a well known theorem, there exist open V_β with $x_\beta \in V_\beta$, $\overline{V_\beta} \subset U_\beta$, $\bigcup V_\beta = S$. The collection of all $\overline{V_\beta}$ has properties indicated in 3.3; hence S is a λ-space. Now apply 3.4.

Remark. It is easy to see that $k\chi(S) = k\psi(S)$ for any locally compact S ; neverheless, I do not know whether $k\chi(S) = k\psi(S)$ holds whenever S is a product of locally compact spaces.

3.6. Corollary. Let R donete the space of rational numbers, J that of irrational ones. Then $e\chi(R) = e\psi(R) = k\chi(J) = k\psi(J) = \mathfrak{b}$, $k\psi(R) = e\psi(J) = \aleph_0$.

Proof. By 2.4, $e\chi(R) = \mathfrak{b}$; hence, R and J being associated, $k\chi(J) = \mathfrak{b}$. Since J is homeomorphic to the product of $\aleph_0$ discrete countable spaces, we have, by 3.5, $k\psi(J) = \mathfrak{b}$, hence $e\psi(R) = \mathfrak{b}$.

Remark. The conjecture seems probable that $k\chi(R) = e\chi(J) = \mathfrak{b}$.