Jiří Rosický

Modular, distributive and simple intervals of the lattice of topologies

Archivum Mathematicum, Vol. 11 (1975), No. 2, 105--114

Persistent URL: `http://dml.cz/dmlcz/104847`

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XI: 105—114 1975

MODULAR, DISTRIBUTIVE AND SIMPLE INTERVALS OF THE LATTICE OF TOPOLOGIES

JIŘÍ ROSICKÝ, Brno

(Received February 25, 1974)

This paper deals with intervals of the lattice of topologies with respect to the lattice properties of modularity, distributivity and simplicity. The behaviour of the lattice of topologies and the lattice of T_1-topologies regarding this properties was studied by several authors (see [2], [9], [18]). Intervals of the lattice of topologies were investigated in some papers, too (see [15], [20]). All results presented in this paper are included in the author's thesis and were communicated at the Summer Session on the Theory of Ordered Sets and General Algebra held at Horní Lipová 1972.

All lattice definitions can be found in [19]. We recall some of them. A mapping from a lattice L into a lattice L' is called a $\vee$-homomorphism (complete $\vee$-homomorphism) if it preserves finite suprema (arbitrary suprema). Dually a $\wedge$-homomorphism is defined.A lattice L is called simple if any homomorphism of L onto a lattice L' is either an isomorphism or L' consists of a single element. Let L be a lattice. We put $[a) = \{x \in L\ x \geqq a\}$, $(a] = \{x \in L\ x \leqq a\}$, $[a, b] = \{x \in L\ a \leqq x \leqq b\}$, where a, $b \in L$. The set-theoretic union (intersection) will be denoted by $\cup(\cap)$, a lattice join (meet) by $\vee(\wedge)$. f/A means the restriction of a mapping $f: X \to Y$ onto a subset $A \subseteq X$. A topology $\mathfrak{T}$ on a set E is a system $\mathfrak{T}$ of subsets of a set E closed under finite intersections and arbitrary joins. All necessary topological definitions are given in [3] or [11].

We shall give some results concerning lattices of topologies now. The system $\mathscr{B}(E)$ of all topologies on a set E ordered by the set inclusion forms a complete lattice. The least element is the indiscrete topology $\{\emptyset, E\}$ and the greatest element the discrete topology $exp\ E$. Meets coincide with set theoretic intersections and the join of two topologies $\mathfrak{T}_1$, $\mathfrak{T}_2$ is the topology with the basis $\{V \cap W / V \in \mathfrak{T}_1,\ W \in \mathfrak{T}_2\}$. The lattice $\mathscr{B}(E)$ is atomic and any topology is a join of atoms. Atoms are precisely topologies $\{\emptyset, X, E\}$, where $\emptyset \neq X \nsubseteq E$. $\mathscr{B}(E)$ is dually atomic and any topology is a meet of dual atoms. Dual atoms, which are called utratopologies, are precisely topologies $\mathfrak{G} \cup exp(E - \{a\})$, where $a \in E$ and $\mathfrak{G}$ is an ultrafilter on E different from the principal ultrafilter generated by a. An ultratopology is called free (principal) if $\mathfrak{G}$ is free (principal). A detailed information on lattices of topologies is given in [13].

The closure of a set $X \subseteq E$ in the topology $\mathfrak{T}$ on E we denote by $Cl_T(X)$, the interior by $Int_{\mathfrak{T}}(X)$. The relative topology induced by $\mathfrak{T}$ on X is denoted by $\mathfrak{T}/X$. The symbol $[X)$ means the principal filter on E generated by $X \subseteq E$.

§ 1. CLOSURE OPERATIONS

Among number of generalizations of the usual notion of topology the concept of closure operation plays an important role. A closure operation u on a set E is a mapping $u : exp\, E \to exp\, E$ having the following properties: $1°\ u\emptyset = \emptyset$, $2°X \subseteq uX$ for any $X \subseteq E$, $3°u(X \cup Y) = uX \cup uY$ for any X, $Y \subseteq E$. Closure operations are investigated in details in [5] and the following notions and results are given there. Some further information on more general structures can be found in [17]. If a closure operation u fulfils the condition $4°\ uuX = uX$ for any $X \subseteq E$, it is called a topological closure operation. By this way we get the usual concept of topology. A subset $U \subseteq E$ is a neighbourhood of a point $x \in E$ in a closure operation u on E if $x \notin u(E - U)$. A system of all neighbourhoods of x is denoted by $\mathfrak{N}_u(x)$. $\mathfrak{N}_u(x)$ is a filter. Conversely, for any system $\{\mathfrak{F}_x\}_{x \in E}$ of filters on E with $x \in X$ for any $X \in \mathfrak{F}_x$ and $x \in E$, there exists a closure operation u on E such that $\mathfrak{N}_u(x) = \mathfrak{F}_x$ for any $x \in E$. A closure operation u is a topological closure operation iff for any $x \in E$ and $U \in \mathfrak{N}_u(x)$ there exists $V \in \mathfrak{N}_u(x)$ with $\mathfrak{N}_u(y)$ for any $y \in V$.

The closure operations on E may be ordered in the following way: $u \leqq v$ iff $uX \supseteq vX$ for any $X \subseteq E$. This ordering is dual to that used in [5]. The system $\mathscr{C}(E)$ of all closure operations on E with ordering is a complete lattice. It holds $\mathfrak{N}_{u \vee v}(x) = \mathfrak{N}_u(x) \vee \mathfrak{N}_v(x)$, $\mathfrak{N}_{u \wedge v}(x) = \mathfrak{N}_u(x) \cap \mathfrak{N}(x)$, where in the right sides of equalities are lattice operations in the lattice of filters. Since the lattice of filters is distributive, $\mathscr{C}(E)$ is a distributive lattice for any E (see [4]). Hence the mapping $Cl : \mathscr{B}(E) \to \mathscr{C}(E)$, $Cl(\mathfrak{T}) = Cl_{\mathfrak{T}}$ for any $\mathfrak{T} \in \mathscr{B}(E)$, is a $\vee$-homomorphism and lattices $\mathscr{B}(E)$ and $Cl\mathscr{B}(E)$ are isomorphic. But, in general, Cl is not a homomorphism. We are going to find the conditions under which the mapping Cl restricted to a subinterval of $\mathscr{B}(E)$ is a homomorphism or an isomorphism. Thereby some investigations from [15] will be completed.

1.1. Theorem. *Let E be a set, $\mathfrak{F}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. The following conditions are equivalent:*

(i) *$Cl/[\mathfrak{T}_1, \mathfrak{T}_2] : [\mathfrak{T}_1, \mathfrak{T}_2] \to [Cl_{\mathfrak{T}_1}, Cl_{\mathfrak{T}_2}]$ is a homomorphism*

(ii) *It holds $\mathfrak{T}_1/M - Int_{\mathfrak{T}_1})(M) = \mathfrak{T}_2/M - Int_{\mathfrak{T}_1}\text{-}(M)$ for any $M \in \mathfrak{T}_2$.*

Proof: Put $\mathring{M} = Int_{\mathfrak{T}_1}(M)$.

Let (i) hold and $M \in \mathfrak{T}_2$. It is $\mathfrak{T}_1/M - \mathring{M} \subseteq \mathfrak{T}_2/M - \mathring{M}$. Let $\emptyset \neq X \in \mathfrak{T}_2/M - \mathring{M}$. There exists $Y \in \mathfrak{T}_2$ with $X = Y \cap (M - \mathring{M})$. Denote $N = Y \cap M$. Let $\mathfrak{T}' = \mathfrak{T}_1 \vee$

$\vee \{\emptyset, M, E\}$, $\mathfrak{T}'' = \mathfrak{T}_1 \vee \{\emptyset, N, E\}$. Let $x \in X$. It is $M \in \mathfrak{N}_{\mathfrak{T}'}(x) \cap \mathfrak{N}_{\mathfrak{T}''})(x) = \mathfrak{N}_{Cl(\mathfrak{T}')} \cap \mathfrak{N}_{Cl(\mathfrak{T}'')}(x) = \mathfrak{N}_{Cl(\mathfrak{T}') \wedge Cl(\mathfrak{T}'')}(x)$. Since $\mathfrak{T}'$, $\mathfrak{T}'' \in [\mathfrak{T}_1, \mathfrak{T}_2]$, it holds $Cl(\mathfrak{T}') \cap Cl(\mathfrak{T}'') = Cl(\mathfrak{T}' \cap \mathfrak{T}'')$. Hence $M \in \mathfrak{N}_{\mathfrak{T}' \cap \mathfrak{T}''}(x)$. Thus there exists $Z_x \in \mathfrak{T}' \cap \mathfrak{T}''$ with $x \in Z_x \subseteq M$. There exist V'_x, W''_x, $W_x \in \mathfrak{T}_1$ with $Z_x = V_x \cap M = W'_x \cup (W''_x \cap N)$. Since $W'_x \cap (M - \mathring{M}) = \emptyset$, it holds $Z_x \cap (M - \mathring{M}) = W''_x \cap N \cap (M - \mathring{M}) \subseteq N \cap (M - \mathring{M}) = Y \cap (M - \mathring{M}) = X$. Put $V = \bigcup_{x \in X} V_x$. It is $V \in \mathfrak{T}_1$,

$$V \cap (M - \mathring{M}) = \bigcup_{x \in X} V_x \cap (M - \mathring{M}) = \bigcup_{x \in X} [V_x \cap (M - \mathring{M})] = \bigcup_{x \in X} [Z_x \cap (M - \mathring{M})] \subseteq$$

$\subseteq X$. Since $x \in V_x$ for each $x \in X$, one gets $V \cap (M - \mathring{M}) = X$. Thus $X \in \mathfrak{T}_1/M - \mathring{M}$, i.e. $\mathfrak{T}_2/M - \mathring{M} \subseteq \mathfrak{T}_1\ M - \mathring{M}$.

Let (ii) hold. Since $Cl/[\mathfrak{T}_1, \mathfrak{T}_2]$ is a $\vee$-homomorphism, it is an isotone mapping. Therefore $Cl(\mathfrak{T}' \cap \mathfrak{T}'') \leqq Cl(\mathfrak{T}') \wedge Cl(\mathfrak{T}'')$ for all $\mathfrak{T}'$, $\mathfrak{T}'' \in [\mathfrak{T}_1, \mathfrak{T}_2]$. It remains to prove that $Cl(\mathfrak{T}') \wedge Cl(\mathfrak{T}'')$ is a topological closure operation for all $\mathfrak{T}'$, $\mathfrak{T}'' \in [\mathfrak{T}_1, \mathfrak{T}_2]$. Let $\mathfrak{T}'$, $\mathfrak{T}'' \in [\mathfrak{T}_1, \mathfrak{T}_2]$ and denote $u = Cl(\mathfrak{T}') \wedge Cl(\mathfrak{T}'')$. We have to show that for $x \in E$ and $U \in \mathfrak{N}_u(x)$ there exists $V \in \mathfrak{N}_u(x)$ such that $U \in \mathfrak{N}_u(y)$ for any $y \in V$. Let $x \in E$ and $U \in \mathfrak{N}_u(x)$. Hence $U \in \mathfrak{N}_{\mathfrak{T}'}(x) \cap \mathfrak{N}_{\mathfrak{T}''})(x)$ and thus there exist $X \in \mathfrak{T}'$, $Y \in \mathfrak{T}''$ with $x \in X \subseteq U$, $x \in Y \subseteq U$. Denote $M = X \cup Y$. It is $X \cap Y \cap (M - \mathring{M}) \in \mathfrak{T}_2/M - \mathring{M} = \mathfrak{T}_1/M - \mathring{M}$. Thus we can find $W \in \mathfrak{T}_1$ such that $W \cap (M - \mathring{M}) = X \cap Y \cap (M - \mathring{M})$. We are going to prove $(W \cap X) \cup \mathring{M} = (W \cap Y) \cup \mathring{M}$. Let $t \in (W \cap X) - \mathring{M}$. Then $t \in W \cap (M - \mathring{M}) \subseteq Y$. Hence $t \in W \cap Y$. Analogously it can be proven that $(W \cap Y) \cup \mathring{M} \subseteq (W \cap X) \cup \mathring{M}$. Denote $V = (W \cap X) \cup \mathring{M}$. We shall prove that V has the desired properties. It is $V \in \mathfrak{T}' \cap \mathfrak{T}''$. Since $Cl(\mathfrak{T}' \cap \mathfrak{T}'') \leqq u$, it holds $u(E - V) \subseteq Cl(\mathfrak{T}' \cap \mathfrak{T}'')(E - V) = E - V$. Therefore $u(E - V) = E - V$. Hence $V \in \mathfrak{N}_u(y)$ for any $y \in V$. Since $V \subseteq U$, $U \in \mathfrak{N}_u(y)$ for any $y \in V$. It remains to prove that $x \in V$. If $x \in M - \mathring{M}$, it holds $x \in W$ for $x \in X \cap Y$ and $W \cap (M - \mathring{M}) = X \cap Y \cap (M - \mathring{M})$. Therefore $x \in V$ holds. The proof is accomplished.

1.2. Theorem: *Let E be a set, $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. The following conditions are equivalent:*

(i) *$Cl/[\mathfrak{T}_1, \mathfrak{T}_2] : [\mathfrak{T}_1, \mathfrak{T}_2] \to [Cl_{\mathfrak{T}_1}, Cl_{\mathfrak{T}_2}]$ is an isomorphism*

(ii) *For any $M \subseteq E$ every point of the set $Int_{\mathfrak{T}_2}(M) - Int_{\mathfrak{T}_1}(M)$ is isolated in the topology $\mathfrak{T}_1/M - Int_{\mathfrak{T}_1}(M)$.*

Proof: Let (i) hold. Let $M \subseteq E$. Denote $M_i = Int_{\mathfrak{T}_i}(M)$, $i = 1, 2$. Let $a \in M_2 - M_1$. Put $\mathfrak{N}_u(t) = \mathfrak{N}_{\mathfrak{T}_1}(t)$ for $t \neq a$ and $\mathfrak{N}_u(a) = \mathfrak{N}_{\mathfrak{T}_1}(a) \vee [M)$. The closure operation u satisfies the inequalities $Cl_{\mathfrak{T}_1} \leqq u \leqq Cl_{\mathfrak{T}_2}$. Since $Cl[\mathfrak{T}_1, \mathfrak{T}_2]$ is an isomorphism, there exists $\mathfrak{T} \in \mathscr{B}(E)$ with $u = Cl(\mathfrak{T})$. Then, for some $V \in \mathfrak{T}$, we have $a \in V \subseteq M$. Let $a \neq t \in V$. Since $V \in \mathfrak{N}_u(t) = \mathfrak{N}_{\mathfrak{T}_1}(t)$, it holds $t \in M_1$. Thus $V \cap (M - M_1) = \{a\}$. Since $V \in \mathfrak{N}_u(a)$, there exists $W \in \mathfrak{T}_1$ such that $a \in W \cap M \subseteq V$. Hence $W \cap (M - M_1) = \{a\}$. Therefore a is isolated in the topology $\mathfrak{T}_1/M - M_1$.

Let (ii) hold. Let $u \in \mathscr{C}(E)$, $Cl_{\mathfrak{T}_1} \leqq u \leqq Cl_{\mathfrak{T}_2}$. We must prove that u is a topological closure operation. Let $x \in E$, $U \in \mathfrak{N}_u(x)$. It is $x \in Int_{\mathfrak{T}_2}(U)$. If $x \in Int_{\mathfrak{T}_1}(U)$, then for

$V = Int_{\mathfrak{T}_1}(U)$ we have $V \in \mathfrak{N}_u(x)$ and $U \in N_u(y)$ for any $y \in V$. Let $x \notin Int_{\mathfrak{T}_1}(U)$. According to (ii) there exists $W \in \mathfrak{T}_1$ with $W \cap (U - Int_{\mathfrak{T}_1}(U)) = \{x\}$. Put $V = W \cap U$. It is $V \in \mathfrak{N}_u(x)$. Let $y \in V$, $y \neq x$. It holds $y \in W \cap Int_{\mathfrak{T}_1}(U) \in \mathfrak{T}_1$. Hence $U \in \mathfrak{N}_u(y)$.

1.3. Corollary: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$. If $Cl/[\mathfrak{T})$ is a homomorphism, then it is an isomorphism.*

It is worthy to mention that if $u \in \mathscr{C}(E)$ and $Cl\mathscr{B}(E) \cap [u)$ is a lattice then $u \in Cl\mathscr{B}(E)$ (see [14]).

§2. MODULAR AND DISTRIBUTIVE INTERVALS OF $\mathscr{B}(E)$

A. K. Steiner showed in [18] that the lattice $\mathscr{B}(E)$ is not modular for $card\, E \geqq 3$. In [2] it is proven that the lattice $\mathscr{K}(E)$ of all T_1-topologies on E is not modular for infinite E. We shall consider modular and distributive intervals of $\mathscr{B}(E)$. A topology $\mathfrak{T}$ on E is called nested if either $X \subseteq Y$ or $Y \subseteq X$ for any $X, Y \in \mathfrak{T}$.

2.1. Theorem: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$. The following conditions are equivalent:*

(i) $(\mathfrak{T}]$ *is a modular lattice*

(ii) $(\mathfrak{T}]$ *is a distributive lattice*

(iii) $\mathfrak{T}$ *is nested or* $\mathfrak{T} = \{\emptyset, X, E - X, E\}$ *for* $X \subseteq E$, $\emptyset \neq X \neq E$.

Proof: We shall prove (i) $\Rightarrow$ (iii). Let $(\mathfrak{T}]$ be modular and $\mathfrak{T}$ be not nested. Then there exist $X, Y \in \mathfrak{T}$ with $X \nsubseteq Y$ and $Y \nsubseteq X$. Since $(\mathfrak{T}]$ is modular, it holds $\{\emptyset, X, E\} = \{\emptyset, X, E\} \vee)\{\emptyset, Y, E\} \cap \{\emptyset, X, X \cup Y, E\}) = (\{\emptyset, X, E\} \vee \{\emptyset, Y, E\}) \cap \{\emptyset, X, X \cup Y, E\} = \{\emptyset, X, X \cup Y, E\}$. Hence $X \cup Y = E$. Analogously we prove $X \cap Y = \emptyset$. Thus $Y = E - X$. Let $\emptyset \neq Z \in \mathfrak{T}$. Then $Z \nsubseteq X$ and $X \nsubseteq Z$ or $Z \nsubseteq Y$ and $Y \nsubseteq Z$. Therefore $Z = X$ or $Z = Y$. Hence $\mathfrak{T} = \{\emptyset, X, E - X, E\}$.

It remains to prove (iii) $\Rightarrow$ (ii). If $\mathfrak{T} = \{\emptyset, X, E - X, E\}$, the lattice $(\mathfrak{T}]$ is clearly distributive. Let $\mathfrak{T}$ be nested. We shall prove $\mathfrak{T}_1 \vee \mathfrak{T}_2 = \mathfrak{T}_1 \cup \mathfrak{T}_2$. It is sufficient to show that $\mathfrak{T}_1 \cup \mathfrak{T}_2$ is a topology. Let $X, Y \in \mathfrak{T}_1 \cup \mathfrak{T}_2$. Since either $X \subseteq Y$ or $Y \subseteq X$, it holds $X \cap Y \in \mathfrak{T}_1 \cup \mathfrak{T}_2$. Let $X_i \in \mathfrak{T}_1 \cup \mathfrak{T}_2$ for every $i \in I$. Let $I_k = \{i \in I / X_i \in \mathfrak{T}_k\}$ for $k = 1, 2$. Since $\mathfrak{T}$ is nested, there exists k such that $\bigcup_{i \in I} X_i = \bigcup_{i \in I_k} X_i \in \mathfrak{T}_k$. Therefore $\mathfrak{T}_1 \cup \mathfrak{T}_2$ is a topology. Hence $(\mathfrak{T}]$ is a sublattice of $exp\,(exp\, E)$ and thus it is distributive.

$\mathfrak{T}$ can be considered as a complete lattice ($\vee$ is $\cup$ and $\wedge$ is the interior of intersection). Evidently $(\mathfrak{T}]$ is the lattice of all subsets of the lattice $\mathfrak{T}$ closed under finite meets and arbitrary joins. Therefore the previous theorem can be compared with the result of Ph. Dwinger (see [6]) asserting that the lattice of all subsets of a complete lattice L closed under arbitrary meets is modular (distributive) iff L is a chain. In the light of the fact that lattices $(\mathfrak{T}]$ are special case of lattices of topologies on complete lattices it seems to be interesting to know when they are complemented.

A topology $\mathfrak{T}$ on E is called a T_D-topology if $Cl_{\mathfrak{T}}\{x\} - \{x\}$ is closed in $\mathfrak{T}$ for every $x \in E$ (see [1]). T_D is between T_0 and T_1. In [12] Larson proved that a T_D-topology $\mathfrak{T}$ is a minimal $\mathfrak{T}_D$-topology iff it is nested.

2.2. Corollary: *Let E be a set, card $E > 2$ and $\mathfrak{T}$ be a T_D-topology on E. Then $\mathfrak{T}$ is a minimal T_D-topology iff $(\mathfrak{T}]$ is a modular lattice.*

In the study of distributive intervals in the lattice of topologies the results of § 1. can be utilized.

2.3. Theorem: *Let E be a set, $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Let $\mathfrak{T}_1/M - Int_{\mathfrak{T}_1}(M) = \mathfrak{T}_2/M - Int_{\mathfrak{T}_1}(M)$ for every $M \in \mathfrak{T}_2$. Then $[\mathfrak{T}_1, \mathfrak{T}_2]$ is a distributive lattice.*
Proof follows from 1.1. because $\mathscr{C}(E)$ is distributive.

Valent and Larson proved in [20] that any finite distributive lattice is isomorphic to an interval in the lattice of all T_1-topologies on a certain set. It follows from the previous theorem that the converse assertion holds.

2.4. Theorem: *Any finite interval in the lattice of all T_1-topologies on an arbitrary set E is distributive.*

Proof: Let E be a set, $[\mathfrak{T}_1, \mathfrak{T}_2]$ a finite interval in the lattice of all T_1-topologies on E. Let $M \in \mathfrak{T}_2$. Denote $\mathring{M} = Int_{\mathfrak{T}_1}(M)$. Suppose that $M - \mathring{M}$ is an infinite set. It is $\{M - \{x\}/x \in M - \mathring{M}\} \subseteq \mathfrak{T}_2 - \mathfrak{T}_1$. Put $\mathfrak{T}_x = \mathfrak{T}_1 \vee \{\emptyset, M - \{x\}, E\}$ for every $x \in M - \mathring{M}$. Let $x, y \in M - \mathring{M}$, $x \neq y$. Suppose that $M - \{x\} \in \mathfrak{T}_y$. Then there exist $V, W \in \mathfrak{T}_1$ with $M - \{x\} = V \cup [M - \{y\})]$. Hence $y \in V \subseteq M$. It is a contradiction to $y \notin \mathring{M}$. Therefore the topologies $\mathfrak{T}_x$, $x \in M - \mathring{M}$, are mutually distinct, what contradicts the finiteness of $[\mathfrak{T}_1, \mathfrak{T}_2]$. We have obtained that $M - \mathring{M}$ is a finite set. Since $\mathfrak{T}_1$ is a T_1-topology, $\mathfrak{T}_1/M - \mathring{M}$ and $\mathfrak{T}_2/M - \mathring{M}$ are discrete. According to 2.3. $[\mathfrak{T}_1, \mathfrak{T}_2]$ is distributive.

Valent and Larson described in [20] linear intervals in the lattice of T_1-topologies. They showed that $[\mathfrak{T}_1, \mathfrak{T}_2]$ being a linear interval in $\mathscr{K}(E)$ implies the existence of a free ultratopology $\mathfrak{T}$ with $\mathfrak{T}_1 = \mathfrak{T}_2 \cap \mathfrak{T}$. Therefore the condition from 2.3. holds in this case again. But in regard to 2.1. this condition is not necessary for the distributivity of $[\mathfrak{T}_1, \mathfrak{T}_2]$. However, in the case $\mathfrak{T}_2$ discrete it turns out this to be necessary even for modularity of the lattice $[\mathfrak{T}_1, \mathfrak{T}_2]$.

2.5. Theorem. *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$. The following conditions are equivalent:*
(i) $[\mathfrak{T})$ *is a modular lattice*
(ii) $[\mathfrak{T})$ *is a distributive lattice*
(iii) $Cl/[\mathfrak{T}) : [\mathfrak{T}) \to [Cl_{\mathfrak{T}})$ *is an isomorphism*
(iv) $\mathfrak{T}/M - Int_{\mathfrak{T}}(M)$ *is a discrete topology for every* $M \subseteq E$.

Proof: According to 1.2. (iv) implies (iii). Since $\mathscr{C}(E)$ is distributive (iii) implies (ii). Clearly (ii) $\Rightarrow$ (i). It remains to prove that (i) $\Rightarrow$ (iv). Denote $\mathring{Z} = Int_{\mathfrak{T}}(Z)$ for every $Z \subseteq E$.

Let $[\mathfrak{T})$ be modular and $M \subseteq E$. Let $x \in M - \mathring{M}$. Let $\mathfrak{T}_1 = \{\emptyset, \mathring{M} \cup \{x\} \cup \cup (E - M), E\}$, $\mathfrak{T}_2 = \{\emptyset, M - \mathring{M}, E\}$, $\mathfrak{T}_3 = \{\emptyset, \{x\}, \mathring{M} \cup \{x\} \cup (E - M), E\}$. Evidently $\mathfrak{T}_1 \vee \mathfrak{T}_2 = \mathfrak{T}_3 \vee \mathfrak{T}_2$. Put $\mathfrak{T}_a = \mathfrak{T} \vee \mathfrak{T}_1$, $\mathfrak{T}_b = \mathfrak{T} \vee \mathfrak{T}_2$, $\mathfrak{T}_c = \mathfrak{T} \vee \mathfrak{T}_3$. Since $\mathfrak{T}_a \leqq \mathfrak{T}_c$, the modularity of $[\mathfrak{T})$ implies that $\mathfrak{T}_a \vee (\mathfrak{T}_b \wedge \mathfrak{T}_c) = (\mathfrak{T}_a \vee \mathfrak{T}_b) \wedge \mathfrak{T}_c$. It holds $(\mathfrak{T}_a \vee \mathfrak{T}_b) \wedge \mathfrak{T}_c = (\mathfrak{T} \vee \mathfrak{T}_1 \vee \mathfrak{T}_2) \wedge (\mathfrak{T} \vee \mathfrak{T}_3) = (\mathfrak{T} \vee \mathfrak{T}_3 \vee \mathfrak{T}_2) \wedge (\mathfrak{T} \vee \mathfrak{T}_3) = \mathfrak{T} \vee \mathfrak{T}_3$. Therefore $\{x\} \in \mathfrak{T}_a \vee (\mathfrak{T}_b \wedge \mathfrak{T}_c)$. Since $\mathfrak{T}_a = \mathfrak{T} \vee \mathfrak{T}_1$, there exist $V \in \mathfrak{T}$, $W \in \mathfrak{T}_b \wedge \mathfrak{T}_c$ with $V \cap \cap [\mathring{M} \cup \{x\} \cup (E - M)] \cap W = \{x\}$. Hence $V \cap W \subseteq M - \mathring{M}$. There exist X_1, X_2, X_3, Y_1, $Y_2 \in \mathfrak{T}$ such that $W = Y_1 \cup [Y_2 \cap M - \mathring{M})] = X_1 \cup (X_2 \cap \{x\}) \cup \cup [X_3 \cap (\mathring{M} \cup \{x\} \cup (E - M))]$. Let $t \in W - \mathring{W}$. Then $t \in Y_2 \cap (M - \mathring{M})$. Hence $t = x$. We get $W = \mathring{W} \cup \{x\}$. Therefore $V \cap W = V \cap (\mathring{W} \cup \{x\}) = (V \cap \mathring{W}) \cup \cup \{x\}$. Since $V \cap W \subseteq M - \mathring{M}$, it holds $V \cap \mathring{W} = \emptyset$. Therefore $V \cap W = \{x\}$. Thus $\{x\} = V \cap [Y_1 \cap (Y_2 \cap (M - \mathring{M}))] \supseteq V \cap (Y_1 \cup Y_2) \cap (M - \mathring{M})$. Since $V \cap \cap (Y_1 \cup Y_2) \in \mathfrak{T}$, it holds $\{x\} \in \mathfrak{T}/M - \mathring{M}$. Therefore (iv) holds.

There is a question, whether it can be found a modular interval of $\mathscr{B}(E)$ which is not distributive.

A topology $\mathfrak{T}$ fulfilling the equivalent conditions of 2.5. will be called an m-topology. We are going to give some examples and properties of this topologies. An MI-topology is a topology without isolated points every dense subset of which is open (see [10]). A topology every dense subset of which is open is called in [3] submaximal. In [3], excercises of § 8. it is proved that supposing $\mathfrak{T}$ is submaximal the relative topology $\mathfrak{T}/Cl_{\mathfrak{T}}(M) - Int_{\mathfrak{T}}(M)$ is discrete for every $M \subseteq E$. Therefore any submaximal topology is an m-topology. For instance any free ultratopology is submaximal.

2.6. Lemma. *Let E be a set and $\mathfrak{T}$ a T_0-topology on E which is an m-topology. Then the interior of any dense set is dense.*

Proof. Let $X \subseteq E$ be dense in $\mathfrak{T}$. Suppose that $\mathring{X} = Int_{\mathfrak{T}}(X)$ is not dense. Then there exists $V_1 \in \mathfrak{T}$ with $V_1 \neq \emptyset$, $V_1 \cap \mathring{X} = \emptyset$. Since X is dense, there exists $x \in V_1 \cap \cap X$. As $\mathfrak{T}$ is an m-topology, there exists $V_2 \in \mathfrak{T}$ with $V_2 \cap (X - \mathring{X}) = \{x\}$. It is $V_1 \cap V_2 \cap X = \{x\}$. We can take $y \in V_1 \cap V_2$, $y \neq x$ because $x \notin \mathring{X}$. Thus $y \notin X$. Since $Int_{\mathfrak{T}}(E - X) = \emptyset$ and $\mathfrak{T}$ is an m-topology, there exists $V_3 \in \mathfrak{T}$ with $V_3 \cap (E - X) = \{y\}$. Let $V \doteq V_1 \cap V_2 \cap V_3$. It is $V \in \mathfrak{T}$ and $\{y\} \subseteq V \subseteq \{x, y\}$. Since $\mathfrak{T}$ is a T_0-topology it must hold $\{x\} \in \mathfrak{T}$ or $\{y\} \in \mathfrak{T}$. This is a contradiction to $x \in \in X - \mathring{X}$, $y \notin X$ and X dense.

A topology is called resolvable if it contains two disjoint dense sets (see [10]).

2.7. Corollary. *Let E be a set and $\mathfrak{T}$ a T_0-topology which is an m-topology. Then $\mathfrak{T}$ is not resolvable.*

The supposition $\mathfrak{T}$ is a T_0-topology is necessary.

2.8. Example. Let E be a set, $A \subseteq E$ and $card\, A = card\,(E - A)$. Then there exists a bijective mapping $f: A \to E - A$. Let $\mathfrak{T}$ be the topology generated by the system $\{\{x, fx\}/x \in A\}$. $\mathfrak{T}$ is not a T_0-topology. It is easy to see that $\mathfrak{T}$ is an m-topology and $A, E - A$ are disjoint dense sets.

2.9. Lemma. *Let E be a set and $\mathfrak{T} \in \mathscr{B}(E)$. The following conditions are equivalent:*
(i) *The system $D_{\mathfrak{T}}$ of all dense sets in $\mathfrak{T}$ is a filter*
(ii) *Any dense set in $\mathfrak{T}$ has the dense interior.*

Proof: Put $\mathring{Z} = Int_{\mathfrak{T}}(Z)$ for $Z \subseteq E$.

Let (i) hold. Let X be dense in $\mathfrak{T}$. $E - (X - \mathring{X})$ is dense. Hence $\mathring{X} = X \cap [E - (X - \mathring{X})]$ is dense.

Let (ii) hold and $X, Y \in D_{\mathfrak{T}}$. It is $\mathring{X} \cap Cl_{\mathfrak{T}}(\mathring{Y}) \subseteq Cl_{\mathfrak{T}}(\mathring{X} \cap \mathring{Y})$. Hence $\mathring{X} = \mathring{X} \cap E = \mathring{X} \cap Cl_{\mathfrak{T}}(\mathring{Y}) \subseteq Cl_{\mathfrak{T}}(Y) \subseteq Cl_{\mathfrak{T}}(\mathring{X} \cap \mathring{Y})$.

Thus $\mathring{X} \cap \mathring{Y} \in D_{\mathfrak{T}}$, i.e. $X \cap Y \in D_{\mathfrak{T}}$. Therefore (i) holds.

Another statements equivalent with the conditions of 2.9. are given in [7].

§ 3. SIMPLE INTERVALS OF $\mathscr{B}(E)$

Hartmanis proved in [9] that the lattice $\mathscr{B}(E)$ is simple for $card\, E \neq 2$ and that the lattice $\mathscr{K}(E)$ is not simple for an infinite set E. It arises a question for which topologies $\mathfrak{T}$ the lattice $[\mathfrak{T})$ is simple.

3.1. Lemma. *Let E be a set and $\mathscr{L}$ a sublattice of $\mathscr{B}(E)$. Let $A \subseteq E$ such that either $A \in \mathfrak{T}$ for every $\mathfrak{T} \in \mathscr{L}$ or $E - A \in \mathfrak{T} \in \mathscr{L}$. Let $\psi_A \mathfrak{T} = \mathfrak{T}/A$ for every $\mathfrak{T} \in \mathscr{L}$. Then a mapping $\psi_A : \mathscr{L} \to \mathscr{B}(A)$ is a homomorphism.*

Proof: Let $A \subseteq E$. It can be easily proved that $\psi_A : \mathscr{L} \to \mathscr{B}(A)$ is a $\vee$-homomorphism which is a homomorphism whenever $E - A \in \mathscr{L}$ for every $\mathfrak{T} \in \mathscr{L}$ (see [16]). Let $A \in \mathfrak{T}$ for every $\mathfrak{T} \in \mathscr{L}$. Then $\psi_A \mathfrak{T} = \mathfrak{T} \cap exp\, A$ for every $\mathfrak{T} \in \mathscr{L}$. Hence ψ_A is a $\cap$-homomorphism and therefore it is a homomorphism.

3.2. Lemma. *Let E be a set and $A \subseteq E$. Then $\mathfrak{T} = exp\,(E - A) \cup [A) \in \mathscr{B}(E)$ and the mapping $\psi_A : [\mathfrak{T}) \to \mathscr{B}(A)$ is an isomorphism.*

Proof. Clearly $\mathfrak{T} \in \mathscr{B}(E)$. Since $A \in \mathfrak{T}$, ψ_A is according to 3.1. a homomorphism. Let $\mathfrak{T} \subseteq \mathfrak{T}_1$, $\mathfrak{T} \subseteq \mathfrak{T}_2$ and $\psi_A \mathfrak{T}_1 = \psi_A \mathfrak{T}_2$. Let $X \in \mathfrak{T}_1$. It is $X \cap A \in \psi_A \mathfrak{T}_1 = \psi_A \mathfrak{T}_2 \subseteq \mathfrak{T}_2$. Since $X - A \in \mathfrak{T} \subseteq \mathfrak{T}_2$, it holds $X \in \mathfrak{T}_2$. Analogously we prove $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$. Thus ψ_A is injective.

Let $\mathfrak{T}' \in \mathscr{B}(A)$. Let $\mathfrak{T}_1$ be the topology on E generated by the system $\mathfrak{T} \cup \mathfrak{T}'$. Clearly $\psi_A(\mathfrak{T}_1) = \mathfrak{T}'$. Thus ψ_A is surjective.

3.3. Theorem. *Let E be a set and $\mathfrak{T} \in \mathscr{B}(E)$. The lattice $[\mathfrak{T})$ is simple iff $\mathfrak{T}$ is either an ultratopology or a topology of the form $exp\,(E - A) \cup [A)$ for $A \subseteq E$, card $A \neq 2$.*

Proof: It $\mathfrak{T}$ is an ultratopology then $[\mathfrak{T})$ is simple. It $\mathfrak{T} = exp\,(E - A) \cup [A)$ *for* $A \subseteq E$, *card* $A \neq 2$ then it follows from Hartmanis' result and from 3.2. that $[\mathfrak{T})$ is simple.

Let $\mathfrak{T} \in \mathscr{B}(E)$ and $[\mathfrak{T})$ be simple. Let $X \in \mathfrak{T}$. We shall show that $exp\,X \subseteq \mathfrak{T}$ or $exp\,(E - X) \subseteq \mathfrak{T}$. Suppose that $exp\,X \nsubseteq \mathfrak{T}$. According to 3.1. $\psi_X : [\mathfrak{T}) \to \mathscr{B}(X)$ is a homomorphism. Since $\psi_X(exp\,E) = exp\,X \neq \psi_X\mathfrak{T}$, the mapping ψ_X is injective. Let $\mathfrak{T}'$ be the topology on E generated by the system $\mathfrak{T} \cup exp\,X$. It holds $\psi_X\mathfrak{T}' = exp\,X = \psi_X(exp\,E)$.Hence $\mathfrak{T}' = exp\,E$. Thus $exp\,(E - X) \subseteq \mathfrak{T}$.

Let $A = \{x \in E/\{x\} \notin \mathfrak{T}\}$. It follows from the above result that the relative topology $\mathfrak{T}/A$ is indiscrete. If $A = \emptyset$, then $\mathfrak{T} = exp\,(E - A) \cup [A)$ is a discrete topology. Let *card* $A = 1$. Then $\mathfrak{T}/M - Int_{\mathfrak{T}}(M)$ is a discrete topology for every $M \subseteq E$. By 2.5. the lattice $[\mathfrak{T})$ is distributive. A distributive lattice is simple iff it contains less than three elements (see [19] Ch. IX, Ex. 15).

Therefore $\mathfrak{T}$ is an ultratopology. Let *card* $A > 1$. It follows from 3.1. that $\psi_A : [\mathfrak{T}) \to \mathscr{B}(A)$ is a homomorphism. Since $\psi_A\mathfrak{T} \neq \psi_A\,exp\,E$, ψ_A is injective. Let $\mathfrak{T}' = \mathfrak{T} \cup \{\emptyset, A, E\}$. As $\psi_A\mathfrak{T} = \psi_A\mathfrak{T}'$, one gets $\mathfrak{T} = \mathfrak{T}'$. Thus $A \in \mathfrak{T}$ and $\mathfrak{T} = exp\,(E - A) \cup [A)$. Since the lattice $\mathscr{B}(A)$ is not simple for card $A = 2$, it follows from 3.2. that *card* $A \neq 2$. The proof is completed.

3.4. Corollary. *Let E be a set and $\mathfrak{T}$ a T_0-topology on E. Then the lattice $[\mathfrak{T})$ is simple iff $\mathfrak{T}$ is either discrete or an ultratopology.*

Finally, we are going to give one result on homomorphisms of the lattice $(\mathfrak{T}]$, where $\mathfrak{T}$ is a T_1-topology.

3.5. Theorem. *Let E be a set and $\mathfrak{T}$ be a T_1-topology on E. The following conditions are equivalent:*

(i) *For any non-injective homomorphism f of $(\mathfrak{T}]$ into a lattice L and any atom $\{\emptyset, X, E\}$ of $(\mathfrak{T}]$ it holds $f\{\emptyset, X, E\} = f\{\emptyset, E\}$.*

(ii) *Any $\vee$-complete homomorphism of $(\mathfrak{T}]$ onto a $\vee$-complete lattice L, is either an isomorphism or L consists of a single element*

(iii) $\mathfrak{T} - \{\{a\}\}$ *is not a topology for any isolated point a of $\mathfrak{T}$.*

Proof: (i) implies (ii) because any element of $(\mathfrak{T}]$ is a join of atoms.

Let (ii) hold and a be an isolated point of $\mathfrak{T}$. In the case *card* $E = 1$ the theorem holds. Let *card* $E > 1$. Suppose that $\mathfrak{T} - \{\{a\}\}$ is a topology. Let $L = \{0, 1\}$, $0 < 1$, be a two-element chain. Let $\mathfrak{S} \subseteq \mathfrak{T}$. Define $f\mathfrak{S} = 0$ for $\mathfrak{S} \subseteq \mathfrak{T} - \{\{a\}\}$ and $f\mathfrak{S} = 1$ otherwise. It is easy to see that $f : (\mathfrak{T}] \to L$ is a complete homomorphism. It is $f\{\emptyset, \{a\}, E\} \neq f\{\emptyset, E\}$. Since *card* $E > 1$ and $\mathfrak{T}$ is a T_1-topology, there exists $\emptyset \neq V \in \mathfrak{T}$ with $a \notin V$. It holds $f\{\emptyset, V, E\} = f\{\emptyset, E\}$. It is a contradiction.

Let (iii) hold. Let L be a lattice and $f: (\mathfrak{T}] \to L$ a non-injective homomorphism. Denote $0 = f\{\emptyset, E\}$. We shall prove an auxiliary assertion

(1) Let $\{\emptyset, X, E\}$, $\{\emptyset, Y, E\}$ be two atoms of $(\mathfrak{T}]$. Let $X \cap Y \neq \emptyset$ and $Y \nsubseteq X$. Let $f\{\emptyset, X, E\} = 0$. Then $f\{\emptyset, Y, E\} = 0$.

It is $f\{\emptyset, X \cap Y, E\} \leqq f(\{\emptyset, X, E\} \vee \{\emptyset, Y, E\}) = f\{\emptyset, X, E\} \vee f\{\emptyset, Y, E\} = f\{\emptyset, Y, E\}$. Thus $f\{\emptyset, X \cap Y, E\} = f\{\emptyset, X \cap Y, E\} \wedge f\{\emptyset, Y, E\} = f)\{\emptyset, X \cap Y, E\} \cap \{\emptyset, Y, E\}) = = 0$ because $Y \nsubseteq X$. In the case *card* $Y = 1$ it is $Y = Y \cap X$ and therefore (1) holds. Let *card* $Y > 1$. There exists $x \in X \cap Y$. Since $\mathfrak{T}$ is a T_1-topology, it holds $Y - \{x\} \in \mathfrak{T}$. It is $\{\emptyset, Y, E\} \subseteq \{\emptyset, X \cap Y, E\} \vee \{\emptyset, Y - \{x\}, E\}$. In the same way as above we obtain $f\{\emptyset, Y, E\} = 0$.

Since f is not injective and every element of $(\mathfrak{T}]$ is a join of atoms, there exists an atom $\{\emptyset, V, E\}$ of $(\mathfrak{T}]$ such that $f\{\emptyset, V, E\} = 0$. Let $\{\emptyset, X, E\}$ be an atom of $(\mathfrak{T}]$ and $X \neq V$. Let *card* $X > 1$. Thus there exists $x \in X$ with $V \neq \{x\}$. Therefore $V \cap \cap (E - \{x\}) \neq \emptyset$. Evidently $E - \{x\} \subseteq V$ implies $E - \{x\} = V$. It follows from (1) that $f\{\emptyset, E - \{x\}, E\} = 0$. Again owing to (1) it holds $f\{\emptyset, X, E\} = 0$. Let *card* $X = 1$. According to (iii) $\mathfrak{T} - \{X\}$ is not a topology. Thus there exist $X_1, X_2 \in \mathfrak{T}$ such that $X = X_1 \cap X_2$ and *card* $X_i > 1$, $i = 1, 2$. Therefore $f\{\emptyset, X_i, E\} = 0$ for $i = 1, 2$. It follows from $\{\emptyset, X, E\} \subseteq \{\emptyset, X_1, E\} \vee \{\emptyset, X_2, E\}$ that $f\{\emptyset, X, E\} = 0$.

Thereby we have shown that (iii) implies (i).

REFERENCES

[1] C. E. Aull and W. J. Thron, *Separation axioms between T_0 and T_1*, Nederl. akad. Wetensch. Proc. Ser. A 65 = Indag. Math. 24 (1962), 26—37.

[2] R. W. Bagley and D. Ellis, *On the topolattice and permutation group of an infinite set*, Math. Japon. 3 (1954), 63—70.

[3] N. Bourbaki, *Topologie générale*, Paris.

[4] A. M. Carstens, *The lattice of pretopologies on an arbitrary set S*, Pac Jour. Math. 29 (1969), 67—71.

[5] E. Čech, *Topological spaces*, Prague 1966.

[6] Ph. Dwinger, *On the lattice of the closure operators of a complete lattice*, Nieuw Arch. Wisk. (4), IV (1956), 112—117.

[7] A. G. El'kin, *Ul'trafil'try i nerazložimyje prostranstava*, Vest. Mosk. Univ., Ser. I, 24, No 5(1969), 51—56.

[8] O. Fröhlich, *Das Halbordnungssystem der topologischen Räume auf einer Menge*, Math. Ann. 156 (1964), 79—95.

[9] J. Hartmanis, *On the lattice of topologies*, Can. Jour. Math. 10 (1958), 547—553.

[10] E. Hewitt, *A problem of set-theoretic topology*, Duke math. J. 10 (1943), 309—333.

[11] H. J. Kowalsky, *Topological spaces*, New York, London 1965.

[12] R. E. Larson, *Minimal T_0-spaces and minimal T_D-spaces*, Pac. Jour. Math. 31 (1969), 451—458.

[13] R. E. Larson and S. J. Zimmerman, *The lattice of topologies*: A survey, to appear.

[14] J. Rosický, *Topologies compatible with ordering*, Publ. Fac. Sci. Univ. Brno, 1971, 9—23.
[15] J. Rosický, *Relative Komplemente im Verband der T_1-Topologien*, Publ. Fac. Sci. Univ. Brno, 518 (1970), 445—460.
[16] J. Rosický, *Embeddings of lattices in the lattice of topologies*, Arch. Math. Brno, to appear.
[17] M. Sekanina, *Sistěmy topologij na dannom množestve*, Czech. Math. Jour. T 15 (90), 1965, 9—29.
[18] A. K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc. 122 (1966), 379—398.
[19] G. Szasz, *Einführung in die Verbandstheorie*, Budapest 1962.
[20] R. Valent and R. E. Larson, *Basic Intervals in the Lattice of Topologies*, Duke Math. J. 39 (1972), 401—411.

J. Rosický
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia