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IDEALS OF N-ALGEBRAS

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Homomorphic mapping of direct products of algebras are investigated from the point of the direct decompositions of mappings in papers [4] and [5]. There is proved that for direct products of so-called pseudo-ordered algebras we can state the converse of the theorem on direct products of surjective homomorphisms. For N-algebras (they are direct products of algebras without zero-divisors) we can state only a weak analog of the converse of this theorem. From it there is clear that the N-algebras play an important role in the theory of direct decompositions of homomorphisms. N-algebras are for example atomic Boolean algebras, lattices generated by chains with the least (or greatest) element, 1-groups in which the minimal condition holds (see [4]), direct products of rings without zero-divisors, linear Ω-algebras and Ω-groups without nilpotent elements (where Ω contains n-ary operation for $n > 1$) – see [1], [6], and other algebras important in applications.

In this paper, there are defined ideals in N-algebras. The definition is similar to the definition of ideal in rings (see [7]) and in linear Ω-algebras (see [6], [1]). These ideals are used for investigation of direct decompositions of homomorphic mappings. This paper is a continuation of papers [4] and [5], all concepts and notations are taken from there.

1.

In the whole paper the symbol $\mathfrak{A}$ denotes a class of algebras with zero 0, binary operation $\oplus$ and a set Ω of n-ary operations fulfilling identities:

(i) $0 \oplus a = a \oplus 0 = a$ for each $A \in \mathfrak{A}$ and arbitrary $a \in A$
(ii) $00 \dots 0\, \omega = 0$ for each $\omega \in \Omega$.

Let $A \in \mathfrak{A}$. We say that A is *without zero-divisors* iff there exists $\Omega' \neq \emptyset$, $\Omega' \subseteq \Omega$ that for each $\omega \in \Omega'$ the arity of is greater than 1 and

(iii) $a_1 a_2 \dots a_n \omega = 0$ iff $a_i = 0$ for at least one $i \in \{1, \dots, n\}$.

Operations from Ω' are called *regular*. Direct products of algebras without zero-divisors are called *N-algebras*.

Let $A \in \mathfrak{A}$ be without zero-divisors. We say that A is *strongly pseudo-ordered* if there exists $\Omega'' \subseteq \Omega'$, $\Omega'' \neq \emptyset$ such that for each $\omega \in \Omega''$ and arbitrary n-tupl $a_1, \dots, a_n \in A$ the following holds:

(iv) $a_1 a_2 \dots a_n \omega = a_i$ for suitable $i \in \{1, \dots, n\}$.

It is clear that each element of strongly pseudo-ordered algebra is idempotent with regard to operations from Ω''.

Operations from Ω are denoted by the same symbols in all algebras of $\mathfrak{A}$. Let $A_\tau \in \mathfrak{A}$ for $\tau \in T$. The direct product of A_τ is denoted by $\prod_{\tau \in T} A_\tau$. By the symbol $\bar{A}_\tau$ (resp. $\overline{\prod_{\tau \in T'} A_\tau}$ for $T' \subseteq T$) we denote a subalgebra of $\prod_{\tau \in T} A_\tau$ such that $pr_\tau \bar{A}_\tau = A_\tau$, $pr_{\tau'} \bar{A}_\tau = 0$ for $\tau' \neq \tau$ (resp. $pr_\tau \overline{\prod_{\tau \in T'} A_\tau} = A_\tau$ for $\tau \in T'$ and $pr_{\tau'} \overline{\prod_{\tau \in T'} A_\tau} = 0$ for $\tau' \in T - T$).

In the whole paper the concept "algebra without zero-divisors" mean *the algebra of $\mathfrak{A}$ without zero-divisors which is not one-elemented.*

2.

In paper [5], there is proved that for N-algebras the theorem analogous to the classical Remark-Krull-Schmidt's theorem (see [7]) is valid, i.e., if A_τ, B_σ are algebras without zerodivisors, $A = \prod_{\tau \in T} A_\tau$, $A = \prod_{\sigma \in S} B_\sigma$, then card T = card S and there exists a permutation π of S such that $A_\tau = B_{\pi(\sigma)}$ for each $\tau \in T$.

Lemma A. *Let A be an N-algebra, $\omega \in \Omega$ and ω be a direct product of regular operations of corresponding direct factors, ω n-ary. Then it holds*

(v) $$a_1 \dots a_{i-1} 0_A a_{i+1} \dots a_n \omega = 0_A$$

for arbitrary $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in A$.

Accordingly, for each N-algebra there exists at least one $\omega \in \Omega$ satisfying (v). The set of all $\omega \in \Omega$ satisfying (v) is denoted by Ω_0.

Proof. Let A_τ is without zero-divisors for all $\tau \in T$, $A = \prod_{\tau \in T} A_\tau$.

Let ω fulfil assumptions of the lemma A. Then

$$pr_\tau(a_1 \dots a_{i-1} 0_A a_{i+1} \dots a_n \omega) = pr_\tau(a_1) \dots pr_\tau(a_{i-1})\, 0\, pr_\tau(a_{i+1}) \dots pr_\tau(a_n)\, \omega = 0.$$

From it follows $a_1 \dots a_{i-1} 0_A a_{i+1} \dots a_n \omega = 0_A$.

Definition 1. A subset B of an N-algebra A is said to be ideal if:

(I) $$a, b \in B \Rightarrow a \oplus b \in B$$

(II) $$a_1, \dots, a_n \in A,\ a_i \in B,\ \omega \in \Omega_0 \Rightarrow a_1 \dots a_n \omega \in B.$$

From the lemma A follows the correctness of the definition 1.

Lemma B. *$0_A \in B$ for each ideal B of an N-algebra A.*

Proof. Let B be an ideal of an N-algebra A, ω be the direct product of regular operations of corresponding direct factors, $a_1, \ldots, a_n \in A$, $a_i \in B$, $aj = 0$ and $i \neq j$, ω n-ary. Then $0_A = a_1 \ldots a_{j-1}\, 0_A a_{j+1} \ldots a_n\omega$, but $a_1 \ldots a_n\omega \in B$ by (II) of the definition 1, then $0_A \in B$.

Theorem 1. *Let $\sim$ be an arbitrary congruence relation on an N-algebra A. The subset of all elements $a \in A$ with $a \sim 0_A$ is an ideal of A.*

Proof. Let φ be a canonical homomorphism of A onto $A/\!\sim$. Then $a \sim 0_A$ if $a \in \ker \varphi$. Denote $B = \ker \varphi$. Let $a, b \in B$, then $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b) = \varphi(0_A) \oplus \oplus\, \varphi(0_A) = \varphi(0_A \oplus 0_A) = \varphi(0_A)$, i.e. $a \oplus b \in B$. Let $a_1, \ldots, a_n \in A$, $a_i \in B$, $\omega \in \Omega_0$. Then $\varphi(a_1 \ldots, a_{i-1}\, a_i a_{i+1} \ldots a_n\omega) = \varphi(a_1) \ldots \varphi(a_{i-1})\, \varphi(0_A)\, \varphi(a_{i+1}) \ldots \varphi(a_n)\, \omega = \varphi(a_1 \ldots a_{i-1}\, 0_A a_{i+1} \ldots a_n\omega) = \varphi(0_A)$, i.e. $a_1 \ldots a_n\omega \in B$. q.e.d.

We can easy prove the theorem:

Theorem 2. *The intersection of arbitrary set of ideals of an N-algebra A is an ideal of A. The ideals of an N-algebra A form the complete lattice with the least element $\{0_A\}$ and the greatest element A with respect to the set inclusion.*

Definition 2. *An N-algebra A is said to be distributive if there holds*

(vi) $$a_1 \ldots a_{i-1}\, 0_A a_{i+1} \ldots a_n\omega = 0_A$$

(vii) $$a_1 \ldots a_{i-1}(b \oplus c)\, a_{i+1} \ldots a_n\omega = (a_1 \ldots a_{i-1} b a_{i+1} \ldots a_n\omega) \oplus (a_1 \ldots a_{i-1}\, c a_{i+1} \ldots a_n\omega)$$

for each $\omega \in \Omega$, b, c, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ and arbitrary $i \in \{1, \ldots, n\}$. The operations $\omega \in \Omega$ of distributive algebra are called distributive.

Definition 3. *An ideal B of an N-algebra A is said to be normal if for any $a \in A$ holds $a \oplus B = B \oplus a$.*

Theorem 3. *Let A be a distributive N-algebra with the associative operation $\oplus$. A partition of A is induced by a congruence relation* on *A if it is a partition by a normal ideal of A.*

Proof. If $\sim$ is a congruence relation on A, then $\sim$ induces a partition by the ideal $B = \ker \varphi$, where φ is the canonical homomorphism of A onto $A/\!\sim$, which follows from the theorem 1.

Let $a_1 \in a \oplus B$, then $a_1 = a \oplus b_1$ for some element $b_1 \in B$. From it we obtain $\varphi(a_1) = \varphi(a) \oplus \varphi(b_1) = \varphi(a)$, i.e. $a_1 \sim a = 0_A \oplus a$. By the lemma $B\, 0_A \in B$, thus $a_1 \in B + a$. The converse inclusion is obtained analogously, thus $a \oplus B = B \oplus a$ and B is a normal ideal.

Conversely: let A be an N-algebra with associative operation $\oplus$, B be *a* normal ideal of A. By the lemma $B\ 0_A \in B$, thus $a \oplus B$ run over the whole algebra A for $a \in A$. Let us consider the partition A/B, i.e. the set of all classes $a \oplus B$ for $a \in A$.

(a) Let $a_1 \oplus a_2 = a_3$, denote $\bar{a}_i = a_i \oplus B$ for $i = 1, 2, 3$. Let $a_1' \in \bar{a}_1$, $a_2' \in \bar{a}_2$, then there exist b_1, $b_2 \in B$ so that $a_1' = a_1 \oplus b_1$, $a_2' = a_2 \oplus b_2$. Then $a_1' \oplus a_2' = (a_1 \oplus b_1) \oplus (a_2 \oplus b_2)$. From normality of B follows the existence of $b_3 \in B$ so that $a_2 \oplus b_2 = b_3 \oplus a_2$, thus $a_1' \oplus a_2' = (a_1 \oplus b_2) \oplus (b_3 \oplus a_2) = a_1 \oplus b' \oplus \oplus a_2 = a_1 \oplus a_2 \oplus b$ for b', $b \in B$, $\oplus$ beeing associative. From it we have $a_1' \oplus a_2' \in \in \bar{a}_3$. Let $a_3' \in \bar{a}_3$, then $a_3' = a_3 \oplus b_3 = a_1 \oplus a_2 \oplus b_3 = (a_1 \oplus 0_A) \oplus (a_2 \oplus b_3)$ for some $b_3 \in B$, thus $a_3' \in \bar{a}_1 \oplus \bar{a}_2$. Hence $\bar{a}_1 \oplus \bar{a}_2 = \bar{a}_3$.

(b) Let A be distributive, $x_1, \ldots, x_n \in A$, $x_i \in B$, ω be a distributive operation and $x_1 \ldots x_n\omega = x$. Then

$$\begin{aligned}\bar{x}_1\bar{x}_2 \ldots \bar{x}_n\omega &= (x_1 \oplus B)(x_2 \oplus B) \ldots (x_n \oplus B)\,\omega = \\ &= x_1(x_2 \oplus B) \ldots (x_n \oplus B)\,\omega \oplus B(x_2 \oplus B) \ldots (x_n \oplus B)\,\omega = \\ &= x_1(x_2 \oplus B) \ldots (x_n \oplus B)\,\omega \oplus B = \\ &= x_1x_2(x_3 \oplus B) \ldots (x_n \oplus B)\,\omega \oplus x_1B(x_3 \oplus B) \ldots (x_n \oplus B)\,\omega \oplus B = \\ &= x_1x_2(x_3 \oplus B) \ldots (x_n \oplus B)\,\omega \oplus B = \ldots = x_1 \ldots x_n\omega \oplus B = \bar{x},\end{aligned}$$

as we obtain from identities (vi) and (vii) in the definition 2. Thus, the partition of A by normal ideal B is induced by a congruence relation.

q.e.d.

3.

Definition 4. *An ideal B of an N-algebra A is called prime if there exists $\omega \in \Omega$ which is the direct product of regular operations such that*

(viii) $a_1a_2 \ldots a_n\omega \in B \Rightarrow a_j \in B$ *for at least one* $j \in \{1, \ldots, n\}$.

Theorem 4. *The homomorphic image of a distributive algebra A without zero-divisors is an algebra without zero-divisors if the kernel of the homomorphism is a normal prime ideal of A.*

Proof. Let A be an distributive algebra without zero-divisors, φ be a homomorphic mapping of A into $C \in \mathfrak{A}$. By the theorem 3 $B = \ker \varphi$ is a normal ideal in A, $\varphi(0_A)$ is a unique zero of an algebra $\varphi(A)$ by the lemma A in [5]. Let B be a prime ideal, ω_0 be an n-ary operation fulfilling (viii) and let $a_1, \ldots, a_n \in A$, $\varphi(a_1) \ldots \varphi(a_n)\,\omega_0 = \varphi(0_A)$. Then $\varphi(a_1 \ldots a_n\omega_0) = \varphi(0_A)$ and from it $a_1 \ldots a_n\omega_0 \in B$, i.e. $a_j \in B$ for at least one j, in other words $\varphi(a_j) = \varphi(0_A)$. Thus ω_0 is a regular operation in $\varphi(A)$, i.e. $\Omega' \neq \emptyset$ for $\varphi(A)$.

Conversely – let $\varphi(A)$ be without zero-divisors, $B = \ker \varphi$, $a_1 \ldots a_n\omega \in B$ and ω be a regular operation in $\varphi(A)$ (by the lemma A in [5] ω is regular in A too). Then $\varphi(a_1 \ldots a_n\omega) = \varphi(0_A)$, from this $\varphi(a_1) \ldots \varphi(a_n)\,\omega = \varphi(0_A)$, i.e. $\varphi(a_i) = \varphi(0_A)$ for at least one $i \in \{1, \ldots, n\}$. Thus $a_i \in B$ and B is a prime ideal.

q.e.d.

With this concepts we can now investigate direct decompositions of homomorphic mappings of N-algebras.

Theorem 5. *Let A_τ, B_τ be distributive algebras without zero-divisors, $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\tau \in T} B_\tau$, φ be a homomorphic mapping of A onto B so that $\bar{A}_\tau \cap \ker \varphi$ is a normal prime ideal of $\bar{A}_\tau$ for each $\tau \in T$. Then $\varphi = \prod_{\tau \in T} \varphi_\tau$, where φ_τ is a homomorphic mapping of A_τ onto $B_{\pi(\tau)}$, where π is a permutation of the index set T.*

Proof. If $\bar{A}_\tau \cap \ker \varphi$ is a prime ideal in $\bar{A}_\tau$, then $\varphi(\bar{A}_\tau)$ is without zero-divisors by the theorem 4 and by the corollary 3 in [5] we obtain the assertion of the theorem 5.

Remark. The theorem 5 is a converse of the theorem 1 in [4] (or theorem in [3], p. 217) for the class of homomorphisms for which $\ker \varphi \cap \bar{A}_\tau$ is a normal prime ideal in $\bar{A}_\tau$, A_τ are distributive. These conditions are fulfilled for arbitrary homomorphism, if A_τ, B_τ are rings without zero-divisors (see [7]) or for homomorphism preserving sup and inf, if A_τ, B_τ are completely ordered groups (see [4]). Other examples are in the theory of Ω-rings.

Theorem 6. *Each ideal of a strongly pseudo-ordered algebra $A \in \mathfrak{A}$ is the prime ideal.*

Proof. Let B be an ideal of strongly pseudo-ordered algebra A, $a_1, \ldots, a_n \in A$, $a_1 \ldots a_n \omega \in B$, but $a_1 \ldots a_n \omega = a_i$ for each n-ary $\omega \in \Omega''$, thus $a_i \in B$ and B is the prime ideal by the definition 4.

We can state now theorems about the homomorphic mappings of the type "into".

Theorem 7. *Let A_τ, $B_\sigma \in \mathfrak{A}$ be distributive algebras without zero-divisors and $A = \prod_{\tau \in T} A_\tau$, $B = \prod_{\sigma \in S} B_\sigma$. Let φ be a homomorphic mapping of A into B with* card $\varphi(A) > 1$. *Let $\bar{A}_\tau \cap \ker \varphi$ be a normal prime ideal in $\bar{A}_\tau$ for each $\tau \in T$. Then there exists $T' \neq \emptyset$, $T' \subseteq T$ such that $\varphi(A) = \varphi(A^*)$, where $A^* = \overline{\prod_{\tau \in T'} A_\tau}$ and $\varphi|_A{}^* = \prod_{\tau \in T'} \varphi_\tau$, where φ_τ is a homomorphic mapping of A_τ onto an algebra $B^{(\tau)}$ without zero-divisors, which is isomorphic with a subalgebra of B.*

Proof. If $\bar{A}_\tau \cap \ker \varphi$ is a normal prime ideal in $\bar{A}_\tau$, then $\varphi(\bar{A}_\tau)$ is without zero-divisors by the theorem 4. Let us denote $\varphi(\bar{A}_\tau) = \overline{B^{(\tau)}}$. Let us denote by T' the subset of T for which $\tau \in T' \Rightarrow \varphi(\bar{A}_\tau) \neq \varphi(0_A)$. From card $\varphi(A) > 1$ it follows $T' \neq \emptyset$. It is clear that $\varphi(A) = \varphi(A^*)$. Let $\tau' \neq \tau''$, $\tau', \tau'' \in T'$ and $\overline{B^{(\tau')}} \cap \overline{B^{(\tau'')}} \neq \{\varphi(0_A)\}$. Then for $b \in (\overline{B^{(\tau')}} \cap \overline{B^{(\tau'')}}) - \{\varphi(0_A)\}$ there exists $a_1 \in \bar{A}_{\tau'}$, $a_2 \in \bar{A}_{\tau''}$ such that $\varphi(a_1) = b$, $\varphi(a_2) = b$. By the lemma A in [4] we have $a_1 \neq 0_A \neq a_2$. Let ω be the direct product of regular operations, then $\varphi(a_1 a_2 \ldots a_2 \omega) = \varphi(0_A)$ because $\tau' \neq \tau''$ and $pr_\tau a_1 = 0$ for $\tau \neq \tau''$ and $pr_\tau a_2 = 0$ for $\tau' \neq \tau''$, but $\varphi(a_1)\, \varphi(a_2) \ldots \varphi(a_2)\, \omega =$

$= bb \dots b\omega \neq \varphi(0_A)$ because $b \neq \varphi(0_A)$ and by the lemma A in [4] $\varphi(A)$ has no zero different from $\varphi(0_A)$. By the theorem 4 $\overline{B^{(\tau)}}$ is without zero-divisors for each $\tau \in T$. From this contradiction there follows $B^{(\tau')} \cap B^{(\tau'')} = \{\varphi(0_A)\}$ for arbitrary $\tau', \tau'' \in T'$, $\tau' \neq \tau''$. From this we obtain $\varphi(A) = \prod_{\tau \in T'} B^{(\tau)}$ and $\varphi|_{A^*} = \prod_{\tau \in T'} \varphi_\tau$, where $\varphi_\tau = pr_\tau \varphi$ and $B^{(\tau)} = \varphi_\tau(A_\tau)$.

q.e.d.

Corollary 8. *Let $A_\tau \in \mathfrak{A}$ be distributive strongly pseudo-ordered algebras for $\tau \in T$, $A = \prod_{\tau \in T} A_\tau$ and $B \in \mathfrak{A}$. Let φ be a homomorphic mapping of A into B with* card $\varphi(A) > 1$. *Then there exists $T' \neq \emptyset$, $T' \subseteq T$ such that $\varphi(A) = \varphi(A^*)$, where $A^* = \overline{\prod_{\tau \in T'} A_\tau}$ and $\varphi|_{A^*} = \prod_{\tau \in T'} \varphi_\tau$, where φ_τ is a homomorphic mapping of A_τ onto a distributive strongly pseudo-ordered algebra isomorphic with a subalgebra of B.*

The proof follow directly from the theorems 6 and 7.

Remark. Direct products of distributive strongly pseudo-ordered algebras are for instance all atomic Boolean algebras and all distributive lattices generated by chains with the least (or greatest) element (see [2] and [4]).

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