Po Fang Hsieh

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ON A REDUCTION OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS AT AN IRREGULAR TYPE SINGULARITY*)

BY PO-FANG HSIEH
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1. INTRODUCTION

Consider a system of nonlinear differential equations of the form

$$x^{\sigma+1}y' = f(x, y, z), \qquad xz' = g(x, y, z), \tag{A}$$

where σ is a positive integer, x is a complex independent variable, f and y are m-dimensional vectors, g and z are n-dimensional vectors, $f(0, 0, 0) = 0$, $g(0, 0, 0) = 0$, f and g are holomorphic in a neighborhood of (0, 0, 0,) say

$$|x| < a, \qquad \|y\| < b, \qquad \|z\| < c \tag{1.1}$$

with $y = \text{col}(y_1, \ldots, y_m)$ and $\|y\| = \max |y_k|$, a, b and c being positive constants. Furthermore, the matrices $f_y(0, 0, 0)$ and $g_z(0, 0, 0)$ are nonsingular. Then, $x = 0$ is called an irregular type singularity of the system (A).

In this paper, we shall assume that:

I. $$g_z(0, 0, 0) = \text{diag}(\mu_1, \ldots, \mu_n) \equiv 1_n(\mu)$$

and

$$\text{Re}\,\mu_k > 0, \qquad (k = 1, \ldots, n). \tag{1.2}$$

II. For any $n + 1$ row vector $(l, q_1, \ldots, l_n)$ of non-negative integers such that $l + \Sigma q_k \geqq 2$, we have

$$\mu_k \neq l + \sum_{j=1}^{n} q_j\mu_j, \qquad k = 1, \ldots, n. \tag{1.3}$$

III. $f_y(0, 0, 0)$ has eigenvalues $\nu_1, \ldots, \nu_s$ with multiplicities $m_1, \ldots, m_s$ $(m_1 + \ldots + m_s = m)$, respectively, and

$$\text{Re}\,\nu_1 \geqq \text{Re}\,\nu_2 \geqq \ldots \geqq \text{Re}\,\nu_s > 0. \tag{1.4}$$

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Under these assumptions, we shall reduce the equations (A) analytically into a form as simple as possible.

In addition to I, II and III, with more restrictive conditions that the coefficients up to the degree of z^σ are all *diagonal* when $f_y(x, 0, z)$ is expanded in powers of z, namely the so-called conditions (A_σ), a similar reduction is made and a general solutions of the reduced equations are studied in [3]. This paper will not assume the conditions (A_σ) as in [3], thus the reduced equations can not be solved by quadrature as in previous case. A much simpler case of the equations (A) is originally studied by *M.* Iwano [4] by the use of two fundamental existence theorems. We shall employ these theorems as well as the method of proof here. But we will use somewhat different approach of successive approximations method which is developed in [2].

2. PRELIMINARY REDUCTION

Without loss of generality, we can assume that $\mathscr{A} = f_y(0, 0, 0)$ is in the Jordan canonical form

$$\mathscr{A} = \sum_{i=1}^{s} \oplus (v_i 1_{m_i} + D_i), \tag{2.1}$$

where 1_m denote the m by m identity matrix and

$$D_i = \begin{pmatrix} 0 & & & 0 \\ \delta_{i2} & \ddots & & \\ & \ddots & \ddots & \\ 0 & & \delta_{im_i} & 0 \end{pmatrix}. \qquad \delta_{ij} = 1 \text{ or } 0. \tag{2.2}$$

Let

$$A_i(x) = \frac{-v_i}{\sigma x^\sigma}, \qquad (i = 1, \dots, s) \tag{2.3}$$

and

$$A_{ij}(x) = A_i(x) - A_j(x), \qquad (i, j = 1, \dots, s;\ i \neq j). \tag{2.4}$$

A sector $\Theta_1 < \arg x < \Theta_2$ is said to have *Property-τ* with respect to $A_i(x)$ if

$$|\arg A_i(x)| < \frac{3\pi}{2} \qquad (\mathrm{mod}\ 2\pi), \tag{2.5}$$

for $\Theta_1 < \arg x < \Theta_2$. Note that for a given set of complex constants $\{v_1, \dots, v_s\}$, we can choose $\arg v_i$ properly that there exists a non-empty sector which contains a preassigned direction in the complex plane and has *Property-τ* with respect to $\{A_1(x), \dots, A_s(x)\}$.

The symbol $f[x; z]$ denotes a polynomial of degree σ in x. $f[x; z]$ is said to have *Property-σ* with respect to x for $\| z \| < c$ if the coefficients of this polynomial are holomorphic functions of z for $\| z \| < c$.

A vector $f(x, y, z)$ which is holomorphic in (x, y, z) for

$$(2.6)\qquad 0 < |x| < a, \qquad \theta_1 < \arg x < \theta_2, \qquad \|y\| < b, \qquad \|z\| < c$$

is said to have *Property-*𝔘 with respect to y and z in (2.6) if its components admit uniformly convergent expansions in powers of y and z for (2.6) and if the coefficients of these expansions are holomorphic and bounded in x for

$$(2.7)\qquad 0 < |x| < a, \qquad \theta_1 < \arg x < \theta_2$$

and admit asymptotic expansions in powers of x as x tends to 0 in (2.7).

For a row vector $p = (p_1, \ldots, p_m)$ of non-negative integer elements, denote $|p| = p_1 + \ldots + p_m$ and $y^p = y_1^{p_1} y_2^{p_2} \ldots y_m^{p_m}$.

We have established in [3] the following.

Theorem 1. *Assume that the Assumptions* I *and* II *are satisfied. Let* $\underline{\theta}_1 < \arg x < \bar{\theta}_1$ *be a sector with Property-*τ *with respect to* $\{A_i(x), A_{ij}(x) \mid i, j = 1, \ldots, s;\ i \neq j\}$ *containing the positive real axis. Then, there is a transformation*

$$(T_1)\qquad y = P(x, Z)\,Y + \Phi(x, Z), \qquad z = \psi(x, Z)$$

such that

(i) $P(x, Z)$, $\Phi(x, Z)$ *and* $\Psi(x, Z)$ *are* m *by* m, m *by* 1 *and* n *by* 1 *matrices, respectively,* $\Phi(x, Z)$ *and* $\Psi(x, Z)$ *are in the form*

$$(2.8)\quad \Phi(x, Z) = \varphi[x, Z] + x^{\sigma+1}\Phi^0(x, Z), \qquad \Psi(x, Z) = \psi[x, Z] + x^{\sigma+1}\psi^0(x, Z)$$

where $\varphi[x, Z]$ *and* $\psi[x, Z]$ *have Property-*σ *with respect to* x *for* $\|Z\| < c_1$. $P(x, Z)$, $\Phi^0(x, Z)$ *and* $\Psi^0(x, Z)$ *have Property-*𝔘 *with respect to* z *in*

$$(2.9)\qquad 0 < |x| < a_1, \qquad \underline{\theta}_1 < \arg x < \bar{\theta}_1, \qquad \|Z\| < c_1$$

for suitably chosen a_1 *and* c_1, *and, in particular*

$$(2.10)\qquad P(0, 0) = 0, \qquad \left.\frac{\partial}{\partial Z}\psi[x, Z]\right|_{\substack{x=0\\ Z=0}} = 1_n$$

(ii) *The equations* (A) *are reduced to*

$$(\mathrm{B})\qquad \begin{cases} x^{\sigma+1}Y' = (\mathscr{A} + C(x, Z))\,Y + \sum_{|p|\geqq 2} Y^p F_p(x, Z) \\ xZ' = 1_n(\mu)\,Z + D(x, Z)\,Y + \sum_{|p|\geqq 2} Y^p G_p(x, Z) \end{cases}$$

where the right hand sides are uniformly convergent for

$$(2.11)\quad 0 < |x| < a_1, \qquad \underline{\theta}_1 < \arg x < \bar{\theta}_1, \qquad \|Y\| < b_1, \qquad \|Z\| < c_1$$

for suitable b_1, $C(x, Z)$, $D(x, Z)$, $F_p(x, Z)$ *and* $G_p(x, Z)$ *are* m *by* m, n *by* m, m *by* 1 *and n by 1 matrices respectively, whose elements having Property-𝔘 with respect to Z* (2.9) *and*

$$C(0,0) = 0, \qquad D(0,0) = 0. \tag{2.12}$$

In particular, $C(x, Z)$ is in block-diagonal form

$$C(x, Z) = \sum_{j=1}^{s} \oplus C_j(x, Z), \tag{2.13}$$

where $C_j(x, Z)$ are m_j by m_j matrices, respectively, and having the forms

$$C_j(x, Z) = c_j[x; Z] + x^{\sigma+1}C_j^0(x, z) \tag{2.14}$$

with $c_j[x; Z]$ having Property-σ with respect to x for $\| Z \| < c_1$ and $C_j^0(x, Z)$ having Property-𝔘 with respect to Z in (2.9).

This Theorem is proved by applying a reduction from a particular solution constructed by Iwano [4] and the block diagonalization similar to that obtained by the author in [2].

3. MAIN REDUCTION

In order to study the main reduction of the equations (A), denote given $p = (p_1, \ldots, p_m)$ according to the multiplicities of v_i, $(i = 1, \ldots, s)$

$$\hat{p}_1 = (p_1, \ldots, p_{m_1}),\ \hat{p}_2 = (p_{m_1+1}, \ldots, p_{m_1+m_2}) \ldots, \hat{p}_s = (p_{m-m_s+1}, \ldots, p_m). \tag{3.1}$$

Namely, $p = (\hat{p}_1, \ldots, \hat{p}_s)$. Let

$$\begin{gathered} \mathcal{R}_j = \{p \mid v_j = \sum_{i=1}^{s} |\hat{p}_i| v_i,\ |p| \geqq 2\}, \qquad (i = 1, \ldots, s), \\ \mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_s. \end{gathered} \tag{3.2}$$

Then, by Assumption III. $\mathcal{R}$ is a *finite* set.

Consider the monomial of the form

$$\{\Omega_{jp}(x) = \Lambda_j(x) - \sum_{i=1}^{s} |\hat{p}_i| \Lambda_i(x) \mid j = 1, \ldots, s; 2 \leqq |p| \leqq M'\}, \tag{3.3}$$

where M' is a sufficiently large positive integer. Since all these monomials have the same degree with respect to x^{-1}, it is easy to verify that in the sector $\underline{\theta}_1 < \arg x < \bar{\theta}_1$, there exists a subsector $\underline{\theta}_2 < \arg x < \bar{\theta}_2$ which has Property-τ with respect to the monomials in (3.3) and contains the positive real axis. Then by virtue of Assumptions III, the sector $\underline{\theta}_2 < \arg x < \bar{\theta}_2$ has Property-τ with respect to all the monomials

$$\{\Omega_{jp}(x) = \Lambda_j(x) - \sum_{i=1}^{s} |\hat{p}_i| \Lambda_i(x) \mid j = 1, \ldots, s; |p| \geqq 2\}. \tag{3.4}$$

The main result of this paper is the following:

Theorem 2. *Assume that Assumption* III *is satisfied. Then, there exists a transformation*

$$(\mathrm{T}_2) \qquad Y = u + \sum_{|p| \geqq 2} u^p A_p(x, v), \qquad Z = v + x^\sigma \sum_{|p| \geqq 2} u^p B_p(x, v),$$

such that

(i) $A_p(x, v)$ *and* $B_p(x, v)$ *are m-column and n-column vectors, respectively, and have Property-*$\mathfrak{A}$ *with respect to* v *in*

$$(3.5) \qquad 0 < |x| < a_2, \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2, \qquad \| v \| < c_2$$

for suitable positive a_2 *and* c_2, *and the right hand sides of* (T_2) *converge uniformly in*

$$(3.6) \quad 0 < |x| < a_2, \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2, \qquad \| u \| < b_2, \qquad \| v \| < c_2$$

for suitable positive b_2;

(ii) (T_2) *reduces* (B) *to*

$$(\mathrm{C}) \qquad \begin{cases} x^{\sigma+1} u' = (\mathscr{A} + C(x, v))\, u + \sum_{p \in \mathscr{R}} u^p H_p(x, v). \\ \quad xv' = 1_n(\mu)\, v, \end{cases}$$

where $H_p(x, v)$ *are m-column vectors and have Property-*$\mathfrak{A}$ *with respect to* v *in* (3.5).

The proof of this theorem is to be given in two parts; formal reduction in § 5 and analytic reduction in §§ 6 – 8.

4. FUNDAMENTAL EXISTENCE THEOREMS

In order to prove Theorem 2, we need two existence theorems. These are first proved by Iwano [4] by the use of Tychonoff type fixed point theory, and later by the author [1] by means of successive approximations. The proof will not be repeated here.

For an α-column vector γ with elements $\{\gamma_j\}$, $1_\alpha(\gamma)$ denotes the diagonal α by α matrix with $\{\gamma_j\}$ as its diagonal elements, while x^γ denotes the α-column vector with $\{x^{\gamma_j}\}$ as its elements. Also, e^γ denotes col $(e^{\gamma_1}, \ldots, e^{\gamma_\alpha})$.

Given a system of nonlinear equations of the form:

$$(4.1) \qquad x^{\sigma+1} \eta' = J(x, \eta, \zeta), \qquad x\zeta' = K(x, \eta, \zeta).$$

Here we assume that:

(i) x is a complex independent variable and σ is a positive integer.

(ii) η and ζ are α- and β-column vectors, respectively.

(iii) $J(x, \eta, \zeta)$ and $K(x, \eta, \zeta)$ are α- and β-column vectors, respectively, whose components have Property-$\mathfrak{A}$ with respect to η and ζ in the domain

$$(4.2) \qquad 0 < |x| < \xi, \qquad \underline{\Theta} < \arg x < \bar{\Theta}, \qquad \| \eta \| < d, \qquad \| \zeta \| < d,$$

where $\underline{\Theta}, \bar{\Theta}$, ξ and d are constants with ξ and d positive.

(iv) The matrices J_η and J_ζ satisfy

$$J_\eta(0, 0, 0) = 1_\alpha(\gamma) + D, \qquad \det 1_\alpha(\gamma) \neq 0, \qquad J_\zeta(0, 0, 0) = 0,$$

where γ is an α-column vector with elements $\{\gamma_j\}$ and D is an α by α nilpotent matrix in lower triangular form.

(v) Equations (4.1) possess a formal solution

$$\eta \sim \sum_{i=0}^{\infty} x^i J_i, \qquad \zeta \sim \sum_{i=0} x^i K_i, \tag{4.3}$$

where J_i and K_i are constant α- and β-column vectors, respectively, and in particular

$$\| J_0 \| < d, \qquad \| K_0 \| < d.$$

Let

$$\Omega_j(x) = -\frac{\gamma_j}{\sigma x^\sigma}, \qquad (j = 1, 2, \ldots, \alpha).$$

The first existence theorem is as follows.

Theorem A. *Assume that in the sector* $\underline{\Theta} < \arg x < \bar{\Theta}$, *there exists a subsector* $\underline{\Theta}^* < \arg x < \bar{\Theta}^*$ *which has Property-τ with respect to* $\{\Omega_1(x), \ldots, \Omega_\alpha(x)\}$. *Then,* (4.1) *have a unique solution* $\{F(x), G(x)\}$ *which is holomorphic and bounded in* x *for*

$$0 < | x | < \xi_0, \qquad \underline{\Theta}^* < \arg x < \bar{\Theta}^*, \tag{4.4}$$

where $0 < \xi_0 \leqq \xi$, *and which admits asymptotic expansions of the forms* (4.3) *as* x *tends to zero in the sector* (4.4).

Let μ be a given n-column vector with elements $\{\mu_k \mid \operatorname{Re} \mu_k > 0\}$. The second existence theorem concerns about a system of equations similar to (4.1), except that the vectorial functions J and K, besides x, η and ζ, depend on an arbitrary function of the form $V(x) = 1_n(x^\mu)\, C$, where C is an arbitrary n-column vector. Namely, the following system:

$$x^{\sigma+1}\eta' = J(x, V(x); \eta, \zeta), \qquad x\zeta' = K(x, V(x); \eta, \zeta). \tag{4.5}$$

Here we assume that

(i′) $J(x, v: \eta, \zeta)$ and $K(x, v; \eta, \zeta)$ are α- and β-column vectors, respectively, which admit uniformly convergent series in powers of η and ζ in the domain

$$0 < | x | < \xi, \qquad \underline{\Theta} < \arg x < \bar{\Theta}, \qquad \| v \| < \delta, \qquad \| \xi \| < d, \tag{4.6}$$

whose coefficients are functions with Property-$\mathfrak{U}$ with respect to v in

$$0 < | x | < \xi, \qquad \underline{\Theta} < \arg x < \bar{\Theta}, \qquad \| v \| < \delta \tag{4.7}$$

with δ a positive constant.

(ii′) The matrices J_η and J_ζ satisfy

$$(4.8)\qquad J_\eta(0, 0; 0, 0) = 1_\alpha(\gamma) + D, \qquad \det 1_\alpha(\gamma) \neq 0, \qquad J_\zeta(0, 0; 0, 0) = 0.$$

(iii′) Equations (4.5) have a formal solution of the form

$$(4.9)\qquad \eta \sim \sum_{|q|=0}^{\infty} V(x)^q J_q(x), \qquad \zeta \sim \sum_{|q|=0}^{\infty} V(x)^q K_q(x),$$

where $q = (q_1, \ldots, q_n)$ with q_k non-negative integers, $J_q(x)$ and $K_q(x)$ are α- and β-column vectors functions, respectively, holomorphic in

$$(4.10)\qquad 0 < |x| < \xi, \qquad \underline{\Theta} < \arg x < \overline{\Theta}$$

and admit asymptotic expansions

$$(4.11)\qquad J_q(x) \cong \sum_{i=0}^{\infty} J_{qi} x^i, \qquad K_q(x) \cong \sum_{i=0}^{\infty} K_{qi} x^i$$

as x tends to zero in (4.10). In particular,

$$\| J_0(x) \| < d, \qquad \| K_0(x) \| < d.$$

Now, the second existence theorem is stated as following:

Theorem B. *Assume that, in the sector* $\underline{\Theta} < \arg x < \overline{\Theta}$, *there exists a subsector* $\underline{\Theta}^* < \arg x < \overline{\Theta}^*$ *which has Property-τ with respect to* $\{\Omega_1(x), \ldots, \Omega_\alpha(x)\}$. *Then, the equations* (4.5) *have a solution of the form* $\{F(x, V(x)), G(x, V(x))\}$ *whenever* x *and* $V(x)$ *are in*

$$(4.12)\qquad 0 < |x| \leqq \xi_0, \qquad \underline{\Theta}^* < \arg x < \overline{\Theta}^*, \qquad \| v \| < \delta_0,$$

where $0 < \xi_0 \leqq \xi$, $0 < \delta_0 \leqq \delta$. *Furthermore, this solution admits uniformly convergent expansions of the form* (4.9) *so that* $F(x, v)$ *and* $G(x, v)$ *are* α- *and* β-*column vector functions with Property-𝔘 with respect to* v *in the domain* (4.12).

5. FORMAL REDUCTION

For a row vector $p = (p_1, \ldots, p_m)$ and a column vector $y = \mathrm{col}(y_1, \ldots, y_m)$, denote by $p \cdot y = p_1 y_1 + \ldots + p_m y_m$.

Let $h(x, v)$ denote the m-column vector consists of the diagonal elements, in its corresponding order, of $\mathscr{A} + C(x, v)$.

Differentiate (T_2) formally, and by (C), we get

$$(5.1)\qquad x^{\sigma+1} Y' = (\mathscr{A} + C(x, v))\, u + \sum_{p \in \mathscr{R}} u^p H_p(x, v) +$$

$$+ \sum_{|p| \geqq 2} u^p \{x^{\sigma+1} A_p'(x, v) + p \cdot h(x, v)\, A_p + \sum_{\substack{|q|=|p| \\ q \neq p}} f_q(x, v)\, A_q + \mathring{R}_p(x, v\,; H_{p'}, A_{p'})\},$$

$$(5.2)\qquad xZ' = 1_n(\mu)\,v + \sum_{|p|\geqq 2} u^p\{x^{\sigma+1}B_p(x,v) + p\,.\,h(x,v)\,B_p + \\ + \sum_{\substack{|q|=|p|\\ q\neq p}} f_q(x,v)\,B_q + \mathring{S}_q(x,v;H_{p'},B_{p'})\},$$

where $f_q(x, v)$ are positive integer multiple of the off-diagonal elements of the matrix $\mathscr{A} + C(x, v)$, $\mathring{R}_p$ and $\mathring{S}_p$ are linear combination of $H_{p'}$ with coefficients depending on $A_{p'}$ and $B_{p'}$ ($|p'| < |p|$) respectively.

On the other hand, substituting (T_2) into (B), we get

$$(5.3)\qquad x^{\sigma+1}Y' = (\mathscr{A} + C(x,v))\,u + \sum_{|p|\leqq 2} u^p\{(\mathscr{A} + C(x,v))\,A_p + \\ + F_p(x,v) + \tilde{R}_p(x,v;A_{p'},B_{p'})\},$$

$$(5.4)\qquad xZ' = 1_n(\mu)\,v + \sum_{|p|\geqq 2} u^p\{x^\sigma 1_n(\mu)\,B_p + G_p(x,v) + \tilde{S}_p(x,v;A_{p'},B_{p'})\},$$

where $\tilde{R}_p$ and $\tilde{S}_p$ are m- and n-column vectors linear in known vectors whose coefficients are polynomials of $A_{p'}$ and $B_{p'}$ ($|p'| < |p|$).

Let $V(x) = 1_n(x^\mu)\,C$, where C is an arbitrary constant n-column vector; namely, $V(x)$ is a general solution of the second equations of (C). Compare the coefficients of u^p in (5.1) with (5.3) and (5.2) with (5.4), and replace v by $V(x)$, we have for $|p| \geqq 2$,

(5.5)
$$x^{\sigma+1}A_p' = \{\mathscr{A} + C(x,V(x)) - p\,.\,h(x,V(x))\,1_n\}\,A_p - \sum_{\substack{|q|=|p|\\ q\neq p}} f_q(x,V(x))\,A_q + \\ + R_p(x,V(x)) + H_p(x,V(x)),$$

(5.6)
$$x^{\sigma+1}B_p' = \{-p\,.\,h(x,V(x))\,1_n + x^\sigma\,1_n(\mu)\}\,B_p - \sum_{\substack{|q|=|p|\\ q\neq p}} f_q(x,V(x))\,B_q + S_p(x,V(x)),$$

where

$$R_p = \tilde{R}_p - \mathring{R}_p + F_p, \qquad S_p = \tilde{S}_p - \mathring{S}_p + G_p.$$

We shall determine A_p and B_p successively from (5.5) and (5.6) in the increasing order of $|p|$, and choose H_p in a manner that A_p and B_p are as simple as possible and they have Property-𝔘 with respect to v in the domain of the form of (3.5).

Let A_{pi}, H_{pi} and R_{pi} ($i = 1, \ldots, s$) be m_i-column vectors according to (3.1) such hat

$$A_p = \mathrm{col}\,(\hat{A}_{p1}, \ldots, \hat{A}_{ps}), \qquad H_p = \mathrm{col}\,(\hat{H}_{p1}, \ldots, \hat{H}_{ps}),$$
$$R_p = \mathrm{col}\,(\hat{R}_{p1}, \ldots, \hat{R}_{ps}).$$

Case I. If p is in $\mathscr{R}_j$ for some j, then choose

$$(5.7)\qquad \hat{A}_{pj}(x,v) \equiv 0.$$

Case II. If p is not in $\mathscr{R}_j$, then, choose

$$\hat{H}_{pj}(x, v) \equiv 0. \tag{5.8}$$

In order to determine A_{pj} in Case II and all the B_p, let

$$\begin{aligned} \mathscr{A}_M &= \operatorname{col}\{A_{pj} \mid p \notin \mathscr{R}_j \quad (j = 1, \ldots, s), \ |p| = M\}, \\ \mathfrak{B}_M &= \operatorname{col}\{B_p \mid |p| = M\}. \end{aligned} \tag{5.9}$$

Then, $\mathscr{A}_M$ and $\mathfrak{B}_M$ satisfy the following equations:

$$x^{\sigma+1}\mathscr{A}'_M = \mathscr{F}_M(x, V(x))\,\mathscr{A}_M + \mathscr{K}_M(x, V(x))\,\mathscr{A}_M + \mathscr{T}_M(x, V(x)), \tag{5.10}$$

$$x^{\sigma+1}\mathfrak{B}'_M = \mathscr{G}_M(x, V(x))\,\mathfrak{B}_M + \mathscr{L}_M(x, V(x))\,\mathfrak{B}_M + \mathscr{S}_M(x, V(x)), \tag{5.11}$$

where the matrices $\mathscr{F}_M$, $\mathscr{K}_M$, $\mathscr{T}_M$, $\mathscr{G}_M$, $\mathscr{L}_M$ and $\mathscr{S}_M$ are of known quantities having Property-$\mathfrak{U}$ with respect to v in (2.9). Furthermore, $\mathscr{F}_M$ and $\mathscr{G}_M$ are non-singular at (0, 0) while $\mathscr{K}_M$ and $\mathscr{L}_M$ are singular at (0, 0).

Put

$$\mathscr{A}_M = \mathscr{A}_{M0}(x) + \sum_{|q|=1}^{\infty} V(x)^q\,\mathscr{A}_{Mq}(x), \tag{5.12}$$

$$\mathfrak{B}_M = \mathfrak{B}_{M0}(x) + \sum_{|q|=1}^{\infty} V(x)^q\,\mathfrak{B}_{Mq}(x). \tag{5.13}$$

Then each of $\mathscr{A}_{M0}$, $\mathscr{A}_{Mq}$, $\mathfrak{B}_{M0}$ and $\mathfrak{B}_{Mq}$ satisfies a certain differential equation. By the fact that $\mathscr{F}_M$, $\mathscr{G}_M$ are non-singular at (0, 0) and $\mathscr{K}_M$, $\mathscr{L}_M$ are singular at (0, 0), we can get $\mathscr{A}_{M0}$, $\mathscr{A}_{Mq}$, $\mathfrak{B}_{M0}$ and $\mathfrak{B}_{Mq}$, successively in $|q|$, as formal power series solutions of their respective equations. By means of Theorem A in § 4, we can find $\mathscr{A}_{M0}(x)$, $\mathscr{A}_{Mq}(x)$, $\mathfrak{B}_{M0}(x)$ and $\mathfrak{B}_{Mq}(x)$ holomorphic in a domain of the form

$$0 < |x| < a'_1, \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2, \qquad (0 < a'_1 < a_1), \tag{5.14}$$

and admit aforementioned formal solutions as their asymptotic expansions as x tends to zero in $\underline{\theta}_2 < \arg x < \bar{\theta}_2$.

Thus (5.10) and (5.11) have formal solutions (5.12) and (5.13), respectively.

Now, by the use of Theorem B in § 4, (5.10) and (5.11) have solutions $\mathscr{A}_M(x, V(x))$, and $\mathfrak{B}_M(x, V(x))$, respectively, whenever $(x, V(x))$ is in

$$\begin{gathered} 0 < |x| < a''_1, \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2, \qquad \|v\| < c'_1, \\ (0 < a''_1 \leqq a'_1,\ 0 < c'_1 \leqq c_1), \end{gathered} \tag{5.15}$$

and admit uniformly convergent expansions (5.12) and (5.13) so that they have Property-$\mathfrak{U}$ with respect to v in (5.15). It is noteworthy that (5.15) is valid for all M when M is large.

Therefore, $A_p(x, V(x))$ and $B(x, V(x))$ are obtained for all p in (5.15). Consequently, $H_{pj}(x, V(x))$ for p not in $\mathscr{R}_j$ is obtained from (5.5). Thus the right hand side of (T_2) are obtained as a formal series.

6. ANALYTIC REDUCTION

Let $\{U(x), V(x)\}$ be a general solution of (C) which is holomorphic in the domain (5.15). As we have seen in § 5, the equations (B) have a formal solution of the form

$$(6.1)\qquad \begin{cases} Y \sim U(x) + \sum_{|p| \geqq 2} U(x)^p A_p(x, V(x)), \\ Z \sim V(x) + x^\sigma \sum_{|p| \geqq 2} U(x)^p B_p(x, V(x)), \end{cases}$$

where $A_p(x, v)$ and $B_p(x, v)$ have Property-$\mathfrak{U}$ with respect to v in (5.15).

In order to prove the uniform convergence of (6.1), let

$$\gamma_j = \mathrm{Re}\, \nu_j, \qquad |\gamma_p| = \gamma_1 |\hat{P}_1| + \gamma_2 |\hat{P}_2| + \ldots + \gamma_s |\hat{P}_s|,$$

and N be a positive integer. Put

$$(6.2)\qquad \begin{cases} P_N(x, u, v) = u + \sum_{|\gamma p| < N} u^p A_p(x, v), \\ Q_N(x, u, v) = v + x^\sigma \sum_{|\gamma p| < N} u^p B_p(x, v). \end{cases}$$

We make a change of variables

$$(6.3)\qquad Y = P_N(x, U(x), V(x)) + \eta, \qquad Z = Q_N(x, U(x), V(x)) + \zeta$$

to equations (C). Since $\{U(x), V(x)\}$ is a formal solution and

$$x^{\sigma+1} \frac{\mathrm{d}}{\mathrm{d}x} P_N(x; U(x), V(x)) = x^{\sigma+1} \frac{\partial}{\partial x} P_N(x, U(x), V(x)) +$$

$$+ \frac{\partial}{\partial U(x)} P_N(x, U(x), V(x)) \{(\mathscr{A} + C(x, V(x)) U(x) + \sum_{p \in \mathscr{R}} U(x)^p H_p(x, V(x))\} +$$

$$+ x^\sigma \frac{\partial}{\partial V(x)} P_N(x, U(x), V(x)) 1_n(\mu) V(x),$$

this expression is determined uniquely as a function of $(x, U(x), V(x))$. Similarly, $x^{\sigma+1} \dfrac{\mathrm{d}}{\mathrm{d}x} Q_N(x, U(x), V(x))$ is determined uniquely as a function of $(x, U(x), V(x))$. Thus, $\{\eta, \zeta\}$ satisfy the differential equations

$$(6.4)\qquad \begin{cases} x^{\sigma+1} \eta' = (\sum_{i=1}^{s} \nu_i 1_{m_i}) \eta + F(x, U(x), V(x); \eta, \zeta), \\ x\zeta' = G(x, U(x), V(x); \eta, \zeta), \end{cases}$$

where $F(x, u, v; \eta, \zeta)$ and $G(x, u, v; \eta, \zeta)$ are, respectively, m- and n-column vector functions holomorphic and bounded in $(x, u, v; \eta, \zeta)$ for a domain of the form

$$(6.5)\qquad \begin{cases} 0 < |x| < \xi_N, \quad \underline{\theta}_2 < \arg x < \bar{\theta}_2, \quad \|u\| < \delta_N, \quad \|v\| < \delta_N, \\ \|\eta\| < d_N, \quad \|\zeta\| < d_N, \end{cases}$$

for suitably chosen positive constants ξ_N, δ_N, and d_N.

Since (6.4) have a formal solution

$$(6.6)\qquad \tilde{\eta} \sim \sum_{|\gamma p| \geqq N} U(x)^p A_p(x, V(x)), \qquad \tilde{\zeta} \sim x^\sigma \sum_{|\gamma p| \geqq N} U(x)^p B_p(x, V(x)),$$

the functions F and G satisfy the inequalities

$$(6.7)\qquad \begin{cases} \| F(x, u, v; \hat{\eta}, \hat{\zeta}) \| \leqq A(\| \hat{\eta} \| + \| \hat{\zeta} \|) + B_N \||| u |||^N, \\ \| G(x, u, v; \hat{\eta}, \hat{\zeta}) \| \leqq A(\| \hat{\eta} \| + \| \hat{\zeta} \|) + B_N ||| u |||^N, \end{cases}$$

for $(x, u, v; \hat{\eta}, \hat{\zeta})$ in (6.5), where A is a positive constant independent of N while B_N is a positive constant may depend on N. Here the norm $||| u |||$ is defined to be

$$(6.8)\qquad ||| u ||| = \max_{j=1}^{s} \{\Phi_j\}, \ \Phi_j = \| \hat{u}_j \|^{1/\gamma_j},$$

with $u = \mathrm{col}\,(\hat{u}_1, \ldots, \hat{u}_s)$, $\hat{u}_j = \mathrm{col}\,(u_{j1}, \ldots, u_{jm_j})$ according to (3.1). Furthermore, F and G satisfy the following Lipschitz condition:

$$(6.9)\qquad \begin{cases} \| F(x, u, v; \hat{\eta}^1, \hat{\zeta}^1) - F(x, u, v; \hat{\eta}^2, \hat{\zeta}^2) \| \leqq A(\| \hat{\eta}^1 - \hat{\eta}^2 \| + \| \hat{\zeta}^1 - \hat{\zeta}^2 \|), \\ \| G(x, u, v; \hat{\eta}^1, \hat{\zeta}^1) - G(x, u, v; \hat{\eta}^2, \hat{\zeta}^2) \| \leqq A(\| \hat{\eta}^1 - \hat{\eta}^2 \| + \| \hat{\zeta}^1 - \hat{\zeta}^2 \|). \end{cases}$$

for $(x, u, v; \hat{\eta}^1, \hat{\zeta}^1)$ and $(x, u, v; \hat{\eta}^2, \hat{\zeta}^2)$ in (6.5).

Let

$$\Lambda(x) = \mathrm{col}\,(\Lambda_1(x), \ldots, \Lambda_1(x), \Lambda_2(x), \ldots, \Lambda_2(x), \ldots, \Lambda_s(x), \ldots, \Lambda_s(x)),$$

where $\Lambda_j(x)$ is defined in (2.3) and appears m_j times. Put

$$(6.10)\qquad \hat{\eta} = 1_m(e^{\Lambda(x)})\, P, \qquad \hat{\zeta} = Q.$$

Then, (6.4) becomes

$$(6.11)\qquad \begin{cases} P' = x^{-\sigma-1}\, 1_m(e^{-\Lambda(x)})\, F(x, U(x), V(x); 1_m(e^{\Lambda(x)})\, P, Q), \\ Q' = x^{-1} G(x, U(x), V(x); 1_m(e^{\Lambda(x)})\, P, Q). \end{cases}$$

For two n-vectors v and $\tilde{v}$ with elements $\{v_k\}$ and $\{\tilde{v}_k\}$, respectively, we denote $[v] < [\tilde{v}]$ if $|v_k| < |\tilde{v}_k|$ for $k = 1, \ldots, n$.

The task of proving uniform convergence of (6.1) becomes proving the following:

Theorem 3. *Let ε be a preassigned positive constant, and N be a positive integer satisfying*

$$(6.12)\qquad N \geqq \frac{4}{\sin 2\sigma\varepsilon} \max\,(2A, \| \nu \|), \qquad (\| \nu \| = \max_{j=1}^{s} |\nu_j|).$$

Then, (6.11) *have a unique solution* $\{\varphi_N(x, U(x), V(x)), \psi_N(x, U(x), V(x))\}$ *such that*

$$(6.13)\qquad [\varphi_N] = K_N\, |||\, U(x)\, |||^N [e^{-\Lambda(x)}], \qquad \|\, \psi_N \,\| = K_N\, |||\, U(x)\, |||^N$$

for suitably chosen positive constant K_N *whenever* $(x, U(x), V(x))$ *is a domain of the form*

$$(6.14)_N\qquad \begin{cases} 0 < |\, x \,| < \xi'_N \omega(\arg x), & \underline{\theta}_2 < \arg x < \bar{\theta}_2, \\ |||\, u \,||| < \delta'_N, & [v] < \delta'_N[\chi(\arg x)]. \end{cases}$$

Here, $\chi(t)$ *is an n-column vector with elements* $\{\chi_k(t)\}$, *and* $\omega(t)$ *and* $\chi_k(t)$ *are positive, continuous and bounded functions of* $\underline{\theta}_2 \leqq t \leqq \bar{\theta}_2$, $\varphi_N(x, u, v)$ *and* $\psi_N(x, u, v)$ *are m- and n-column vectors, respectively, whose components are holomorphic and bounded functions in* $(6.14)_N$, δ'_N *and* ξ'_N *are suitably chosen positive constants, and*

$$(6.15)\qquad \xi'_N \max_{\underline{\theta}_2 \leqq t \leqq \bar{\theta}_2} \omega(t) < 1.$$

The convergence of (6.1) follows from this Theorem in the following manner. Owing to the transformations (6.3) and (6.10), the functions

$$(6.16)\qquad \begin{cases} U(x) + \sum_{|\gamma p| < N} U(x)^p A_p(x, V(x)) + 1_m(e^{\Lambda(x)})\, \varphi_N(x, U(x), V(x)), \\ V(x) + x \sum_{|\gamma p| < N} U(x)^p B_p(x, V(x)) + \psi_N(x, U(x), V(x)) \end{cases}$$

are a solution of (B) provided that $(x, U(x), V(x))$ is in the domain $(6.14)_N$. Let N' be an integer greater than N. Then,

$$(6.17)\qquad \begin{cases} 1_m(e^{-\Lambda(x)}) \sum_{N \leqq |\gamma p| < N'} U(x)^p A_p(x, V(x)) + \varphi_{N'}(x, U(x), V(x)), \\ x^\sigma \sum_{N \leqq |\gamma p| < N'} U(x)^p B_p(x, V(x)) + \psi_{N'}(x, U(x), V(x)) \end{cases}$$

are a solution of (6.11), satisfying (6.13) if $(x, U(x), V(x))$ belongs to the common part of the domains $(6.14)_N$ and $(6.14)_{N'}$. Hence, by the uniqueness of solution, (6.17) must coincide with $\{\varphi_N(x, U(x), V(x)), \psi_N(x, U(x), V(x))\}$. Thus the solution of (B) expressed by (6.16) is independent of N, provided that N satisfies (6.12). We denote this solution by $\{\tilde{\varphi}(x, U(x), V(x)), \tilde{\psi}(x, U(x), V(x))\}$. Then, by analytic continuation, the functions $\tilde{\varphi}(x, u, v)$ and $\tilde{\psi}(x, u, v)$ are defined in the domain

$$(6.18)\qquad \begin{cases} 0 < |\, x \,| < \xi_0 \omega(\arg x), & \underline{\theta}_2 < \arg x < \bar{\theta}_2, \\ |||\, u \,||| < \delta_0, & [v] < \delta_0[\chi(\arg x)], \end{cases}$$

where $\xi_0 = \sup \xi'_N$, $\delta_0 = \sup \delta'_N$. By the facts that ω and χ are positive, continuous and bounded functions, (3.6) and (6.18) are equivalent in the sense that $(6.14)_N$ is contained in (3.6) if δ'_N and ξ'_N are chosen suitably, and vice versa.

On the other hand, since $u = 0$ is an interior point of (3.6) in which $\tilde{\varphi}(x, u, v)$ and $\tilde{\psi}(x, u, v)$ are defined, therefore, by Cauchy's Theorem, $\tilde{\varphi}(x, U(x), V(x))$ and $\tilde{\psi}(x, U(x), V(x))$ can be expanded into uniformly convergent power series of $U(x)$ whenever $(x, U(x), V(x))$ is in (3.6). Clearly, from Theorem 3, we know that $\tilde{\varphi}$ and $\tilde{\psi}$ admit the asymptotic expansions (6.1). By the uniqueness of the asymptotic expansions, these asymptotic expansions must coincide with the uniformly convergent expansions. This proves the uniform convergence of the formal series (6.1).

Thus, the transformation (T_2) converges uniformly in (3.6).

7. A FUNDAMENTAL LEMMA

In order to prove Theorem 3, a fundamental lemma is needed and to be established here.

Since $-\Lambda_j(x)$ are the dominating terms of the monomials $\Omega_{jp}(x)$ defined in (3.4) when $|p|$ is large, let θ_{j-} and θ_{j+} be directions in x-plane along which $\mathrm{Re}\{-\Lambda_j(x)\} = 0$ and situated immediately above and below the positive real axis. We can choose $\arg \nu_j$ so that

$$\theta_{j-} = \frac{1}{\sigma}\left(\arg \nu_j + \frac{\pi}{2}\right), \qquad \theta_{j+} = \frac{1}{\sigma}\left(\arg \nu_j - \frac{\pi}{2}\right). \tag{7.1}$$

Put

$$\theta_- = \min_{j=1}^{s} \{\theta_{j-}\}, \qquad \theta_+ = \max_{j=1}^{s} \{\theta_{j+}\}. \tag{7.2}$$

Then we can assume, without loss of generality, that the angles θ_- and θ_+ satisfy

$$0 \leqq \theta_{j-} - \theta_- \leqq \frac{\pi}{\sigma} - 6\varepsilon, \qquad 0 \leqq \theta_+ - \theta_{j+} \leqq \frac{\pi}{\sigma} - 6\varepsilon, \tag{7.3}$$

and the angles $\underline{\theta}_2, \bar{\theta}_2$ appeared in Theorem 2 satisfy

$$\theta_+ - \left(\frac{\pi}{\sigma} + 6\varepsilon\right) < \underline{\theta}_2 < \bar{\theta}_2 < \theta_- + \frac{\pi}{\sigma} - 6\varepsilon \tag{7.4}$$

for a preassigned sufficiently small positive constant ε.

We define a continuous function $L(t)$ for $\underline{\theta}_2 \leqq t \leqq \bar{\theta}_2$ by

$$L(t) = \begin{cases} \sigma(t - \theta_- + 4\varepsilon), & \theta_- - 2\varepsilon \leqq t \leqq \bar{\theta}_2, \\ \dfrac{\pi}{2}, & \theta_+ + 2\varepsilon \leqq t \leqq \theta_- - 2\varepsilon, \\ \sigma(t - \theta_+ - 4\varepsilon) + \pi, & \underline{\theta}_2 \leqq t \leqq \theta_+ + 2\varepsilon. \end{cases} \tag{7.5}$$

Then, by virtue of (7.4), $L(t)$ satisfies the inequality

$$\sigma\varepsilon \leqq L(t) \leqq \pi - \sigma\varepsilon \quad \text{for} \quad \underline{\theta}_2 \leqq t \leqq \bar{\theta}_2. \tag{7.6}$$

Now $\omega(t)$ and $\chi_k(t)$ $(k = 1, 2, \ldots, n)$ are defined as

(7.7) $$\omega(t) = \exp \int_{\theta_0}^{t} \cot L(\tau)\, d\tau$$

and

(7.8) $$\chi_k(t) = \exp \{(\mathrm{Re}\, \mu_k) \int_{\theta_0}^{t} \cot L(\tau)\, d\tau + (\mathrm{Im}\, \mu_k)(\theta_0 - t)\},$$

where θ_0 is a fixed angle satisfying $\underline{\theta}_2 \leqq \theta_0 \leqq \bar{\theta}_2$. Then, these are strictly positive, bounded and continuous functions in $\underline{\theta}_2 \leqq t \leqq \bar{\theta}_2$.

According to (6.8), let

(7.9) $$U(x) = \mathrm{col}\,(\hat{U}_1(x), \hat{U}_2(x), \ldots, \hat{U}_s(x)),$$

where $\hat{U}_j(x)$ is an m_j-column vector

(7.10) $$\hat{U}_j(x) = \mathrm{col}\,(U_{j1}(x), U_{j2}(x), \ldots, U_{jm_j}(x)).$$

Then,

(7.11) $$|||\, U(x)\, ||| = \max_{j=1}^{s} \{\Phi_j(x)\}, \qquad \Phi_j(x) = \|\, \hat{U}_j(x)\, \|^{1/\gamma_j}$$
$$\text{with} \qquad \|\, U_j(x)\, \| = \max_{l=1}^{m_j} \{|\, U_{jl}(x)\,|\}.$$

We shall establish the following fundamental

Lemma F. *Let $\{U(x), V(x)\}$ be a general solution of the equations* (C). *Let x_1, u^1, v^1 be arbitrary values in a domain of the form*

(7.12) $$0 < |\, x\,| < \xi\omega(\arg x), \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2,$$
$$[v] < \delta[\chi(\arg x)], \qquad |||\, u\, ||| < \delta,$$

where $\chi(t)$ is the n-column vector with elements $\{\chi_k(t)\}$, and ξ and δ are constants to be determined. Choose the integration constants included in $\{U(x), V(x)\}$ so that $U(x_1) = u^1$ and $V(x_1) = v^1$. Then, there exists a curve Γ_{x_1} connecting the point x_1 with the origin such that:

(i) *The curve Γ_{x_1} is entirely contained in the domain*

(7.13) $$0 < |\, x\,| < \xi\omega(\arg x), \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2,$$

except for the origin;

(ii) *As x moves on the curve Γ_{x_1}, we have the following inequalities*:

(7.14) $$[V(x)] < \delta[\chi(\arg x)], \qquad \underline{\theta}_2 < \arg x < \bar{\theta}_2,$$

(7.15) $$\frac{d}{ds}|||\, U(x)\,||| \geqq \frac{\sin 2\sigma\varepsilon}{2} |\,x\,|^{-\sigma-1}\, |||\, U(x)\,|||\,.$$

(7.16) $$\frac{d}{ds}(|||\, U(x)\,|||^N e^{-\mathrm{Re}\,\Lambda_j(x)}) \geqq \frac{N \sin 2\sigma\varepsilon}{4} |\,x\,|^{-\sigma-1}\, |||\, U(x)\,|||^N e^{-\mathrm{Re}\,\Lambda_j(x)},$$
$$(N \sin 2\sigma\varepsilon \geqq 4\|\, v\,\|;\quad j = 1, \dots, s),$$

with s the arc length of Γ_{x_1} measured from the origin to the variable point x.

To prove the lemma, let the polar coordinates of the variable point x be (ϱ, t). Then the curve Γ_{x_1} is defined as follows:

If $\underline{\theta}_2 < \arg x_1 < \theta_+ + 2\varepsilon$ or $\theta_- - 2\varepsilon < \arg x_1 < \bar{\theta}_2$, Γ_{x_1} consists of a curvilinear part Γ':

(7.17) $$\varrho = |\,x_1\,| \exp \int_{\arg x_1}^{t} \cot L(\tau)\, d\tau$$
$$\text{for } \arg x_1 \leqq t \leqq \theta_+ + 2\varepsilon \text{ or } \theta_- - 2\varepsilon \leqq t \leqq \arg x_1,$$

and of a rectilinear part Γ'':

(7.18) $$0 \leqq \varrho \leqq |\,x_1\,| \exp \int_{\arg x_1}^{t} \cot L(\tau)\, d\tau, \qquad t = \theta_+ + 2\varepsilon \qquad \text{or} \qquad \theta_- - 2\varepsilon.$$

If $\theta_+ + 2\varepsilon \leqq \arg x_1 \leqq \theta_- - 2\varepsilon$, Γ_{x_1} consists of a rectilinear part Γ'' only:

(7.19) $$0 \leqq \varrho \leqq |\,x_1\,|, \qquad t = \arg x_1\,.$$

From the definitions of Γ_{x_1} and the functions ω and χ, the curve Γ_{x_1} is contained entirely in (7.13) except for the origin. Also, $V(x)$ satisfies (7.14) as x moves on Γ_{x_1}.

In order to show (7.15), let

(7.20) $$\hat{H}_{pj}(x, v) = \mathrm{col}\,(H_{pj1}, \dots, H_{pjm_j}).$$

By (7.11)

(7.21) $$\begin{aligned}\frac{d}{ds}|||\, U(x)\,||| &= \frac{d}{ds}\Phi_j = \frac{d}{ds}(\|\, \hat{U}_j(x)\,\|^{1/\gamma_j}) \qquad \text{(for some } j\text{)}\\ &= \frac{1}{\gamma_j}\|\, \hat{U}_j\,\|^{1/\gamma_j}\, \|\, \hat{U}_j\,\|^{-1} \frac{d}{ds}\|\, \hat{U}_j\,\|\,.\end{aligned}$$

But

(7.22) $$\begin{aligned}\|\, \hat{U}_j\,\|^{-1}\frac{d}{ds}\|\, \hat{U}_j\,\| &= \frac{d}{ds}(\log \|\, \hat{U}_j\,\|) = \frac{d}{ds}(\mathrm{Re}\log |\, U_{jl}(x)\,|) \qquad \text{(for some } l\text{)}\\ &= \mathrm{Re}\frac{d}{ds}(\log U_{jl}(x)) = \mathrm{Re}\left(\frac{1}{U_{jl}(x)} x^{\sigma+1} \frac{dU_{jl}}{dx} \frac{1}{x^{\sigma+1}} \frac{dx}{ds}\right)\end{aligned}$$

$$= \mathrm{Re}\left[\left\{v_j + \delta_{l-1}\frac{U_{j,l-1}}{U_{jl}} + \sum C_{lk}(x, V(x))\frac{U_{jk}(x)}{U_{jl}(x)} + \right.\right.$$

$$\left.\left. + \sum \frac{U(x)^p}{U_{jl}(x)} H_{pjl}(x, V(x))\right\} \frac{1}{x^{\sigma+1}}\frac{\mathrm{d}x}{\mathrm{d}s}\right].$$

where $\delta_{l-1} = 0$ if $l = 1$ or $m_i + 1$, $(i = 1, \ldots, s - 1)$, C_{lk} is the (l, k)-element of the matrix $C(x, V(x))$.

Since $\| V(x) \|$ is uniformly bounded in the domain (7.13), by (2.12), we can choose ξ and δ such that $\max | C_{lk}(x, V(x)) |$ is as small as one wishes in (7.12). Also, without loss of generality, δ_{l-1} and $\| H_p(x, V(x)) \|$ can be assumed as small as one wishes.

Furthermore,

$$\left| \frac{U(x)^p}{U_{jl}(x)} \right| \leqq \frac{\| \hat{U}_1(x) \|^{|\hat{p}_1|} \| \hat{U}_2(x) \|^{|\hat{p}_2|} \ldots \| \hat{U}_s(x) \|^{|\hat{p}_s|}}{\| U_j(x) \|} = \tag{7.23}$$

$$= \left(\frac{\Phi_1(x)}{\Phi_j(x)}\right)^{\gamma_1|\hat{p}_1|} \left(\frac{\Phi_2(x)}{\Phi_j(x)}\right)^{\gamma_2|\hat{p}_2|} \cdots \left(\frac{\Phi_s(x)}{\Phi_j(x)}\right)^{\gamma_s|\hat{p}_s|} \leqq 1,$$

for p and j such that $\hat{H}_{pj}(x, v) \not\equiv 0$; namely, $\gamma_j = \gamma_1 | \hat{p}_1 | + \ldots + \gamma_s | \hat{p}_s |$. Thus, we have for x on Γ_{x_1},

$$| \delta_{l-1} | + m \,.\, \max | C_{lk}(x, V(x)) | + \sum_{p \in \mathscr{R}} \left| \frac{U(x)^p}{U_{jl}(x)} \right| | H_{pjl}(x, V(x)) | < \tag{7.24}$$

$$< \frac{\| v \|' \sin 2\sigma\varepsilon}{2},$$

where $\| v \|' = \min \{| v_j |\}$, $(j = 1, \ldots, s)$.

Since, for x on Γ',

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \mp e^{i(L(t)+t)}, \tag{7.25}$$

according as t satisfies $\arg x_1 \leqq t \leqq \theta_+ + 2\varepsilon$ or $\theta_- - 2\varepsilon \leqq t \leqq \arg x_1$. Hence, by (7.22), (7.24) and (7.25), we get

$$\| U_j \|^{-1} \frac{\mathrm{d}}{\mathrm{d}s} \| U_j \| \geqq$$

$$\geqq \pm \varrho^{-\sigma-1} \mathrm{Re}\,\{| v_j | \cos(L(t) - \sigma t + \arg v_j)\} - \frac{\| v \|' \sin 2\sigma\varepsilon}{2\varrho^{\sigma+1}} \geqq$$

$$\geqq \frac{\| v_j \| \sin 2\sigma\varepsilon}{\varrho^{\sigma+1}} - \frac{\| v \|' \sin 2\sigma\varepsilon}{2\varrho^{\sigma+1}} = \frac{(2 | v_j | - \| v \|'\}) \sin 2\sigma\varepsilon}{2\varrho^{\sigma+1}}.$$

Substitute this into (7.21), since $2| v_j | - \| v \|' \geqq \gamma_j$, we get (7.15) for x on Γ'.

For x on Γ'', note that

$$\operatorname{Re}\left(\frac{1}{x^{\sigma+1}} v_j \frac{dx}{ds}\right) = \operatorname{Re}\left(\frac{e^{-\sigma t}}{\varrho^{\sigma+1}} v_j\right) =$$

$$= \frac{|v_j|}{\varrho^{\sigma+1}} \cos(\arg v_j - \sigma t) \geqq \frac{|v_j| \sin 2\sigma\varepsilon}{\varrho^{\sigma+1}}.$$

By the use of (7.22), (7.24) and (7.25) again, (7.15) follows immediately for x on Γ''. Consequently, (7.15) is proved on Γ_{x_1}.

In order to show (7.16), notice that

$$\operatorname{Re}\{\Lambda_j(x)\} = \frac{-|v_j|}{\sigma\varrho^{\sigma}} \cos(\arg v_j - \sigma t). \tag{7.26}$$

We have, by the use of (7.25),

$$\frac{d}{ds}\{\operatorname{Re} \Lambda_j(x)\} \geqq \frac{|v_j| \sin 2\sigma\varepsilon}{\varrho^{\sigma+1}} \tag{7.27}$$

for x on Γ_{x_1}.

Let N be a positive integer such that

$$N \sin 2\sigma\varepsilon \geqq 4\|v\|. \tag{7.28}$$

Then, by (7.15), (7.26) and (7.28),

$$\frac{d}{ds}(|||U(x)|||^N e^{-\operatorname{Re}\Lambda_j(x)}) =$$

$$= \left(N |||U(x)|||^{-1} \frac{d}{ds} |||U(x)||| - \frac{d}{ds} \operatorname{Re} \Lambda_j(x)\right) |||U(x)|||^N e^{-\operatorname{Re}\Lambda_j(x)} \geqq$$

$$\geqq \left(\frac{N \sin 2\sigma\varepsilon}{2|x|^{\sigma+1}} - \frac{\|v\|}{|x|^{\sigma+1}}\right) |||U(x)|||^N e^{-\operatorname{Re}\Lambda_j(x)} \geqq$$

$$\geqq \frac{N \sin 2\sigma\varepsilon}{4|x|^{\sigma+1}} |||U(x)|||^N e^{-\operatorname{Re}\Lambda_j(x)}.$$

Thus, Lemma F is proved.

8. PROOF OF THEOREM 3

Let (x_1, u^1, v^1) be an aribitrary point in the domain $(6.14)_N$ and $\{U(x), V(x)\}$ be the holomorphic solution of (C) satisfying $U(x_1) = u^1$ and $V(x_1) = v^1$. Notice first that the equations (6.11) are equivalent to the system of integral equations:

$$
(8.1)\qquad \begin{cases} P(x_1, u^1, v^1) = \displaystyle\int_0^{x_1} x^{-\sigma-1}\, 1_m(e^{-A(x)})\, F\{x, U(x), V(x); \\ \qquad\qquad 1_m(e^{A(x)})\, P(x, U(x), V(x)), Q(x, U(x), V(x))\}\, dx, \\ Q(x_1, u^1, v^1) = \displaystyle\int_0^{x_1} x^{-1} G\{x, U(x), V(x); \\ \qquad\qquad 1_m(e^{A(x)})\, P(x, U(x), V(x)), Q(x, U(x), V(x))\}\, dx, \end{cases}
$$

where the path for the first integration is taken along Γ_{x_1} defined in § 7, and that for the second integration is taken along $\overline{Ox_1}$.

The successive approximations of (8.1) are defined to be the sequence of functions $\{P^{(\chi)}(x_1, u^1, v^1), Q^{(\chi)}(x_1, u^1, v^1)\}$ $(\chi = 0, 1, 2, \ldots)$ given recursively by the formulas:

$$
(8.2)\qquad P^{(0)}(x_1, u^1, v^1) \equiv 0, \qquad Q^{(0)}(x_1, u^1, v^1) \equiv 0,
$$

and

$$
(8.3)\qquad \begin{cases} P^{(\chi+1)}(x_1, u^1, v^1) = \displaystyle\int_0^{x_1} x^{-\sigma-1}\, 1_m(e^{-A(x)})\, F\{x, U(x), V(x); \\ \qquad\qquad 1_m(e^{A(x)} P^{(\chi)}(x, U(x), V(x)), Q^{(\chi)}(x, U(x), V(x))\}\, dx, \\ Q^{(\chi+1)}(x_1, u^1, v^1) = \displaystyle\int_0^{x_1} x^{-1} G(x, U(x), V(x); \\ \qquad\qquad 1_m(e^{A(x)})\, P^{(\chi)}(x, U(x), V(x)), Q^{(\chi)}(x, U(x), V(x))\}\, dx, \\ \qquad\qquad (\chi = 0, 1, 2, \ldots). \end{cases}
$$

Here, the path for (8.3) is the same as that for (8.1).

By the definition of Γ_{x_1}, $P^{(1)}(x_1, u^1, v^1)$ and $Q^{(1)}(x_1, u^1, v^1)$

$$
(8.4)\qquad \begin{cases} [P^{(1)}(x_1, u^1, v^1)] \leqq \dfrac{4B_N}{N \sin 2\sigma\varepsilon} \,|||\, u^1 \,|||^N [e^{-\mathrm{Re}\, A(x_1)}], \\ \| Q^{(1)}(x_1, u^1, v^1) \| \leqq \dfrac{2B_N}{N \sin 2\sigma\varepsilon} \,|||\, u^1 \,|||^N. \end{cases}
$$

In fact, the first inequality follows from (6.7) and (7.16) while the second inequality utilizes, in addition, (6.15). Thus we choose

$$
(8.5)\qquad K_N = \frac{8B_N}{N \sin 2\sigma\varepsilon}
$$

and δ'_N satisfies

$$
(8.6)\qquad \max\,(2K_N \delta_N'^N, \quad \delta'_N \max_{\theta_2 \leqq t \leqq \bar{\theta}_2} \| \chi(t) \|) < d_N.
$$

Hence, we have

$$(8.7)\qquad \begin{cases} [P^{(1)}(x_1, u^1, v^1)] \leqq \dfrac{K_N}{2} ||| u^1 |||^N [e^{-\mathrm{Re}\,\Lambda(x_1)}], \\ \| Q^{(1)}(x_1, u^1, v^1) \| \leqq \dfrac{K_N}{2} ||| u^1 |||^N, \end{cases}$$

and $P^{(1)}$, $Q^{(1)}$ are holomorphic for (x_1, u^1, v^1) in $(6.14)_N$. This can be seen with an argument similar to that in [1].

By means of mathematical induction and by the use of (6.7), (6.9), (7.6) and (6.15), we can prove that $P^{(\chi)}(x_1, u^1, v^1)$ and $Q^{(\chi)}(x_1, u^1, v^1)$ given in (8.3) are all well defined, holomorphic for (x_1, u^1, v^1) in $(6.14)_N$ and satisfy

$$(8.8)\qquad \begin{cases} [P^{(\chi)}(x_1, u^1, v^1) - P^{(\chi-1)}(x_1, u^1, v^1)] \leqq \dfrac{K_N}{2^\chi} ||| u^1 |||^N [e^{-\mathrm{Re}\,\Lambda(x_1)}], \\ [P^{(\chi)}(x_1, u^1, v^1)] \leqq K_N\left(\dfrac{1}{2} + \dots + \dfrac{1}{2^\chi}\right) ||| u^1 |||^N [e^{-\mathrm{Re}\,\Lambda(x_1)}], \\ \| Q^{(\chi)}(x_1, u^1, v^1) - Q^{(\chi-1)}(x_1, u^1, v^1) \| \leqq \dfrac{K_N}{2^\chi} ||| u^1 |||^N, \\ \| Q^{(\chi)}(x_1, u^1, v^1) \| \leqq K_N\left(\dfrac{1}{2} + \dots + \dfrac{1}{2^\chi}\right) ||| u^1 |||^N, \qquad (\chi = 1, 2, 3, \dots), \end{cases}$$

for (x_1, u^1, v^1) in $(6.14)_N$.

From these, we can prove that $P^{(\chi)}$ and $Q^{(\chi)}$ converge uniformly and absolutely to $\varphi_N(x_1, u^1, v^1)$ and $\psi_N(x_1, u^1, v^1)$, respectively, which are holomorphic in $(6.14)_N$ and satisfy (6.13). Furthermore, φ_N and ψ_N are solutions of (6.11). Thus, Theorem 3 is proved.

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Po-Fang Hsieh

Department of Mathematics
Western Michigan University
Kalamazoo, Michigan 49001
U.S.A.