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REMARKS ON QUASIVARIETIES OF ALGEBRAS

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A class $\mathcal{K}$ of Ω-algebras (for arbitrarily fixed type or species Ω) is called a quasivariety if $\mathcal{K}$ can be defined by a set of identical implications,

$$\forall x_1, \ldots, x_n(w_1 = w_1' \wedge \ldots \wedge w_n = w_m' \rightarrow w = w')$$

where w's are Ω-words in $x_1, \ldots, x_n$. Every variety is an example of quasivariety. But there are other examples, too, the class of cancellative semigroups being a familiar one. Many other examples can be constructed by the following simple general method. Let S be a finite set of finite Ω-algebras. Then the class $Q(S)$ of algebras embeddable in cartesian products of families of algebras from S is a locally finite quasivariety. This follows from [2]. (A class $\mathcal{K}$ of Ω-algebras is locally finite if finitely generated subalgebras of algebras of $\mathcal{K}$ are finite.)

In general $\mathcal{K}$ is a quasivariety if and only if [3] $\mathcal{K}$ is closed under the formation of (i) subalgebras (ii) cartesian products (iii) direct limits of mono direct systems (direct systems in which all morphisms are mono) (iv) direct limits of epi direct systems. (We remark in passing that (iii) is the categorical way of saying that $\mathcal{K}$ is of local character, i.e., $\mathcal{K}$ contains an algebra A if every finitely generated subalgebra of A is in $\mathcal{K}$.) Under a special circumstance the most awkward of the above closure properties, namely (iv), can be omitted. Let us say that $\mathcal{K}$ *has finite basis property for equations* if within $\mathcal{K}$ every system of equations in finite number of variables is equivalent to a finite system in those variables. This is equivalent to saying that every congruence over a finitely generated algebra $A \in \mathcal{K}$ is finitely generated as a subalgebra of $A \times A$. In still other terms our finite basis property can be expressed by saying that every finitely generated subalgebra of an algebra of $\mathcal{K}$ is finitely presented. Clearly locally finite classes have the finite basis property for equations. But there are other examples, too, the class of abelian groups being a familiar one. It follows from Theorem 70 of [1] that the class of commutative monoids also has the finite basis property for equations. Subquasivarieties of a quasivariety with the finite basis property for equations have a simple characterization given by.

Theorem 1. *Let $\mathcal{K}$ be a quasivariety with the finite basis property for equations. Then a subclass $\mathcal{K}'$ of $\mathcal{K}$ is a quasivariety if and only if $\mathcal{K}'$ is closed under the formation of subalgebras and cartesian products and is of local character.*

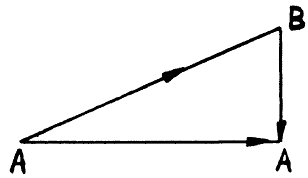
The above is a generalization of Lemma 1 of [2] and is proved by an essentially the same argument. We omit the proof. The result applies to commutative monoids because of Theorem 70 of [1]. We state this observation in the form of

Corollary 1. *A class of commutative monoids is a quasivariety if and only if it is closed with respect to submonoids and cartesian products and is of local character.*

The rest of this note concerns the situation in which a quasivariety has only finitely many subquasivarieties.

Theorem 2. *Let a quasivariety $\mathscr{K}$ be generated by finitely many finite algebras. Let all the subdirectly irreducible algebras of $\mathscr{K}$ be projective in $\mathscr{K}$. Then the lattice $\mathscr{L}_{qv}(\mathscr{K})$ of subquasivarieties of $\mathscr{K}$ is a finite, distributive lattice.*

Proof. Within isomorphism let S be the set of all subdirectly irreducible algebras of $\mathscr{K}$. Then, by the first assumption of the theorem, S is a finite set of finite algebras. Let $\mathscr{K}'$ be a subquasivariety of $\mathscr{K}$. Clearly every algebra of $\mathscr{K}'$ can be represented as a subcartesian product of algebras from S; let $S(\mathscr{K}')$ be the set of all algebras in S that occur in such representations of algebras of $\mathscr{K}'$. We show that $S(\mathscr{K}') \subseteq \mathscr{K}'$. Let $A \in S(K')$. Then there is a subcartesian product B with A as a factor such that $B \in \mathscr{K}'$. Since A is projective the diagram



commutes for some homomorphism $A \to B$, where $B \to A$ is the usual projection map and $A \to A$ is the identity map. It follows that A is embeddable in B. Since $\mathscr{K}'$ is a quasivariety and $B \in \mathscr{K}'$ we conclude that $A \in \mathscr{K}'$. This proves $S(\mathscr{K}') \subseteq \mathscr{K}'$. It follows from this, in view of the definition of $S(\mathscr{K}')$, that $\mathscr{K}' = Q(S(\mathscr{K}'))$. The function $S(\mathscr{K}')$ from the lattice $\mathscr{L}_{qv}(\mathscr{K}')$ into the ring of subsets of $S(= S(\mathscr{K}))$ is, therefore one-to-one. In fact $S(\mathscr{K}')$ is a lattice homomorphism. This follows fairly easily from the fact, mentioned in the beginning of this note, that $Q(S')$ is a quasivariety for every finite set S' of finite algebras. We leave the very easy details and conclude the proof of the teorem.

Remark 1. In the notation of the above proof it is clear that if $A_1, A_2 \in S(\mathscr{K})$, $A_2 \in S(\mathscr{K}')$ and A_1 is embeddable in A_2 then $A_1 \in S(\mathscr{K}')$. We can express this by saying that $S(\mathscr{K}')$ is closed under embeddability. Since $Q(S') \in \mathscr{L}_{qv}(\mathscr{K})$ for all $S' \subseteq \subseteq S(\mathscr{K})$ it follows from the proof of the last theorem that $\mathscr{L}_{qv}(\mathscr{K})$ is isomorphic to the ring of subsets of $S(\mathscr{K})$ that are closed under embeddability.

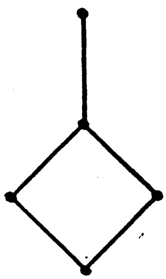
Our next theorem obtains the conclusion of Theorem 2 under somewhat different assumptions.

Theorem 3. *Let a quasivariety $\mathscr{K}$ have only finitely many subquasivarieties and let all subdirectly irreducible algebras in $\mathscr{K}$ be projective in $\mathscr{K}$. Then $\mathscr{L}_{qv}(\mathscr{K})$ is distributive.*

Proof. The theorem is proved on lines of the proof of Theorem 2 except that we need proof of the following crucial point on the basis of our present assumptions: For every $S' \subseteq S(\mathscr{K})$ the class $Q(S')$ is a quasivariety, where $S(\mathscr{K})$ is, as before, the set of subdirectly irreducible algebras of $\mathscr{K}$. This follows from Theorem 2 of [2] which states that if $\mathscr{K}$ is a quasivariety with finitely many subquasivarieties then every subclass $Q(\mathscr{K}'), \mathscr{K}' \subseteq \mathscr{K}$, is a quasivariety.

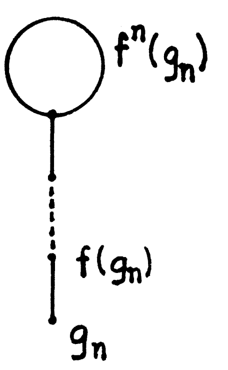
Remark 2. The converse of each of the last two theorems is false. More specifically, there are quasivarieties $\mathscr{K}$ such that $\mathscr{L}_{qv}(\mathscr{K})$ is finite and distributive but not all the subdirectly irreducible algebras of $\mathscr{K}$ are projective. We give an example. Let $\mathscr{V}$ be the variety of left normal semigroups, i.e., semigroups satisfying the identities $x^2 = x$, $xyz = xzy$. The variety $\mathscr{V}$ was shown in [4] to be $Q(\{\Sigma_2^-, \Sigma_2^\circ, \Sigma_3^-\})$, where

Σ_2^-, Σ_2°, Σ_3^- are semigroups defined as follows: Σ_2^- is the two-element semigroup satisfying the identity $xy = x$, Σ_3^- is obtained from Σ_2^- by adding a zero and Σ_2° is the two element semilattice. It was further shown in [4] that $\mathscr{L}_{qv}(\mathscr{V})$ has the graph

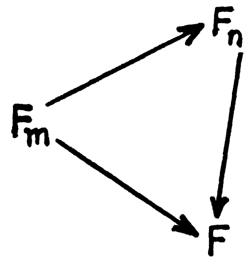


The lattice $\mathscr{L}_{qv}(\mathscr{V})$ is thus finite and distributive. We complete this remark by showing that Σ_3^- is subdirectly irreducible but not projective. Clearly a proper subcartesian factor of Σ_3^- must have two elements and hence should be Σ_2^- or Σ_2°. Since Σ_3^- has a zero while Σ_2^- does not Σ_2^- cannot be a subcartesian factor of Σ_3^-. Hence if Σ_3^- is subdirectly reducible, then Σ_3^- must be a subcartesian product of Σ_2^-. Since Σ_3^- is not a semilattice this is impossible to shat Σ_3^- is subdirectly irreducible. To show that Σ_3^- is not projective let Σ_2^- have elements a, b and let Σ_3^- in addition have 0 as the zero element. The set $\{\langle a, a\rangle, \langle b, b\rangle, \langle o, a\rangle, \langle o, b\rangle\}$ forms a subcartesian product of Σ_3^- and Σ_2^-; call it Σ. The projectivity of Σ_3^- would imply the embeddability of Σ_3^- into Σ. However Σ_3^- can be easily seen not to be embeddable in Σ.

Remark 3. In the last part of Remark 2 we used the fact (also used and proved in the proof of Theorem 2) that if A is projective in a class $\mathscr{K}$ of algebras then (*) A is embeddable in every subcartesian product in $\mathscr{K}$ of which A is a factor. Let us call A semiprojective in $\mathscr{K}$ if A satisfies (*). Equivalently, A is semiprojective in $\mathscr{K}$ if for every epimorphism $B \to A$, $B \in \mathscr{K}$, A is embeddable in B. It is easy to see that semiprojectivity is indeed a weaker property than projectivity. We give an example which we shall find of use later. Let ω_n be the variety of unary algebras $\langle A, f\rangle$ satisfying $f^{n+1}(x) = f^n(x) = f^n(y)$ identically. Let F_n be the free algebra with one generator g_n. We show that F_m is semiprojective in ω_n but not projective if $2 \leq m < n - 2$. First note that if $m \leqq n$ then F_m is isomorphic to a subalgebra of F_n, namely, the subalgebra generated by $f^{n-m}(g_n)$. For this and other easy assertions which we will make without proof it may be helpful to refer to the graph



of F_n. Let Θ: $B \to F_m$ be an epimorphism with $B \in \omega_n$, $m \le n$. Let $b \in B$ be such that $\Theta(b) = g_m$. Then b generate in B an lagebra isomorphic to F_l for some l, $n \ge \ge l \geq m$. Since F_m is embeddable in F_l we see that F_m is embeddable in B. This proves that F_m is semiprojective in ω_n for $m < n$. Assume now that $2 \le m < n-2$. Let α: $F_n \to F_2$, β: $F_m \to F_2$ be epimorphisms; the (unique) existence of such epimorphisms is clear. Let γ: $F_m \to F_n$ be any homomorphism. Then $\gamma(g_m) = f^r(g_n)$ for some integer $r > n - m > 2$. Since $\alpha(f^r(g_n)) = f(g_2)$ we see that $\alpha\gamma(F_m) = = \{f(g_2)\} \ne \beta(F_m) = \{g_2, f(g_2)\}$. Hence for given epimorphisms $F_n \to F_2$, $F_m \to F_2$ there exists no homomorphism $F_m \to F_n$ which makes the diagram commute. Thus F_m is not projective in ω_n. It is easy to see that F_m is subdirectly



irreducible in ω_n for $m \leq n$. We have thus shown that semiprojectivity of A in $\mathscr{K}$ does not imply projectivity of A in $\mathscr{K}$ even when $\mathscr{K}$ is a variety and A is subdirectly irreducible in $\mathscr{K}$.

Remark 4. Theorem 2, Theorem 3, Remark 1 and Remark 2 hold if the condition of projectivity is replaced by that of semiprojectivity.

Theorem 4. *Let all subclasses of a variety $\mathscr{K}$ that are closed under the formation of subalgebras and cartesian products be subvarieties. Then all subdirectly irreducible algebras of $\mathscr{K}$ are semiprojective.*

Proof. Let A be subdirectly irreducible in $\mathscr{K}$. Let $B \to A$ be an epimorphism with $B \in \mathscr{K}$. Consider $Q(\{B\})$. By assumption $Q(\{B\})$ is a variety. Hence $A \in Q(\{B\})$. Since A is subdirectly irreducible this implies that A is embeddable in B. This proves the theorem.

Remark 5. In the last theorem "semiprojective" cannot be replaced by "projective". To show this consider the variety ω_n of Remark 4. It follows from [5] that an algebra in ω_n is subdirectly irreducible if and only if it is isomorphic to F_m, $m \leq n$. From this and the fact that F_m is embeddable in F_n for $m \le n$ we see that $\omega_n = Q(\{F_n\})$. Let $\mathscr{K} \subseteq \omega_n$ be closed under the formation of subalgebras and cartesian products. From $\omega_n = Q(\{F_n\})$ and the semiprojectivity of F_m for $m \le n$ it follows that $\mathscr{K} = \omega_m$ for some $m \le n$ and hence is a variety. However, as noted in Remark 4, if $n > 4$, then not all the subdirectly irreducible algebras of ω_n are projective. Hence in Theorem 4 "semiprojective" cannot be improved to 'projective'. Nor can "variety" be replaced by quasivariety in the last theorem. Thus the variety $\mathscr{V}$ of Remark 2 has [4] the property that if $\mathscr{K} \subseteq \mathscr{V}$ is closed in $\mathscr{V}$ under the formation of subalgebras and cartesian products, then $\mathscr{K}$ is a quasivariety. Yet, as shown in Remark 2, not all subdirectly irreducible algebras of $\mathscr{V}$ are semiprojective.

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