Jiří Rosický

Embeddings of lattices in the lattice of topologies

Archivum Mathematicum, Vol. 9 (1973), No. 2, 49--56

Persistent URL: http://dml.cz/dmlcz/104791

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENNSIS,
IX: 49—56, 1973

EMBEDDINGS OF LATTICES IN THE LATTICE OF TOPOLOGIES

J. Rosický, Brno

(Received December 15, 1971)

R. Duda put the problem (Coll. Math. XXIII, 2 (1971), Problem 749) whether any lattice can be realized as a sublattice of the lattice of all topologies (or even of all T_1-topologies) on a certain set. We even prove that for any lattice L there exists a set E and an embedding ψ of L in the lattice of all topologies on E such that ψx is a completely Hausdorff topology for every $x \in L$. This embedding we get in two steps. Firstly, there exists a set E and a sublattice L' of the lattice of all topologies on E isomorphic to L, which follows from the well-known Whitman's result that any lattice is isomorphic to a sublattice of the lattice of all partitions on a certain set. Secondly, we construct a completely Hausdorff topology $\mathfrak{T}$ on E such that $\psi_2(\mathfrak{S}) = \mathfrak{S} \vee \mathfrak{T}$ for $\mathfrak{S} \in L'$ defines an embedding of L' in the lattice of all topologies on E finer then $\mathfrak{T}$.

This construction is given in § 3. In § 3. it is also shown that there exists a lattice L for which no embedding ψ of L in the lattice of all topologies on a set exists such that ψx is a metrizible topology for every $x \in L$. In addition we give in § 2. another but far simpler proof that any lattice can be embedded in the lattice of all $\mathfrak{T}_1$-topologies on some set.

The author expresses his deep gratitude to M. Sekanina for his encouragement and advice during the preparation of this paper.

§ 1. BASIC NOTIONS

Definitions concerning lattices can be found in [12]. We recall some of them. A mapping φ from a lattice L into a lattice L' is defined to be a $\vee$-homomorphism if $\varphi(a \vee b) = \varphi a \vee \varphi b$ for every $a, b \in L$. Dually we define a $\wedge$-homomorphism. An embedding is an injective homomorphism. A lattice L is called simple if any homomorphism of L onto a lattice L' is either an isomorphism or L' consists of a single element. Let L be a lattice. We put $[a) = \{x \in L / x \geqq a\}$, $(a] = \{x \in L / x \leqq a\}$. The set-theoretic union (intersection) will be denoted by $\cup(\cap)$, a lattice join (meet) by $\bigvee(\bigwedge)$. All necessary topological definitions are given in [4]. We identify a topology with the system of its open sets. The closure of a set X in a topology $\mathfrak{T}$, we denote by $Cl_{\mathfrak{T}}(X)$. A topology $\mathfrak{T}$ on E is called completely Hausdorff if for any two distinct points $a, b \in E$ there exists a continuous function f from $\mathfrak{T}$ to the real line with $fa \neq fb$. Any completely Hausdorff topology is Hausdorff.

We shall give some results concerning lattices of topologies. Let $\mathscr{B}(E)$ be the system of all topologies on a set E ordered by the set-inclusion. $\mathscr{B}(E)$ is a complete lattice. The least element is the indiscrete topology $\{\varnothing, E\}$ and the greatest element is the discrete topology exp E. Meets coincide with set intersections and the join of two topologies $\mathfrak{T}_1$, $\mathfrak{T}_2$ is the topology with the basis $\{V \cap W / V \in \mathfrak{T}_1,\ W \in \mathfrak{T}_2\}$. $\mathscr{B}(E)$ is atomic and any topology is a join of atoms. Atoms are precisely topologies $\{\varnothing, X, E\}$, where $\varnothing \neq X \subsetneqq E$ (see Vaidyanathaswamy [13]). $\mathscr{B}(E)$ is dually atomic and any topology is a meet of dual atoms. Dual atoms are precisely topologies $\mathfrak{G} \cup \cup\ exp\ (E - \{a\})$, where $a \in E$ and $\mathfrak{G}$ is an ultrafilter on E different from the principal ultrafilter generated by a (see Fröhlich [1] or Sekanina [10]). Let $\mathscr{K}(E)$ be the lattice of all $\mathfrak{T}_1$-topologies on E. $\mathscr{K}(E)$ is a complete sublattice of $\mathscr{B}(E)$. The least element in $\mathscr{K}(E)$ is the cofinite topology $\mathfrak{K}(E) = \{X \subseteq E / E - X \text{ is finite}\} \cup \{\varnothing\}$. It holds $(E) = [\mathfrak{K}(E))$. Hence $\mathscr{K}(E)$ is dually atomic. The dual atoms of $\mathscr{K}(E)$ are free ultraspaces, i.e. ultraspaces for which $\mathfrak{G}$ is a free ultrafilter. A topology is called principal if the union of an arbitrary family of its closed sets is closed. Principal topologies form a sublattice of the lattice of topologies (Steiner [11]). More detailed information on lattices of topologies can be found in Larson, Zimmerman [6].

§2. ONE CONSTRUCTION OF EMBEDDINGS OF LATTICES IN THE LATTICE OF $\mathfrak{T}_1$-TOPOLOGIES

It was already mentioned that the starting point of our investigation is the following well-known Whitman's result.

2.1 Theorem. (see [14]): *Any lattice is isomorphic to a sublattice of the lattice of all partitions on a certain set.*

The lattice of all partitions on a set E will be denoted by $\mathscr{P}(E)$. We recall that $\mathfrak{R}_1 \leqq \mathfrak{R}_2$ for $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(E)$ iff for every $X \in \mathfrak{R}_1$ there exists $Y \in \mathfrak{R}_2$ such that $X \subseteq Y$.

From this Whitman's result it follows that any lattice can be embedded in the lattice of topologies. A topology is called a partition topology if every its open set is closed. Let $\mathscr{P}^\circ(E)$ be the system of all partition topologies on E.

2.2. Theorem (see [13]): *$\mathscr{P}^\circ(E)$ is a sublattice of $\mathscr{B}(E)$.*

Proof: Evidently the intersection of two partition topologies is a partition topology. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{P}^\circ(E)$. It is easy to show that $V \cap W$ is open-closed in $\mathfrak{T}_1 \vee \mathfrak{T}_2$ for every $V \in \mathfrak{T}_1$ and $W \in \mathfrak{T}_2$. Any partition topology is a principal topology. Thus $\mathfrak{T}_1 \vee \mathfrak{T}_2$ is a principal topology for principal topologies form a sublattice of $\mathscr{B}(E)$. $\mathfrak{T}_1 \vee \mathfrak{T}_2$ has a basis $\{V \cap W / V \in \mathfrak{T}_1,\ W \in \mathfrak{T}_2\}$ composed of open-closed sets and therefore it is a principal topology.

But $\mathscr{P}^\circ(E)$ is not a complete sublattice of $\mathscr{B}(E)$ as it is stated in [13]. Even the following theorem holds.

2.3. Theorem: *Let E be an infinite set. Then the smallest complete sublattice of $\mathscr{B}(E)$ containing $\mathscr{P}^\circ(E)$ is $\mathscr{B}(E)$ itself.*

Proof: Let $\mathscr{L}$ be the smallest complete sublattice of $\mathscr{B}(E)$ containing $\mathscr{P}^\circ(E)$. At first we prove that any $\mathfrak{T}_1$-topology belongs to $\mathscr{L}$. It is sufficient to show that any free ultratopology belongs to $\mathscr{L}$. Let $\mathfrak{T} = \mathfrak{G} \cup exp\ (E - \{a\})$ be a free ultratopology. $\mathfrak{G} \cup \{E - X / X \in \mathfrak{G}\}$ is a base of $\mathfrak{T}$ composed of open-closed sets. Hence $\mathfrak{T} = \bigvee_{X \in \mathfrak{G}} \{\varnothing, X, E - X, E\}$ and $\{\varnothing, X, E - X, E\} \in \mathscr{P}^\circ(E)$ for every $X \in \mathfrak{G}$. Therefore $\mathfrak{T} \in \mathscr{L}$.

Now we prove that any atom of $\mathscr{B}(E)$ belongs to $\mathscr{L}$. Let $\varnothing \neq X \subsetneqq E$. If $E - X$ is finite, then $\{\varnothing, X, E\} = \{\varnothing, X, E - X, E\} \cap \mathfrak{K}(E) \in \mathscr{L}$. If X and $E - X$ are infinite, then $\{\varnothing, X, E\} = \{\varnothing, X, E - X, E\} \cap (\mathfrak{K}(E) \vee \{\varnothing, X, E\}) \in \mathscr{L}$ because $\mathfrak{K}(E) \vee \{\varnothing, X, E\}$ is a $\mathfrak{T}_1$-topology. Let X be finite. There exist infinite sets $X_1, X_2 \subseteq E$ such that

$E - X_1$, $E - X_2$ are infinite and $X = X_1 \cap X_2$, $E = X_1 \cup X_2$. Thus $\{\varnothing, X, X_1, X_2, E\} = \{\varnothing, X_1, E\} \vee \{\varnothing, X_2, E\} \in \mathscr{L}$. Hence $\{\varnothing, X, E\} = \{\varnothing, X, X_1, X_2, E\} \wedge (\mathfrak{K}(E) \vee \vee \{\varnothing, X, E\}) \in \mathscr{L}$.

Since any topology is a join of atoms, $\mathscr{L} = \mathscr{B}(E)$ holds.

2.4. Theorem (see [9]): *The lattice $\mathscr{P}^\circ(E)$ of all partition topologies on E is isomorphic to the dual of the lattice $\mathscr{P}(E)$ of all partitions on E. This isomorphism α is defined by this way: $\alpha\mathfrak{R} = \{\bigcup X_i / X_i \in \mathfrak{R}\}$ for every $\mathfrak{R} \in \mathscr{P}(E)$.*

2.5. Corollary (see [6]): *Any lattice is isomorphic to a sublattice of the lattice of all topologies on a certain set.*

Proof follows from 2.1., 2.2. and 2.4.

2.6. Lemma: *Let E, F be sets, $\mathfrak{R}$ a partition on F, $\xi\colon E \to \mathfrak{R}$ an injective mapping. Let $\xi_{\mathfrak{R}}(\mathfrak{T}) = \{\bigcup_{x \in X} \xi(x) / X \in \mathfrak{T}\}$ for every $\mathfrak{T} \in \mathscr{B}(E)$. Then the mapping $\xi_{\mathfrak{R}}\colon \mathscr{B}(E) \to \mathscr{B}(F)$ is an embedding.*

Proof is evident.

Let E be a set and $\mathfrak{m}$ an infinite cardinal number. Put $\mathfrak{K}(E, \mathfrak{m}) = \{X \subseteq E / \mathit{card}\, (E - X) < \mathfrak{m}\} \cup \{\varnothing\}$. It is $\mathfrak{K}(E, \mathfrak{m}) \in \mathscr{B}(E)$. It holds $\mathfrak{K}(E, \mathfrak{m}) \subseteq \mathfrak{K}(E, \mathfrak{n})$ for $\mathfrak{m} \leq \mathfrak{n}$. It is $\mathfrak{K}(E, \aleph_0) = \mathfrak{K}(E)$. Larson in [5] proved that $\mathfrak{K}(E, \mathfrak{m})$ and the indiscrete topology are exactly topologies which are the least or the greatest element with respect to some topological property.

2.7. Lemma: *Let E, F be sets, $\mathit{card}\, E = \mathfrak{m}$ and $\mathit{card}\, F = \mathfrak{n}$. Let $\mathfrak{n}$ be regular, $\mathfrak{n} \geq \aleph_0$, $\mathfrak{n} > 2^{\mathfrak{m}}$. Let $\mathfrak{R}$ be a partition on E such that $\mathit{card}\, \mathfrak{R} = \mathfrak{n}$ and $\mathit{card}\, X = \mathfrak{n}$ for every $X \in \mathfrak{R}$. Let $\xi\colon E \to \mathfrak{R}$ be an injective mapping. Let $\psi\mathfrak{T} = \xi_{\mathfrak{R}}(\mathfrak{T}) - \mathfrak{K}(F, \mathfrak{n})$ for every $\mathfrak{T} \in \mathscr{B}(E)$. Then $\psi\colon \mathscr{B}(E) \to [\mathfrak{K}(F, \mathfrak{n}))$ is an embedding.*

Proof: It follows from 2.6. that ψ is a $\vee$-homomorphism. For verifying that ψ is a homormophism it is sufficient to show that $\psi\mathfrak{T}_1 \wedge \psi\mathfrak{T}_2 \leq \psi(\mathfrak{T}_1 \wedge \mathfrak{T}_2)$ for every $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$. At first we prove some property of a topology $\psi\mathfrak{T}$.

Let $\mathfrak{T} \in \mathscr{B}(E)$, $\varnothing \neq X \in \psi\mathfrak{T}$. It is $X = \bigcup_{i \in I} V_i \cap W_i$, where $\varnothing \neq V_i \in \xi_{\mathfrak{R}}(\mathfrak{T})$, $\varnothing \neq W_i \in \mathfrak{K}(F, \mathfrak{n})$ for every $i \in I$ and further $V_i \neq V_j$ for $i \neq j$. It holds $W_i = F - X_i$, where $\mathit{card}\, X_i < \mathfrak{n}$ for every $i \in I$. Hence $X = \bigcup_{i \in I} (V_i - X_i)$. Since $V_i \neq V_j$ for $i \neq j$ and $\mathit{card}\, E = \mathfrak{m}$, it is $\mathit{card}\, I \leqq \mathit{card}\, \xi_{\mathfrak{R}}(\mathfrak{T}) = \mathit{card}\, \mathfrak{T} \leqq 2^{\mathfrak{m}} < \mathfrak{n}$. Hence card $\bigcup_{i \in I} X_i < \mathfrak{n}$ for $\mathfrak{n}$ is regular. From $\bigcup_{i \in I} V_i - \bigcup_{i \in I} X_i \subseteq X \subseteq \bigcup_{i \in I} V_i$ it follows that there exists $V \in \xi_{\mathfrak{R}}(\mathfrak{T})$ and $Y \subseteq F$ with $\mathit{card}\, Y < \mathfrak{n}$ such that $X = V - Y$.

Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $X \in \psi\mathfrak{T}_1 \cap \psi\mathfrak{T}_2$. There exist $V_k \in \mathfrak{T}_k, Y_k \subseteq F$ with $\mathit{card}\, Y_k < \mathfrak{n}$ for $k = 1, 2$ such that $X = V_1 - Y_1 = V_2 - Y_2$. Since the symmetric difference $V_1 \div V_2$ is contained in $Y_1 \cup Y_2$, it holds $\mathit{card}\, (V_1 \div V_2) < \mathfrak{n}$. It is $V_k = \bigcup_{x \in U_k} \xi(x)$, where $U_k \in \mathfrak{T}_k$ for $k = 1, 2$. Hence $V_1 = V_2$ because $\mathit{card}\, \xi(x) = \mathfrak{n}$ for every $x \in E$. Thus $V_1 = V_2 \in \mathfrak{T}_1 \wedge \mathfrak{T}_2$ and from $X = V_1 - Y_1$ it follows that $X \in \psi(\mathfrak{T}_1 \wedge \mathfrak{T}_2)$.

It remains to prove that ψ is injective. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{B}(E)$, $\psi\mathfrak{T}_1 = \psi\mathfrak{T}_2$. Let $X \in \mathfrak{T}_1$. Then $\bigcup_{x \in X} \xi(x) \in \xi_{\mathfrak{R}}(\mathfrak{T}_1) \subseteq \psi\mathfrak{T}_1 = \psi\mathfrak{T}_2$. There exists $V \in \psi_{\mathfrak{R}}(\mathfrak{T}_2)$ and $Y \subseteq F$ with $\mathit{card}\, Y < \mathfrak{n}$ such that $\bigcup_{x \in X} \xi(x) = V - Y$. Further $V - Y = \bigcup_{x \in U} \xi(x) - Y$ for a certain $U \in \mathfrak{T}_2$. Since $\mathit{card}\, \xi(x) = \mathfrak{n}$ for every $x \in E$, it holds $X = U$. Therefore $\mathfrak{T}_1 \subseteq \mathfrak{T}_2$. Analogously we can prove $\mathfrak{T}_2 \subseteq \mathfrak{T}_1$.

2.8. Theorem: *Let $\mathfrak{n}$ be an infinite cardinal number. Any lattice can be embedded in the lattice $[\mathfrak{K}(F, \mathfrak{n}))$ for a certain set F.*

Proof: Let L be a lattice. According to 2.5. there exists a set E such that L can be embedded in $\mathscr{B}(E)$. Let $\mathfrak{m} = card\ E$, $\mathfrak{p} = max\ \{\mathfrak{n}, 2^{\mathfrak{m}}\}$. Let $\mathfrak{p}^+$ be the successor of $\mathfrak{p}$. Since $\mathfrak{p}^+$ is regular, it follows from 2.7. that there exists an embedding ψ: $\mathscr{B}(E) \to [\mathfrak{K}(F, \mathfrak{p}^+))$, where F is a set of cardinality $\mathfrak{p}^+$. Since $[\mathfrak{K}(F, \mathfrak{p}^+))$ is a sublattice of $[\mathfrak{K}(F, \mathfrak{n}))$, the proof is accomplished.

2.9. Corollary: *Any lattice is isomorphic to a sublattice of the lattice of all $\mathfrak{T}_1$-topologies on a certain set.*

The constructed embedding ψ maps elements of a lattice L to topologies structure of which is to be easily clearyfied. For instance they are locally connected and disconnected $\mathfrak{T}_1$-topologies.

§ 3. REPRESENTATIONS OF LATTICES BY MORE SPECIAL TOPOLOGIES

Let $\mathfrak{R}$ be a partition on a set E, $\alpha\mathfrak{R}$ the partition topology from 2.4. Let $\mathscr{P}^0_{\mathfrak{R}}(E) = \mathscr{P}^\circ(E) \cap (\alpha\mathfrak{R}]$. Evidently $\mathscr{P}^0_{\mathfrak{R}}(E)$ is a sublattice of $\mathscr{B}(E)$.

3.1. Lemma: *Let E, F be sets and $\mathfrak{R}$ a partition on F with $card\ E = card\ \mathfrak{R}$. Then the lattices $\mathscr{P}^\circ[E)$ and $\mathscr{P}^0_{\mathfrak{R}}(F)$ are isomorphic.*

Proof: There exists a bijective mapping $\xi : E \to \mathfrak{R}$. Let $\xi_{\mathfrak{R}} : \mathscr{B}(E) \to \mathscr{B}(F)$ be the embedding from 2.6. Evidently $\xi_{\mathfrak{R}}(\mathscr{B}(E)) = (\alpha\mathfrak{R}]$ holds. Since $\xi_{\mathfrak{R}}(T)$ is a partition topology iff $\mathfrak{T}$ is, $\xi_{\mathfrak{R}}/\mathscr{P}^\circ(E) : \mathscr{P}^\circ(E) \to \mathscr{P}^0_{\mathfrak{R}}(F)$ is an isomorphism.

3.2. Lemma: *Let E be a set, $\mathfrak{S} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$ with $card\ \mathfrak{R} > 1$. Let $\varphi : \mathscr{P}^0_{\mathfrak{R}}(E) \to \mathscr{B}(E)$, $\varphi\mathfrak{T} = \mathfrak{S} \vee \mathfrak{T}$ for every $\mathfrak{T} \in \mathscr{P}^0_{\mathfrak{R}}(E)$, be a homomorphism. Then φ is injective* iff $\alpha\mathfrak{R} \nsubseteq \mathfrak{S}$.

Proof: Supposing $\alpha\mathfrak{R} \subseteq \mathfrak{S}$, $\varphi\mathfrak{T} = \mathfrak{S}$ holds for every $\mathfrak{T} \in \mathscr{P}^0_{\mathfrak{R}}(E)$. Since *card* $\mathfrak{R} > 1$, φ is not injective.

Assume that $\alpha\mathfrak{R} \nsubseteq \mathfrak{S}$. Then $\varphi\{\varnothing, E\} = \mathfrak{S} \neq \mathfrak{S} \vee \alpha\mathfrak{R} = \varphi(\alpha\mathfrak{R})$ and $\{\varnothing, E\}, \alpha\mathfrak{R} \in \mathscr{P}^0_{\mathfrak{R}}(E)$. Ore proved in [7] that the lattice of all partitions on a set is simple. Hence it follows from 2.4. and 3.1. that the lattice $\mathscr{P}^0_{\mathfrak{R}}(E)$ is simple. Thus φ is injective.

3.3. Definition: Let E be a set, $\mathfrak{S} \in \mathscr{B}(E)$, $\mathfrak{R} \in \mathscr{P}(E)$.

Let $M \in \alpha\mathfrak{R}$. Let $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(M)$, $\mathfrak{R}/M \leqq \mathfrak{R}_1 \wedge \mathfrak{R}_2$; $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}$. Let $\mathfrak{B}(\mathfrak{S}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2) = \{\langle\{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2}\rangle / \{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2} \subseteq \mathfrak{S}, \bigcup_{X \in \mathfrak{R}_1} (Z^1_X \cap X) = \bigcup_{X \in \mathfrak{R}_2} (Z^2_X \cap X)\}$ be a set of pairs of subsystems of $\mathfrak{S}$. Let $\pi = \langle\{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2}\rangle \in \mathfrak{B}(\mathfrak{S}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)$. Put $A_1(\pi) = \bigcup_{X \in \mathfrak{R}_1} (Z^1_X \cap X) = \bigcup_{X \in \mathfrak{R}_2} (Z^2_X \cap X)$, $A_2(\pi) = \bigcup_{X \in \mathfrak{R}_1} Z^1_X \cup \bigcup_{X \in \mathfrak{R}_2} Z^2_X$, $A(\pi) = A_1(\pi) \cup (A_2(\pi) - M)$. Let $\mathfrak{A}(\mathfrak{S}, \mathfrak{R}, M) = \{A(\pi) / \mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(M), \mathfrak{R}/M \subseteq \mathfrak{R}_1 \wedge \mathfrak{R}_2, \mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}, \pi \in \mathfrak{B}(\mathfrak{S}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)\}$.

Let $\mathfrak{A}(\mathfrak{S}, \mathfrak{R}) = \bigcup_{M \in \alpha\mathfrak{R}} \mathfrak{A}(\mathfrak{S}, \mathfrak{R}, M)$.

3.4. Definition: Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$. Put $\mathfrak{T}^0_{\mathfrak{R}} = \mathfrak{T}$. Suppose that the topologies $\mathfrak{T}^\xi_{\mathfrak{R}}$ are defined for every ordinal $\xi < \alpha$. For an isolated α let $\mathfrak{T}^\alpha_{\mathfrak{R}}$ be the topology generated by the system $\mathfrak{T}^{\alpha-1}_{\mathfrak{R}} \cup \mathfrak{A}(\mathfrak{T}^{\alpha-1}_{\mathfrak{R}}, \mathfrak{R})$. For a limit α let $\mathfrak{T}^\alpha_{\mathfrak{R}} = \bigvee_{\xi < \alpha} \mathfrak{T}^\xi_{\mathfrak{R}}$. We have constructed the transfinite sequence $\mathfrak{T}^0_{\mathfrak{R}} \subseteq \ldots \subseteq \mathfrak{T}^\xi_{\mathfrak{R}} \subseteq \ldots$ of topologies on E. Evidently there exists an ordinal γ such that $\mathfrak{T}^{\xi}_{\mathfrak{R}} = \mathfrak{T}^{\gamma}_{\mathfrak{R}}$ for any $\xi > \gamma$. Let $\mathfrak{T}^{*}_{\mathfrak{R}} = \mathfrak{T}^{\gamma}_{\mathfrak{R}}$.

Let $\varphi\mathfrak{S} = \mathfrak{S} \vee \mathfrak{T}^{*}_{\mathfrak{R}}$ for every $\mathfrak{S} \in \mathscr{P}^{0}_{\mathfrak{R}}(E)$. We get a mapping $\varphi = \varphi(\mathfrak{T}. \mathfrak{R}) : \mathscr{P}^{0}_{\mathfrak{R}}(E) \to$ $\to \mathscr{B}(E)$.

3.5. Lemma: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$. The mapping $\varphi(\mathfrak{T}, \mathfrak{R})$: $: \mathscr{P}^{0}_{\mathfrak{R}}(E) \to \mathscr{B}(E)$ is a homomorphism.*

Proof: We shall prove that for every ordinal β and for every $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{P}^{0}_{\mathfrak{R}}(E)$ it holds $(\mathfrak{T}_1 \vee \mathfrak{T}^{\beta}_{\mathfrak{R}}) \cap (\mathfrak{T}_2 \vee \mathfrak{T}^{\beta}_{\mathfrak{R}}) \subseteq (\mathfrak{T}_1 \cap \mathfrak{T}_2) \vee \mathfrak{T}^{\beta+1}_{\mathfrak{R}}$.

Let β be an ordinal and $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{P}^{0}_{\mathfrak{R}}(E)$. Let $V \in (\mathfrak{T}_1 \vee \mathfrak{T}^{\beta}_{\mathfrak{R}}) \cap (\mathfrak{T}_2 \vee \mathfrak{T}^{\beta}_{\mathfrak{R}})$. From 2.4. it follows that there exist partitions $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$ on E such that $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2 \geq \mathfrak{R}$ and $\mathfrak{T}_i = \alpha\overline{\mathfrak{R}_i}$ for $i = 1, 2$. Evidently $V = \bigcup_{X \in \overline{\mathfrak{R}}_1} (Z^1_X \cap X) = \bigcup_{X \in \overline{\mathfrak{R}}_2} (Z^2_X \cap X)$, where $Z^i_X \in \mathfrak{T}^{\beta}_{\mathfrak{R}}$ for every $X \in \overline{\mathfrak{R}}_i$, $i = 1, 2$. Let $M \in \overline{\mathfrak{R}}_1 \vee \overline{\mathfrak{R}}_2$. It is $M \in \alpha\mathfrak{R}$. Let $\mathfrak{R}_1 = \overline{\mathfrak{R}}_1/M$, $\mathfrak{R}_2 = \overline{\mathfrak{R}}_2/M$ be partitions induced by $\overline{\mathfrak{R}}_1, \overline{\mathfrak{R}}_2$ on M. It holds $\mathfrak{R}/M \leq \mathfrak{R}_1 \wedge \mathfrak{R}_2$ and $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}$. Further $\bigcup_{X \in \mathfrak{R}_1} (Z^1_X \cap X) = V \cap M = \bigcup_{X \in \mathfrak{R}_2} (Z^2_X \cap X)$. Hence $\pi_M =$ $= \langle \{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in \mathfrak{R}_2} \rangle \in \mathfrak{B}(\mathfrak{T}^{\beta}_{\mathfrak{R}}, \mathfrak{R}, M, \mathfrak{R}_1, \mathfrak{R}_2)$. Thus $A(\pi_M) \in \mathfrak{A}(\mathfrak{T}^{\beta}_{\mathfrak{R}}, \mathfrak{R}) \subseteq \mathfrak{T}^{\beta+1}_{\mathfrak{R}}$. It holds $A(\pi_M) \cap M = A_1(\pi_M) \cap M = V \cap M$. Therefore $V = \bigcup_{M \in \overline{\mathfrak{R}}_1 \vee \overline{\mathfrak{R}}_2} (A(\pi_M) \cap M)$. Since $\mathfrak{T}_1 \cap \mathfrak{T}_2 = \alpha(\overline{\mathfrak{R}}_1 \vee \overline{\mathfrak{R}}_2)$, it holds $V \in (\mathfrak{T}_1 \cap \mathfrak{T}_2) \vee \mathfrak{T}^{\beta+1}_{\mathfrak{R}}$.

Since $\mathfrak{T}^{*}_{\mathfrak{R}} = \mathfrak{T}^{\gamma}_{\mathfrak{R}} = \mathfrak{T}^{\gamma+1}_{\mathfrak{R}}$ for a certain ordinal γ, it holds $\varphi\mathfrak{T}_1 \cap \varphi\mathfrak{T}_2 = (\mathfrak{T}_1 \vee \mathfrak{T}^{*}_{\mathfrak{R}}) \cap$ $\cap (\mathfrak{T}_2 \vee \mathfrak{T}^{*}_{\mathfrak{R}}) \subseteq (\mathfrak{T}_1 \cap \mathfrak{T}_2) \vee \mathfrak{T}^{*}_{\mathfrak{R}} = \varphi(\mathfrak{T}_1 \cap \mathfrak{T}_2)$. Therefore φ is a homomorphism because according to the definition φ is a $\vee$-homomorphism.

3.6. Lemma: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R}$ a partition on E such that every element of $\mathfrak{R}$ is dense in $\mathfrak{T}$. Let $M \in \alpha\mathfrak{R}$ and $A(\pi) \in \mathfrak{A}(\mathfrak{T}, \mathfrak{R}, M)$. Let $V \in \mathfrak{T}$, $\mathfrak{T} \in \mathfrak{R}$ and $V \cap$ $\cap A(\pi) \cap T = \varnothing$. Then $V \cap A_2(\pi) = \varnothing$ holds.*

Proof: Let $T \cap M = \varnothing$. Then $V \cap A_2(\pi) \cap T = V \wedge A(\pi) \cap T = \varnothing$ because $A_1(\pi) \subseteq M$. Since T is dense in $\mathfrak{T}$ and $V \cap A_2(\pi) \in \mathfrak{T}$, it holds $V \cap A_2(\pi) = \varnothing$.

Let $T \cap M \neq \varnothing$. Then $T \subseteq M$. It is $\pi = \langle \{Z^1_X\}_{X \in \mathfrak{R}_1}, \{Z^2_X\}_{X \in X_2} \rangle \in \mathfrak{B}(\mathfrak{T}, \mathfrak{R}, M,$ $\mathfrak{R}_1, \mathfrak{R}_2)$ for suitable $\mathfrak{R}_1, \mathfrak{R}_2 \in \mathscr{P}(M)$ with $\mathfrak{R}/M \leqq \mathfrak{R}_1 \wedge \mathfrak{R}_2$ and $\mathfrak{R}_1 \vee \mathfrak{R}_2 = \{M\}$. There exist $X_1 \in \mathfrak{R}_1$, $X_2 \in \mathfrak{R}_2$ such that $T \subseteq X_1 \cap X_2$.

Let $X \in \mathfrak{R}_1 \cup \mathfrak{R}_2$. According to the construction of joins in the lattice $\mathscr{P}(E)$ there exist $T_i \in \mathfrak{R}_1 \cup \mathfrak{R}_2$ for $i = 1, \ldots, n$ such that $T_1 = X_1$, $T_n = X$ and $T_i \cap T_{i+1} \neq \varnothing$ for $i = 1, \ldots, n-1$. It holds $\varnothing = V \cap A(\pi) \cap T \supseteq V \cap A_1(\pi) \cap T \supseteq V \cap (Z^1_{T_1} \cap$ $\cap T_1) \cap T = V \cap Z^1_{T_1} \cap T$. Since T is dense in $\mathfrak{T}$ and $V \cap Z^1_{T_1} \in \mathfrak{T}$, it holds $V \cap$ $\cap Z^1_{T_1} = \varnothing$. Suppose that $V \cap Z^s_{T_k} = \varnothing$, where $k < n$, $T_k \in \mathfrak{R}_s$, $s = 1, 2$. Let $T' =$ $= T_k \cap T_{k+1}$. It is $Z^s_{T_k} \cap T' = \bigcup_{X \in \mathfrak{R}_s} (Z^s_X \cap X) \cap T' = \bigcup_{X \in \mathfrak{R}_r} (Z^r_X \cap X) \cap T' =$ $= Z^r_{T_{k+1}} \cap T'$, where $r \in \{1, 2\}$, $T_{k+1} \in \mathfrak{R}_r$. Hence $\varnothing = V \cap Z^s_{T_k} \cap T_k \supseteq V \cap Z^s_{T_k} \cap$ $\cap T' = V \cap Z^r_{T_{k+1}} \cap T'$. Since T' is dense in $\mathfrak{T}$, it holds $V \cap Z^r_{T_{k+1}} = \varnothing$. It can be concluded that $V \cap Z^t_X = \varnothing$, where $X \in \mathfrak{R}_t$.

Therefore $V \cap A_2(\pi) = \varnothing$ and the proof is accomplished.

3.7. Lemma: *Let E be a set, $\mathfrak{T} \in \mathscr{B}(E)$ and $\mathfrak{R} \in \mathscr{P}(E)$. Let every element of $\mathfrak{R}$ be dense in $\mathfrak{T}$. Then every element of $\mathfrak{R}$ is dense in $\mathfrak{T}^{*}_{\mathfrak{R}}$.*

Proof: We shall use the transfinite induction. Suppose that every element of $\mathfrak{R}$

is dense in $\mathfrak{T}_{\mathfrak{R}}^{\xi}$ for every ordinal $\xi < \beta$. If β is limit, every element of $\mathfrak{R}$ is evidently dense in $\mathfrak{T}_{\mathfrak{R}}^{\beta}$. Let β be isolated. The system of all finite intersections of elements of $\mathfrak{T}_{\mathfrak{R}}^{\beta-1} \cup \mathfrak{A}(\mathfrak{T}_{\mathfrak{R}}^{\beta-1}, \mathfrak{R})$ forms a basis of $\mathfrak{T}_{\mathfrak{R}}^{\beta}$. Let Y be such an intersection. We shall show that $Y \neq \varnothing$ implies $Y \cap T \neq \varnothing$ for every $T \in \mathfrak{R}$. Thereby the proof will be accomplished.

It is $Y = W \cap \bigcap_{i=1}^{n} A(\pi_i)$, where $W \in \mathfrak{T}_{\mathfrak{R}}^{\beta-1}$ and $A(\pi_i) \in \mathfrak{A}(\mathfrak{T}_{\mathfrak{R}}^{\beta-1}, \mathfrak{R})$ for $i = 1, \dots, n$. There exist $M_i \in \alpha\mathfrak{R}$ such that $A(\pi_i) \in \mathfrak{A}(\mathfrak{T}_{\mathfrak{R}}^{\beta-1}, \mathfrak{R}, M_i)$ for $i = 1, \dots, n$. Suppose that $T \in \mathfrak{R}$ exists with $Y \cap T = \varnothing$. There exist $X_i \in \mathfrak{T}_{\mathfrak{R}}^{\beta-1}$ with $X_i \cap T = A(\pi_i) \cap T$ for every $i = 1, \dots, n$. It is $\varnothing = Y \cap T = W \cap \bigcap_{i=1}^{n} A(\pi_i) \cap T = (W \cap \bigcap_{i=1}^{n-1} X_i) \cap \cap A(\pi_n) \cap T$. Since $W \cap \bigcap_{i=1}^{n-1} X_i \in \mathfrak{T}_{\mathfrak{R}}^{\beta-1}$, it follows from 3.6. that $V \cap \bigcap_{i=1}^{n-1} X_i \cap \cap A_2(\pi_n) = \varnothing$. Suppose that $W \cap \bigcap_{i=1}^{n-k} X_i \cap \bigcap_{i=n-k+1}^{n} A_2(\pi_i) = \varnothing$. Then $\varnothing = W \cap \cap \bigcap_{i=n-k+1}^{n} A_2(\pi_i) \cap \bigcap_{i=1}^{n-k} X_i \cap T = (W \cap \bigcap_{i=n-k+1}^{n} A_2(\pi_i) \cap \bigcap_{i=1}^{n-k-1} X_i) \cap A(\pi_{n-k}) \cap T$. 3.6. implies $W \cap \bigcap_{i=1}^{n-k-1} X_i \cap \bigcap_{i=n-k}^{n} A_2(\pi_i) = \varnothing$. We can conclude that $W \cap \bigcap_{i=1}^{n} A_2(\pi_i) = \varnothing$. Since $A(\pi_i) \subseteq A_2(\pi_i)$ for every i, it holds $Y = \varnothing$.

Let $\mathfrak{m}$ be a cardinal number. A topology $\mathfrak{T}$ is called $\mathfrak{m}$-resolvable if it contains $\mathfrak{m}$ pairwise disjoint dense sets. A 2-resolvable topology is called briefly resolvable. The concept of resolvable topologies was introduced by Hewitt ([3]). He proved that every metrizible topology devoid of isolated points is resolvable.

3.8. Lemma: *There exists an $\mathfrak{m}$-resolvable completely Hausdorff topology for any cardinal number $\mathfrak{m}$.*

Proof: Let I be a set, *card* $I = \mathfrak{m}$. Let Q be the set of all rational numbers and $\mathfrak{S}$ the usual topology of Q. Put $E = \prod_{i \in I} Q_i$, $\mathfrak{T} = \prod_{i \in I} \mathfrak{S}_i$, where $Q_i = Q$ and $\mathfrak{S}_i = \mathfrak{S}$ for every $i \in I$. Evidently $\mathfrak{T}$ is completely Hausdorff. Since $\mathfrak{S}$ is resolvable, there exist sets A_i, $B_i = Q_i - A_i$ dense in $\mathfrak{S}_i$ for every $i \in I$. Let $\mathfrak{B} = \{\prod_{i \in I} X_i / X_i = A_i$ or $X_i = B_i\}$. Every element of $\mathfrak{B}$ is dense in $\mathfrak{T}$ and elements of $\mathfrak{B}$ are pairwise disjoint. Since card $\mathfrak{B} = 2^{\mathfrak{m}}$, the topology $\mathfrak{T}$ is $2^{\mathfrak{m}}$-resolvable and therefore it is $\mathfrak{m}$-resolvable.

3.9. Theorem: *For every lattice L there exists a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that ψx is a completely Hausdorff topology for every $x \in L$.*

Proof: Let L be a lattice. According to 2.1 and 2.4. L can be embedded in the lattice of all partition topologies on some set F. According to 3.8. there exists a *card* F-resolvable completely Hausdorff topology $\mathfrak{T}$. Let E be the underlying set of $\mathfrak{T}$. There exists a partition $\mathfrak{R}$ on E every element of which is dense in $\mathfrak{T}$ and *card* $\mathfrak{R} =$ *card* F. From 3.1. it follows that L can be embedded in $\mathscr{P}_{\mathfrak{R}}^{0}(E)$. Let $\varphi = \varphi(\mathfrak{T}, \mathfrak{R}) : \mathscr{P}_{\mathfrak{R}}^{0}(E) \to \mathscr{B}(E)$ be the mapping from 3.4. According to 3.5. φ is a homomorphism. 3.7. implies that every element of $\mathfrak{R}$ is dense in $\mathfrak{T}_{\mathfrak{R}}^{*}$. It follows from 3.2. that φ is injective. Since $\varphi\mathscr{P}_{\mathfrak{R}}^{0}(E) \subseteq [\mathfrak{T}_{\mathfrak{R}}^{*}] \subseteq [\mathfrak{T}]$, every topology from $\varphi\mathscr{P}_{\mathfrak{R}}^{0}(E)$ is completely Hausdorff. The proof is ready.

Since the topology of the rationals numbers is 0-dimensional, every topology ψx is totally disconnected. Even in the same way as the previous theorem we can prove the following one.

3.10. Theorem: *Let $\mathscr{C}$ be a class of topologies with the properties:* $1° \mathfrak{T} \in \mathscr{C} \cap \mathscr{B}(F)$, $\mathfrak{T}' \in \mathscr{B}(F)$, $\mathfrak{T} \subseteq \mathfrak{T}' \Rightarrow \mathfrak{T}' \in \mathscr{C}$

$2°$ *$\mathscr{C}$ contains an $\mathfrak{m}$-resolvable topology for any cardinal number $\mathfrak{m}$.*

Then for any lattice L there exists a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that $\psi L \subseteq \mathscr{C}$.

Analogously as in 3.8. we can show that $\mathscr{C}$ fulfils $2°$ whenever it is closed under products and contains a resolvable topology.

A question arises whether any lattice can be represented by topologies more special than completely Hausdorff. We shall show that for metrizible topologies it is not true.

3.11. Lemma: *Let E be a set and $\mathscr{L}$ a sublattice of $\mathscr{B}(E)$. Let $A \subseteq E$ with $E - A \in \mathfrak{T}$ for every $\mathfrak{T} \in \mathscr{L}$. Then a mapping $\psi_A: \mathscr{L} \to \mathscr{B}(A)$, $\psi_A \mathfrak{T} = \mathfrak{T}/A$ is the relative topology for every $\mathfrak{T} \in L$, is a homomorphism.*

Proof: Evidently ψ_A is isotone. Hence $\psi_A\mathfrak{T}_1 \vee \psi_A\mathfrak{T}_2 \subseteq \psi_A(\mathfrak{T}_1 \vee \mathfrak{T}_2)$ holds for every $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{L}$. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{L}$ and $X \in \psi_A(\mathfrak{T}_1 \vee \mathfrak{T}_2)$. There exist $V_i \in \mathfrak{T}_1$, $W_i \in \mathfrak{T}_2$ for $i \in I$ such that $X = \bigcup_{i \in I} (V_i \cap W_i) \cap A$. Hence $X = \bigcup_{i \in I} [(V_i \cap A) \cap (W_i \cap A) \in \in \psi_A\mathfrak{T}_1 \vee \psi_A\mathfrak{T}_2$.

It holds $\psi_A(\mathfrak{T}_1 \cap \mathfrak{T}_2) \leq \psi_A\mathfrak{T}_1 \cap \psi_A\mathfrak{T}_2$. Let $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathscr{L}$, $X \in \psi_A\mathfrak{T}_1 \cap \psi_A\mathfrak{T}_2$. There exists $V \in \mathfrak{T}_1$ and $W \in \mathfrak{T}_2$ with $X = V \cap A = W \cap A$. It is $(E - A) \cup V \in \mathfrak{T}_1$ and $(E - A) \cup W \in \mathfrak{T}_2$. Since $(E - A) \cup V = (E - A) \cup (V \cap A) = (E - A) \cup \cup (W \cap A) = (E - A) \cup W$, it holds $X \in \psi_A(\mathfrak{T}_1 \cap \mathfrak{T}_2)$.

Let $\mathfrak{m}$ be an infinite cardinal number. A topology $\mathfrak{T}$ on a set E is called $\mathfrak{m}$-generated if it has the following property: $X \in \mathfrak{T}$ iff $X \cap A \in \mathfrak{T}/A$ for every $A \subseteq E$ with *card* $A < < \mathfrak{m}$ (see Herrlich [2]).

3.12. Theorem: *Let $\mathfrak{m}$ be an infinite cardinal number and L be a simple lattice with the least element a. Let there exist a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that ψa is an $\mathfrak{m}$-generated Hausdorff topology. Then card $L \leq 2^{2^{\mathfrak{n}}}$, where $\mathfrak{n} = 2^{2^{\mathfrak{m}}}$.*

Proof: In the case *card* $L = 1$ the theorem holds. Let *card* $L > 1$. Then there exists $b \in L$ with $a < b$. Thus $\psi a \subset \psi b$. There exists $X \subseteq E$ with $X \in \psi b$ and $X \notin \psi a$. Since ψa is $\mathfrak{m}$-generated, there exists $B \subseteq E$ with *card* $B < \mathfrak{m}$ such that $X \cap B \notin \notin \psi a/B$. It is *card* $(B - X) < \mathfrak{m}$ and $Cl_{\psi b}(B - X) \subsetneqq Cl_{\psi a}(B - X)$. Let $C = Cl_{\psi a}(B - - X)$. Since ψa is Hausdorff, every filter on E has at most one limit point in ψa. It implies *card* $C \subseteq 2^{2^{\mathfrak{m}}} = \mathfrak{n}$. Since $E - C \in \psi a \subseteq \psi x$ for every $x \in L$, it follows from 3.11. that $\psi_C : \psi L \to \mathscr{B}(C)$, $\psi_C \mathfrak{T} = \mathfrak{T}/C$ for every $\mathfrak{T} \in \psi L$, is a homomorphism. Since $Cl_{\psi b}(B - X) \subsetneqq C$, it holds $\psi_C\psi b \neq \psi_C\psi a$. Since L is simple, the mapping $\psi_C\psi$ is injective. Therefore *card* $L \leq$ *card* $\mathscr{B}(C)$. Pospíšil proved in [8] that *card* $\mathscr{B}(C) = 2^{2^{card\ C}}$ whenever C is infinite. We have obtained that *card* $L \leq 2^{2^{\mathfrak{n}}}$.

3.13. Corollary: *There exists a lattice L for which no set E exists such that there exists an embedding $\psi : L \to \mathscr{B}(E)$ having the property that ψx is a metrizible topology for every $x \in L$.*

Proof: Evidently any metrizible topology is $\aleph_0$-generated. The result follows from 3.12. and from the existence of simple lattices of an arbitrary cardinality (e.g. the lattice of partitions is always simple).

There is a problem whether for any lattice L there exists a set E and an embedding $\psi : L \to \mathscr{B}(E)$ such that ψx is a (completely) regular $\mathfrak{T}_1$-topology.

REFERENCES

[1] Fröhlich O., *Das Halbordnungssystem der topologischen Räume auf einer Menge*, Math. Annalen 156 (1964), 79—95.

[2] Herrlich H., *Topologische Reflexionen und Coreflexionen*, Lecture Notes 78 (Berlin 1968).

[3] Hewitt E., *A problem of set-theoretic topology*, Duke Math. Jour. 10 (1943), 309—333.

[4] Kuratowski K., *Topology*, Russian translation, Moscow 1966.

[5] Larson R. E., *Minimum and maximum topological spaces*, Bull. Acad. Pol. Sci. XVIII (1970), 707—710.

[6] Larson R. E. and Zimmerman S. J., *The lattice of topologies: A survey*, to appear.

[7] Ore O., *Theory of equivalence relations*, Duke Math. Jour. 9 (1942), 573—627.

[8] Pospíšil B., *Sur le nombre des topologies d'une ensemble donné*, Čas. pro pěst. mat. 67 (1938), 100—102.

[9] Rayburn M. C., Jr., *On the lattice of compactifications and the lattice of topologies*, Ph. D. dissertation, Univ. of Kentucky, 1969.

[10] Sekanina M., *Sistěmy topologij na dannom množestve*, Czech. Math. Jour. T 15 (90), 1965, 9—29.

[11] Steiner A. K., *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc. 122 (1969), 379—398.

[12] Szasz G., *Einführung in die Verbandstheorie*, Budapest 1962.

[13] Vaidyanathaswamy R., *Treatise on set topology*, Madras 1947.

14] Whitman P. M., *Lattices, equivalence relations, and subgroups*, Bull. Amer. Math. Soc. LII (1946), 507—522.

J. Rosický
Department of Mathematics, J. E. Purkyně University
Brno, Janáčkovo nám. 2a
Czechoslovakia