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ON PRODUCTS OF SEMI-DYNAMICAL SYSTEMS

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INTRODUCTION

Semi-dynamical systems (s.d.s.) are continuous fllows defined only in the "future". Natural examples of s.d.s. are furnished by functional differential equations for which existence and uniqueness conditions hold [7], and by Volterra Integral Equations [8]. Not all the results in dynamical systems extend to s.d.s. Indeed even the basic properties of *positive* prolongations do not hold in s.d.s. [4]; the 'past' does affect the 'future'. Moreover, many new interesting notions (e.g., a singular point, a start point, escape time) arise in s.d.s. For a family of s.d.s., the product s.d.s. is defined in a natural way. This paper deals with product s.d.s. with reference to singular points.

After stating the basic concepts, a product s.d.s. is defined, notion of proper/improper singular point introduced, and conditions obtained for existence of an improper singular point. It is shown that in the presence of singular points in at least two factor s.d.s. (path) connectedness of the set of proper singular points is equivalent to that of the product space. The case where only one of the factor s.d. s. contains a singular point is also discussed. Finally it is shown that in the presence of an improper singular point, the (path) connectedness of either of the product space, the set of proper singular points, and the set of singular points implies that of the rest. Since in a Hausdorff space, notions of path connectedness and arc-wise connectedness are equivalent [9], similar theorems can be stated for arc-wise connectedness.

1. **Definitions** Let X be a topological space and R^+ the set of nonnegative reals with usual topology. Then a continuous map π from $X \times R^+$ into X is said to define a *semi dynamical system* (s.d.s.) if $\pi(x, 0) = x$ (identity axiom) and $\pi(\pi(x, t), s) = \pi(x, t + s)$ (semigroup axiom) hold for each x in X and t, s in R^+. As usual (e.g., [1], [5]) we denote $\pi(x, t)$ by xt, the set $\{xt: x\varepsilon M \subset X, t\varepsilon K \subset R^+\}$ by MK. Positive trajectory, critical points, etc., are defined as in dynamical systems.

A semi-dynamical system is said to have unicity if $xt = yt$ implies $x = y$ for all x, y in X and t in R^+. Define maps E and L from X into extended non-negative reals by $E(x) = \sup \{t \geqq 0: yt = x$ for some y in $X\}$ and $L(x) = \sup \{t \geqq 0: yt = x$ for a unique y in $X\}$, $x\varepsilon X$. $E(x)$ is called the *escape time* of x, and $L(x)$ the *extent of unicity* [6, p. 168]. A point x is said to be a *start point* if its excape time vanishes. Some properties of start points are discussed in [3]. A point x which is not a start point is said to be *singular* if its extent of unicity is zero.

2. **Proposition** In a semi-dynamical system (X, π), the set of start points has an empty interior. Equivalently, $U - S$ is non-empty whenever U is a non-empty open set and S the set of start points. Moreover, X is (path) connected [9] if and only if $X - S$ is (path) connected.

3. **Proposition** Let (X_α, π_α), $\alpha \varepsilon I$ be a family of s.d.s. Let $X = \Pi X_\alpha$ be the produc. space. Let π be a map from $X \times R^+$ into X defined by $\pi(x, t) = \{x_\alpha t\}$, $x = \{x_\alpha\}$ Then (X, π) is a s.d.s., called the *direct product* (or simply *product*) of the family (X_α, π_α), $\alpha \varepsilon I$ of s.d.s.

4. **Remark** Let (X_α, π_α), $\alpha \varepsilon I$ and (X, π) be as above. Clearly singular points exist in product s.d.s. if and only if some factor s.d.s. contains singular points. If $x \varepsilon X$, $x = \{x_\alpha\}$ is not a start point and x_α is singular for some α, then x will be singular; however, if x is singular, none of x_α need be. Consequently, we have the following.

5. **Definition** Let (X_α, π_α), $\alpha \varepsilon I$ and (X, π) be as above. Let $x \varepsilon X$, $x = \{x_\alpha\}$ be singular. Then, relative to the factorization ΠX_α of X, x is said to be proper singular if x_α is singular for some α; otherwise, call x to be improper singular.

6. **Notation** Throughout the rest of the paper, (X_α, π_α), $\alpha \varepsilon I$ denotes a family of s.d.s. and (X, π) the product s.d.s. The sets of start points and singular points in (X_α, π_α) will be denoted by S_α, P_α respectively; S and P will denote the corresponding sets in (X, π). The set of proper singular points will be denoted by P^*, so that $P - P^*$ denotes the set of improper singular points. For any β in I, S^β denotes the set of start points in the product s.d.s. of the family (X_α, π_α), $\alpha \varepsilon I - \{\beta\}$ of s.d.s. Finally E and L denote the maps defined in § 1 above.

7. **Theorem** *Let $t > 0$. Let $K(t) = \{\alpha \varepsilon I\colon 0 < L(x_\alpha) \leqq t \leqq E(x_\alpha) - L(x_\alpha)$ for some x_α in $X_\alpha\}$. The set of improper singular points is non empty if and only if there exists $T > 0$ such that*

(a) *$K(T)$ is infinite.*

(b) *for each $\alpha \varepsilon I - K(T)$, there exists an x_α in X_α such that $T - E(x_\alpha) \leqq 0 < L(x_\alpha)$. (For any α in $K(T)$, condition obviously holds).*

Proof. Let there exist T as stated in the theorem. Let $\{\alpha_1, \alpha_2, \ldots\}$ be a countable infinite subset of $K(T)$, and for each n pick x_{α_n} of the definition of $K(T)$. We may suppose $(\{L(x_{\alpha_n}))$ to be decreasing. Let $\{s_n\}$, $0 < s_n < L(x_{\alpha_n})$ be a sequence converging to zero. Pick $y \varepsilon X$, $y = \{y_\alpha\}$, such that (i) for any n, $y_{\alpha_n}(L(x_{\alpha_n}) - s_n) = x_{\alpha_n}$ and (ii) for $\alpha \neq \alpha_n$, $T - E(x_\alpha) \leqq 0 < L(x_\alpha)$. Clearly y is an improper singular point.

Proof of the converse is left to the reader.

In the presence of an improper singular point, the set of singular points is dense everywhere. The following theorem indicates when the set of (proper) singular points is dense everywhere.

8. **Theorem** [2, p. 285]. *The following are equivalent:*

(i) *At least one of the following holds:*

(a) *P_β is dense in X_β for some β in I.*

(b) *Infinitely many factor s.d.s. contain singular points.*

(ii) *The set of proper singular points is dense everywhere.*

(iii) *The set of singular points is dense in X.*

In what follows, the set $\{x\}$ containing a single point will be denoted by (x).

9. **Theorem** *Let singular points exist in at least two factor s.d.s. Then the set of proper singular points is (path) connected if and only if the product space is (path) connected.*

Proof. Let X be (path) connected. Let z, z', $z = \{z_\alpha\}$, $z' = \{z'_\alpha\}$ be proper singular points so that $z_\beta \varepsilon P_\beta$, $z'_\gamma \in P_\gamma$ for some β, γ in I.

If $\beta \neq \gamma$, let $K_1 = (z_\beta) \times (\prod_{\alpha \neq \beta} X_\alpha - S^\beta)$ and $K_2 = (z'_\gamma) \times (\prod_{\alpha \neq \gamma} X_\alpha - S^\gamma)$. Since K_1, K_2 are (path) connected and $K_1 \cap K_2 \neq \varnothing$, therefore, $K_1 \cup K_2 \subset P^*$. Thus z, z' lie is a (path) connected set $K_1 \cup K_2 \subset P^*$.

If $\beta = \gamma$, pick $\mu \neq \beta$ such that P_μ is non-empty. Let $x_\mu \varepsilon P_\mu$. Consider the (path) connected sets.

$K_1 = (z_\beta) \times (\underset{\alpha \neq \beta}{\Pi} X_\alpha - S_\beta)$, $K_2 = (z_\beta') \times (\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$ and $K_3 = (x_\mu) \times (\underset{\alpha \neq \mu}{\Pi} X_\alpha - S^\mu)$.

Since $K_1 \cap K_3 \neq \varnothing$ and $K_2 \cap K_3 \neq \varnothing$, therefore, $K_1 \cup K_2 \cup K_3$ is also (path) connected. Moreover, $z \varepsilon K_1, z' \varepsilon K_2$ and $K_1 \cup K_2 \cup K_3 \subset P^*$. Hence P^* is (path) connected.

Next, let P^* be (path) connected. It is sufficient to prove that $(X_\alpha - S_\alpha)$ is (path) connected for each $\alpha \in I$. Let $z_\beta, z_\beta' \in X_\beta - S_\beta$ for any $\beta \in I$. Pick $\mu \neq \beta$ such that P_μ is non-empty. Let $x_\mu \in P_\mu$. For each $\alpha \neq \beta, \mu$ pick $x_\alpha \in X_\alpha$. Choose $y, y' \in X, y = \{y_\alpha\}, y' = \{y_\alpha'\}$ $y_\mu' = y_\mu = x_\mu, y_\beta = z_\beta, y_\beta' = z_\beta'$ and $y_\alpha = x_\alpha T = y_\alpha'$ if $\alpha \neq \beta, \mu$ where $T > 0$ is arbitrary but fixed. Clearly $y, y' \in P^*$. Since P^* is (path) connected, proj_β (P^*) is (path) connected. But $z_\beta, z_\beta' \in \text{proj}_\beta (P^*) = X_\beta - S_\beta$, etc.

10. **Remark** If the set of proper singular points is (path) connected, the set of singular points in any factor s.d.s. is not necessarily so.

11. **Theorem** *Let P_β be non-empty for unique β in I. The set of singular points is (path) connected if and only if both the following conditions hold:*

(a) *P_β is (path) connected.*

(b) *X_α is (path) connected for each $\alpha \in (I - \beta)$.*

We need a lemma.

12. **Lemma.** *Let P_β be non empty. Then*

$$(P_\beta \times \underset{\alpha \neq \beta}{\Pi} X_\alpha) - S = P_\beta \times (\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$$

Proof. Let $x \in X$, $x = \{x_\alpha\} = x_\beta \times x^\beta$ where $x^\beta \in \underset{\alpha \neq \beta}{\Pi} X_\alpha$. Since $x^\beta \in S^\beta$ implies $x \in S$ and $x_\beta \in P_\beta$ implies $x_\beta \notin S_\beta$, therefore, $x \in ((P_\beta \times \underset{\alpha \neq \beta}{\Pi} X_\alpha) - S)$ iff ($x_\beta \in P_\beta$ and $x^\beta \notin S^\beta$) iff ($x_\beta \in P_\beta$ and $x^\beta \in (\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$) iff $x \in P_\beta \times (\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$.

Proof of Theorem 11. Hypothesis implies that each singular point in (X, π), is proper. Now $P = P^* = (P_\beta \times \underset{\alpha \neq \beta}{\Pi} X_\alpha) - S = P_\beta \times (\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$ therefore, P is (path) connected if and only if both P_β and $(\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$ are (path) conected. But (path) connectedness of $(\underset{\alpha \neq \beta}{\Pi} X_\alpha - S^\beta)$ is equivalent ot that of $\underset{\alpha \neq \beta}{\Pi} X_\alpha$ (Prop. 2), i.e., of X_α for each $\alpha \neq \beta$.

13. **Theorem** *Let there exist an improper singular point. The following are equivalent:*

(a) *X is (path) connected.*

(b) *The set of proper singular pionts is (path) connected.*

(c) *The set of singular pionts is (path) connected.*

Proof. It is easy to see that existence of an improper singular point implies that the set $\{\alpha \in I: P_\alpha \neq \varnothing\}$ is infinite, and so, (a), (b) are equivalent. We prove that (b) implies (c) which, in turn, implies (a). Since $P^* \subset P \subset X$ and P^* is dense [Th. 8] in X, connectedness of p^* implies that of P, and connectedness of P implies that of X.

Let P^* be path connected. Let $z \in X$, $z = \{z_\alpha\}$ be an improper singular point. Pick $\beta \in I$ such that z_β has its extent of unicity $L(z_\beta)$ less than the escape time. Then for unique $z_\beta' \in X_\beta$ condition $z_\beta' L(z_\beta) = z_\beta$ holds. Choose $y \in X$, $y = \{y_\alpha\}$ by taking $y_\beta = z_\beta'$ and $y_\alpha = z_\alpha$ for each $\alpha \neq \beta$. Clearly y is a proper singular point. But y, z can be joined by a path in P, i.e., an improper singular point can be joined by a path, in P, to some proper singular point, etc.

Now let P be path connected. Let x_β, $x'_\beta \in X_\beta - S_\beta$ be arbitrary for any $\beta \in I$. Let $y \in P - P^*$, $y = \{y_\alpha\}$. Choose z, $z' \in X$, $z = \{z_\alpha\}$, $z' = \{z'_\alpha\}$ such that $z_\beta = x_\beta$, $z'_\beta = x'_\beta$ and $z_\alpha = y_\alpha = z'_\alpha$ whenever $\alpha \neq \beta$. Then $z, z' \in P$. Let f: $[0, 1] \to P$ be a path joining z and z'. Clearly $\text{proj}_\beta \circ f$: $[0, 1] \to X_\beta - S_\beta$ is a path joining x_β and x'_β. Hence, etc.

14. **Remark** If there exists an improper singular point, then, in general, none of the implications in "X is (path) connected iff $P - P^*$ is (path) connected" holds. Examples can easily be constructed to verify this statement.

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